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CONTINUOUS DEPENDENCE FOR ITO EQUATIONS WITH RESPECT TO THE DRIFT INVOLVING LIE BRACKETS

by

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Introduction

The problem we are concerned is a nonstandard continuous dependence for stochastic differential equations with respect to the drift coefficients. Roughly speaking it can be stated as follows. We are given a finite set of smooth functions $g_i(t,x):[0,T]\times R^n \longrightarrow R^n$, $i=1,\ldots,m$, and denote $\mathcal{L}(g_1,\ldots,g_n)$ the Lie algebra generated by them, where $[g_i,g_j](t,x) = ((\partial g_j/\partial x)g_i - (\partial g_i/\partial x)g_j)(t,x)$. Take $h_1,\ldots,h_l \in \mathcal{L}(g_1,\ldots,g_m)$ and denote $y(\cdot)$ the solution of the Ito equation

$$(*)\qquad dy = f(t,y)dt + [\sum_{i=1}^{\ell} u_i(t)h_i(t,y)]\,dt + \sum_{k=1}^{d} \sigma_k(t,y)dw_k(t),\quad y(0)=x_0, t\in[0,T],$$

where f, $\sigma_k:[0,T]\times R^n \longrightarrow R^n$ and $u_i:[0,T]\longrightarrow R$ are fixed. Along with (*) we consider

$$(**)\qquad dx = f(t,x)dt + [\sum_{i=1}^{m} v_i(t)g_i(t,x)]\,dt + \sum_{k=1}^{d} \sigma_k(t,x)dw_k(t),\ x(0)=x_0, t\in[0,T]$$

where f, g_i, σ_k are the given functions.

The problem we answer is to approximate the solution in (*) by solutions in (**) using the ~~usual~~ metrics $d_1(x(\cdot),y(\cdot)) = (E \max_{t\in[0,T]} |x(t)-y(t)|^2)^{1/2}$ and $d_2(x(\cdot),y(\cdot)) = (\max_{t\in[0,T]} E|x(t)-y(t)|^2)^{\frac{1}{2}}$. It can be done by defining an appropriate sequence of functions $\{v_i^h(\cdot)\}$ such that the corresponding solutions $x^h(\cdot)$ in (**) fulfils the goal; the sequence $\{v^h(\cdot)\}_{h>0}$ is unbounded with respect to h and since the pointwise convergence of the drift term in (**) to the drift term in (*) is meaningless we need a nonstandard approach.

This result is connected with the controllability properties of deterministic control systems as it appears in [1] and [2] and the techniques used here originate in [3]. In the stochastic case it completes the result in [4] by

considering Lie brackets in the drift part.
The use of the two metrics d_1 and d_2 is motivated by the fact that d_2 allows a more accurate estimate between the solutions, while d_1 insures the existence of a convergent sequence with probability one (see Theorem and Remark 4).

In particular, roughly speaking, it follows that if $g_i = \sigma_i$, $i = 1,\ldots,m$, $m = d$, then the support of the measure π_u on $C([0,T];R^n)$ generated by the solution $y_u(.)$ in (**) is an invariant under the transformations of the drift f performed in (*); it equals the support of the measure π_0 generated by the solution in (*) which corresponds to $U_i = 0$, $i = 1,\ldots,l$. It can be seen using Remark 4 and Girsanov's transformation of the probability measure in (**).

In deterministic case ($\sigma_k = 0$, $k = 1,\ldots,d$) the use of the metric d_2 is more relevant and it gives the possibility to study controllability of the system (**) along a fixed trajectory via the enlarged system (*). It can be stated more precisely as follows. Denote $\tilde{x}(t)$, $t \in [0,T]$, the solution in (**) and (*) which corresponds to $u_i = 0$, $i = 1,\ldots,l$, and $v_i = 0$, $i = 1,\ldots,m$, respectively. Suppose that dim span $\{\widetilde{ad}^{(k)} f(h)\ (0,x_0):$
$: h \in \mathcal{L}(g_1,\ldots,g_m)$, $k = 0,1,2,\ldots\} = n$ where
$\widetilde{ad}\, f(h)(t,x) = [f,h]\ (t,x) + \partial h/\partial t\ (t,x)$, and
$[f,h]$, $\mathcal{L}(g_1,\ldots,g_m)$ are defined as above. Then for each $t \in (0,T]$ there exists a sphere $S(\tilde{x}(t), \rho_t)$ centered at $\tilde{x}(t)$, such that the initial point x_0 is steered to any point in $S(\tilde{x}(t), \rho_t)$ in time t by using bounded controls $u(t) \in \mathcal{U}$ and trajectories in (*); the same property holds for the reduced system (**) but the control we have to use cannot be restricted to belong to the same set U. In our setting the controllability of the system (*) along $\tilde{x}(.)$ at time $t = T$ is preserved even if we restrict ourselves to the class of periodic controls $(u_i(0) = u_i(T)$, $i = 1,\ldots,l$, $v_j(0) = v_j(T)$ $j=1,\ldots m)$.
The result in Theorem remains the same in the case that the Wiener process $w(.)$ in (*) and (**) is replaced by a continuous square integrable martingale for which the quadratic variation matrix $V(t) = \langle M(t), M(t)\rangle$ has the form $V(t) = \int_0^t H(s,\omega)ds$ with H a bounded measurable matrix valued process. The invariance of the support of the measure π_u can be proved under the assumption that H is nonsingular and $H^{-1}(s,\omega)$ is bounded and measurable.

considering Lie brackets in the drift part.

1. Formulation of the problem and main result

Let $T > 0$ be fixed. Denote $C_b^{l,p}([0,T] \times R^n)$ the space consisting of real functions which are continously differentiable up to order l with respect to $t \in [0,T]$, up to order p with respect to $x \in R^n$ and are bounded along with all their derivatives; if the boundedness condition is supressed we denote it by $C^{l,p}([0,T] \times R^n)$. We are given $f, g_i, \sigma_k : [0,T] \times R^n \to R^n$ which are continuous and $g_i \in C^{1,\infty}([0,T] \times R^n)$, f, $\sigma_k \in C^{0,2}([0,T] \times R^n)$. Define $g_I(t,x) = [g_{i_0}, g_{i_1}](t,x)$, $|I| = 2$, if $I = \{i_0, i_1\}$, $i_0, i_1 \in \{1, \ldots, m\}$, where $[g_i, g_j](t,x) = \partial g_j/\partial x(t,x) g_i(t,x) - \partial g_i/\partial x(t,x) g_j(t,x)$; generally $g_I(t,x) = [g_{i_0}, g_{I_1}](t,x)$, $|I| = L + 1$, if $I = \{i_0, \ldots, i_L\}$, where $I_1 = \{i_1, \ldots, i_L\}$, $i_j \in \{1, \ldots, m\}$. For each $u_i(\cdot) \in C([0,T];R)$, $i = 1, \ldots, m$, $u_I(\cdot) \in C^1([0,T];R)$, $2 \le |I| \le L+1$, we consider the following stochastic differential equation

$$dy = [f(t,y) + \sum_{i=1}^{m} u_i(t) g_i(t,y) + \sum_{|I|=2}^{L+1} u_I(t) g_I(t,y)]dt + \sum_{k=1}^{d} \sigma_k(t,y) dw_k(t) \tag{1}$$

$y(0) = x_0$, $x_0 \in R^n$, $t \in [0,T]$,

where $w(t)$, $t \in [0,T]$, is a standard Wiener process over the filtered probability space $\{\Omega, \mathcal{F}, P; \mathcal{F}_t\}$.

Let N be a natural number and denote $h = T/N$. We associate with (1) the following stochastic differential equation

$$dx = [f(t,x) + \sum_{i=1}^{m} u_i(t) g_i(t,x) + \sum_{i=1}^{m} v_i^h(t) g_i(t,x)]dt + \sigma(t,x) dw(t), \tag{2}$$

$x(0) = x_0$, $t \in [0,T]$, where $\sigma = (\sigma_1 \ldots \sigma_d)$, f, g_i, σ_k, u_i are as in (1) and $v_i^h(\cdot) \in C^1([0,T];R)$, $v_i^h(0) = v_i^h(T) = 0$

We need the following conditions to be fulfilled

1.1 $\quad \partial f/\partial x_j$, $\partial \sigma_k/\partial x_j \in C_b^{0,1}([0,T] \times R^n)$, $g_i \in C_b^{1,L+1}([0,T] \times R^n)$

1.2 $(\partial g_I/\partial x)f^*$, $(\partial g_I/\partial x)\sigma_k^* \in C_b[0,T] \times R^n)$, $(\partial^2 g_I/\partial x_i \partial x_j) a_{ij} \in C_b([0,T] \times R^n)$ for any $1 \leq |I| \leq L$, $k = 1,\dots,d$, $i,j \in \{1,\dots,n\}$

where $a = \sigma\sigma^*$, "v^*" is the transposed of v, and a vector or a matrix belongs to $C_b^{1,p}$ if all their components fulfil it.

In the deterministic case ($\sigma_k = 0$, $k = 1,\dots,d$) the smoothness of f, g_i with $\partial g_i/\partial x \in C_b^{1,L}([0,T] \times R^n)$ and a linear growth condition $|h(t,x)| \leq C(1 + |x|)$ $x \in R^n$, $t \in [0,T]$, ($h = f, g_i$ respectively) are enough to get (1.1) and (1.2) satisfied because in this case we multiply g_i by a function $p(\cdot) \in C_b^\infty(R^n)$, $p(x) = 1$, $x \in S(0,\rho)$, where the sphere S is sufficiently large to contain the solution in (1).

Remark 1

If we replace (1.2) by

1.3. $(g_i f^*)$, $(g_i \sigma_k^*)$; $g_i^j a \in C_b^{0,2}([0,T] \times R^n)$ then (1.1), (1.3) imply (1.1) and (1.2).

Theorem

Assume that (1.1) and (1.2)are fulfilled for (1) and let $y(\cdot)$ be the solution in (1) corresponding to $u_i(\cdot)$, $u_I(\cdot)$, $2 \leq |I| \leq L+1$, and $x_0 \in S(0,\rho)$.

Then there exist $v_i^h(\cdot) \in C^1([0,T];R)$, $v_i^h(0) = v_i^h(T) = 0$, $i = 1,\dots,m$, depending on $u_i(\cdot)$, $u_I(\cdot)$, such that the solution $x^h(\cdot)$ in (2) fulfils $d_2(x^h(\cdot), y(\cdot)) = \left(\max_{t \in [0,T]} E|x^h(t) - y(t)|^2\right)^{\frac{1}{2}} \leq C_\rho \sqrt{h}$, for some constant $C_\rho > 0$ uniformly with respect to $u_i(\cdot)$, $u_I(\cdot)$ in bounded sets in $C([0,T];R)$ for $u_i(\cdot)$ and in $C^1([0,T];R)$ for $u_I(\cdot)$; d_2 can be replaced by d_1 if $L = 1$.

Remark 2

If we consider $u_i(\cdot)$ and $u_I(\cdot)$ in (1) as functions of (t,y) and fulfilling

$u_i(\cdot) \in C_b^{0,1}([0,T] \times R^n)$, $u_I(\cdot) \in C_b^{1,2}([0,T] \times R^n)$ then the theorem remains unchanged except that in this case $v_i^h(\cdot)$ are functions of (t,x) fulfilling $v_i^h(\cdot) \in C_b^{1,1}([0,T] \times R^n)$, $v_i^h(0,x) = v_i^h(T,x) = 0$, $x \in R^n$.

In addition there exists a sequence $h \to 0$ such that

$\lim_{h \to 0} E \max_{t \in [0,T]} |x^h(t) - y(t)|^2 = 0$ holds uniformly with respect to $u_i(\cdot)$, $u_I(\cdot)$, $2 \leq |I| \leq L+1$, in bounded sets in $C_b^{0,1}([0,T] \times R^n)$ for $u_i(\cdot)$ and $C_b^{1,2}([0,T] \times R^n)$ for $u_I(\cdot)$. It can be seen by repeating the proofs in Lemmas 1 and 2 in the next section.

Some auxiliary results and proof of the Theorem

We associate with (1) the maximal number L of the Lie brackets contained in (1) and call it the order of the system. To prove Theorem we need to approximate the solution in (1) by one determined by a system which has an order less than L. It is done in the next Lemma. In the following we shall define the approximate equation.

Denote $\tilde{f}(t,y) = f(t,y) + \sum_{i=1}^{m} u_i(t) g_i(t,y) + \sum_{|I|=2}^{L} u_I(t) g_I(t,y)$ and (1) is rewritten as

$$\text{3)} \qquad dy = \left\{ \tilde{f}(t,y) + \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} u_{ij}(t)[g_i, b_j](t,y) \right\} dt + \sigma(t,y)dw(t),$$

$y(0) = x_0$, $t \in [0,T]$ where

$$\sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} u_{ij}(t)[g_j, b_y](t,y) = \sum_{|I|=L+1} u_I(t) g_I(t,y)$$

Let N be a natural number. We consider a partition Π_0 of $[0,T]$ determined by the intervals $[k,h, (k+1)h)$, $k = 0,1,\ldots,N-1$, with $|\Pi_0| = h = T/N$. For each $k \in \{0,1,\ldots,N-1\}$, let A_{ij}^k, $i = 1,\ldots,m$, $j = 1,\ldots,\tilde{m}$, be a partition of $[kh, (k+1)h)$ with $|A_{ij}^k| = h_1 = h/m\tilde{m}$. Denote P^l the space consisting of scalar polynomial functions defined on $[0,1]$ and fulfilling $\int_0^1 t^k p(t)dt = 0$, $k = 0,1,\ldots,l$.

Let $p_1(\cdot), p_2(\cdot) \in P^0$ be such that $p_i(0) = p_i(1) = 0$, $dp_i/dt(0) = dp_i/dt(1)$ and $\int_0^1 p_2(t)\tilde{p}_1(t)dt = 1$, where $\tilde{p}_1(t) = \int_0^1 p_1(s)ds$. These functions will be fixed in the sequel and they could be chosen as polynomials of third and fourth degree respectively.

Let $p_i^k(t,h) : [kh,(k+1)h] \to R$, $i = 1,2$, be defined by

$$p_i^k(t,h) = p_i(t-kh/h_1), t \in A_{11}^k = [kh, kh + h_1), \ldots,$$

$$p_i^k(t,h) = p_i(t-(kh+(m\tilde{m}-1)h_1)/h_1), \ t \in A_{m\tilde{m}}^k = [kh,(k+1)h), \ k = 0,1,\ldots,N-1..$$

Obviously $p_i^k(\cdot) \in C^1([kh,(k+1)h];R)$ and $p_i^k(kh) = p_i^k((k+1)h) = 0$. With (3) and the partition π_0 we associate the following differential equation of order L-1.

$$dx = \Big\{ \tilde{f}(t,x) + \sum_{k=0}^{N-1} \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} (m\tilde{m}/\sqrt{h_1})[p_1^k(t,h)u_{ij}(t)g_i(t,x) + p_2^k(t,h)b_j(t,x)]dt + \tag{4}$$

$$+ \sigma(t,x)dw(t), \ x(0) = x_0, \ t \in [0,T].$$

Denote $x^h(t)$, $t \in [0,T]$ the solution in (4).

Remark 3

By definition, the equation (4) is of order L-1 and the coeficients $p_i(t,h) = (m\tilde{m}/\sqrt{h_1})\, p_i^k(t,h)$, $t \in [kh, (k+1)h]$, $k = 0,1,\ldots N-1$, are in $C^1([0,T];R)$ with $p_i(0,h) = p_i(T,h) = 0$, $i = 1,2$, but they are unbounded with respect to h.

Denote $\tilde{p}_i(t,h) = \int_0^t p_i(s,h)ds$, $i = 1,2$,

$$M^h(t) = \int_0^t \Big\{ \sigma(s,x^h(s)) - \sigma(s,y(s)) + \tag{5}$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} [\tilde{p}_1(s,h)(\partial/\partial x)(u_{ij}g_i)\sigma)(s,x^h(s)) +$$

$$+ \tilde{p}_2(s,h)((\partial b_j)/(\partial x)\sigma)(s,x^h(s))] \Big\} dw(s), \ t \in [0,T]$$

By $\eta(r)$ we denote a random or a deterministic vector fulfilling

$(E\,|\eta(r)|^2)^{\frac{1}{2}} \leq Cr$ for some fixed constant $C > 0$.

Lemma 1

Assume that (1.1) and (1.2) are fulfilled. Let $x^h(\cdot)$ be the solution in (4) and $y(\cdot)$ fulfils (3). Then there exists a martingale $M^h(t)$, $t \in [0,T]$ (see (5)) such that

$$x^h(t'') - x^h(t') = y(t'') - y(t') + M^h(t'') - M^h(t') + (t''-t')\,\eta(\sqrt{h})$$

$t',t'' \in \{0,h,2h,\ldots,(N-1)h = T\}$, $t' < t''$, where

$(E\,|M^h(t'') - M^h(t')|^2)^{\frac{1}{2}} \leq \sqrt{t''-t'}\,\eta(\sqrt{h})$

uniformly with respect to $u_i(\cdot)$, $u_I(\cdot)$, $2 \leq |I| \leq L+1$ in bounded sets in $C([o,T];R)$ for u_i and $C^1([0,T];R)$ for u_I.

Proof.

By definition

$$x^h(h_1) = x_0 + \int_0^{h1} \tilde{f}(t,x^h(t))dt + (m\tilde{m}/\sqrt{h_1}) \int_0^{h1} [p_1^0(t,h)u_{11}(t)g_1(t,x^h(t)) +$$

$$+ p_2^0(t,h)b_1(t,x^h(t))]dt + \int_0^{h1} \sigma(t,x^h(t))dw(t) = x_0 + T_1 + T_2 + T_3.$$

By hypothesis $\tilde{f}$ and σ are Lipschitz continuous with respect to $x \in R^n$ and computation shows

$$6) \qquad (E \max_{t\in[0,h]} |x^h(t) - y(t)|^2)^{\frac{1}{2}} \leq (1-Ch)^{-1}\,\eta(\sqrt{h})$$

where $C > 0$ is the Lipschitz constant for $\tilde{f}$ and $\tilde{\sigma}$. Using (6) we get

$$7) \qquad T_1 = \int_0^{h1} \tilde{f}(t,y(t))dt + \int_0^{h1} [\tilde{f}(t,x^h(t)) - \tilde{f}(t,y(t))]dt =$$

$$= \int_0^{h1} \tilde{f}(t,y(t))dt + h_1\,\eta(\sqrt{h})$$

$$8) \qquad T_3 = \int_0^{h1} \sigma(t,y(t))dw(t) + \int_0^{h1} [\sigma(t,x^h(t)) - \sigma(t,y(t))]dw(t) =$$

$$= \int_0^{h1} \sigma(t,y(t))dw(t) + M_1'(h_1), \text{ where } E\,|M_1'(h_1)|^2 \leq h_1\,\eta(h)$$

denote $\tilde{x}(s) = x^h(sh_1)$, $s \in [0,1]$, $a = \sigma\sigma^*$,

$$(\mathcal{L}u)(t,x) = [(\partial/\partial t + \sum_{i=1}^{n} \tilde{f}_i(t,x)(\partial/\partial x_i) + 1/2 \sum_{i,j=1}^{n} a_{ij}(t,x)(\partial^2/\partial x_i \partial x_j)]u(t,x)$$

With these notations and using $p_i(\cdot) \in \mathcal{P}^0[0,1]$, $i = 1,2$ we get

9) $$T_2 = m\tilde{m}\sqrt{h_1} \int_0^1 [p_1(s)u_{11}(sh_1)g_1(sh_1,\tilde{x}(s)) + p_2(s)b_1(sh_1,\tilde{x}(s))]ds =$$

$$= m\tilde{m}h_1^{3/2}[\int_0^1 p_1(s)ds \int_0^s \mathcal{L}(u_{11}g_1)(s_1h_1,\tilde{x}(s_1))ds_1 +$$

$$+ \int_0^1 p_2(s)ds \int_0^s (\mathcal{L}b_1)(s_1h_1,\tilde{x}(s_1))ds_1] +$$

$$+ m\tilde{m}h_1[\int_0^1 p_1(s)ds \int_0^s (p_1(s_1)(\partial)/(\partial x)(u_{11}g_1)((u_{11}g_1)(s_1h_1,\tilde{x}(s_1)) +$$

$$+ p_2(s_1)(\partial)/(\partial x)(u_{11}g_1)b_1(s_1h_1,\tilde{x}(s_1))ds_1 +$$

$$+ \int_0^1 p_2(s)ds \int_0^s (p_1(s_1)(\partial b_1)/(\partial x)(u_{11}g_1)(s_1h_1,\tilde{x}(s_1)) +$$

$$+ p_2(s_1)(\partial b_1)/(\partial x)b_1(s_1h_1,\tilde{x}(s_1)))ds_1] +$$

$$+ (m\tilde{m})/(\sqrt{h_1}) \int_0^{h_1} [\tilde{p}_1^0(t,h)(\partial)/(\partial x)(u_{11}g_1)\sigma(t,x^h(t)) + \tilde{p}_2^0(t,h)(\partial b_1)/(\partial x)\sigma(t,x^h(t))]dw(t) =$$

$$= T_2' + T_2'' + M_1''(h_1)$$

By hypothesis (see (1.2)) we have

10) $|\mathcal{L}(u_{11}g_1)(t,x^h(t))| + |\mathcal{L}b_1(t,x^h(t))| \leq C_1$, $t \in [0,T]$, where $C_1 > 0$ is a constant wich doesn't depend on h, and using (10) in (9) we obtain

11) $$T_2' = h_1\eta(\sqrt{h}),\ E\,|M_1''(h_1)|^2 \leq h_1\eta(h)$$

Since $\int_0^1 p_2(s)\tilde{p}_1(s)ds = - \int_0^1 p_1(s)\tilde{p}_2(s)ds = 1$ and

$\int_0^1 p_i(s)(\tilde{p}_i(s))^j ds = 0$, $i,j = 1,2$, we get

12) $$T_2'' = m\tilde{m}h_1\, u_{11}(0)[g_1, b_1](0,x_0) + h_1\eta(\sqrt{h}) =$$

$$= \int_0^h u_{11}(t)[g_1, b_1](t,y(t))dt + h_1\eta(\sqrt{h})$$

Using (11) and (12) in (9) it follows

13) $$T_2 = \int_0^h u_{11}(t))[g_1,b_1](t,y(t))dt + h_1\eta(\sqrt{h}) + M_1''(h_1)$$

and from (7), (8) and (13) we get

14) $$x^h(h_1) = x_0 + \int_0^{h_1} \tilde{f}(t,y)(t))dt + \int_0^{h_1} u_{11}(t)[g_1,b_1](t,y(t))dt +$$

$$+ \int_0^{h_1} \tilde{\sigma}(t,y)(t))dw(t) + h_1\,\eta(\sqrt{h}) + M_1(h_1)$$ where

$M_1(h_1) = M_1'(h_1) + M_1''(h_1)$ fulfils

$$E\,|M_1(h)|^2 \le h_1\,\eta(h)$$

On the next interval $[h_1, 2h_1]$ we repeat the computations for $[0,h_1]$

By definition

15) $$x^h(2h_1) = x^h(h_1) + \int_{h_1}^{2h_1} \tilde{f}(t,x^h(t))dt + (m\tilde{m})/(\sqrt{h_1}) \int_{h_1}^{2h_1} [p_1^0(t,h)u_{12}(t)g_1(t,x^h)(t)) +$$

$$+ p_2^0(t,h)b_2(t,x^h(t)]dt + \int_{h_1}^{2h_1} \sigma(t,x^h(t))dw(t) = x^h(h_1) + \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3$$

and we get easily

16) $$\tilde{T}_1 = \int_{h_1}^{2h_1} \tilde{f}(t,y)(t))dt + \int_{h_1}^{2h_1} [\tilde{f}(t,x^h(t)) - \tilde{f}(t,y(t))]dt =$$

$$= \int_{h_1}^{2h_1} f(t,y(t))dt + h_1\,\eta(\sqrt{h})$$

17) $$\tilde{T}_3 = \int_{h_1}^{2h_1} \sigma(t,y(t))dw(t) + \int_{h_1}^{2h_1} [\sigma(t,x^h(t)) - \sigma(t,y(t))]dw(t) =$$

$$= \int_{h_1}^{2h_1} \sigma(t,y(t))dw(t) + M_2'(h_1)$$

where

$$E\,|M_2'(h_1)|^2 \le h_1\,\eta(h),\; M_2'(h_1) = \int_{h_1}^{2h_1} [\sigma(t,x^h(t)) - \sigma(t,y(t))]dw(t)$$

Similarly, repeating the computations in (9) - (11) we get

18) $$\tilde{T}_2 = \tilde{T}_2' + \tilde{T}_2'' + M_2''(h_1)$$

where

19) $$\tilde{T}_2' = h_1\,\eta(h),$$

$$M_2''(h_1) = (m\tilde{m})/(\sqrt{h_1}) \int_{h_1}^{2h_1} [\tilde{p}_1^0(t,h)(\partial)/(\partial x)(u_{11}g_1)\sigma(t,x^h(t)) +$$

$$+ \tilde{p}_2^0(t,h)(\partial b_2)/(\partial x)\,\sigma(t,x^h(t)]dw(t)$$

and $E\,|M_2''(h_1)|^2 \le h_1\,\eta(h)$

Also, we have

20) $$\tilde{T}_2'' = m\tilde{m}h_1\, u_{12}(h_1)[g_1,b_2](h_1,x^h(h_1)) + h_1\,\eta(\sqrt{h}) =$$

$$= m\tilde{m}h_1\, u_{12}(0)[g_1,b_2](0,x_0) + h_1\,\eta(\sqrt{h}) =$$

$$= \int_0^h u_{12}(t)[g_1,b_2](t,y(t))dt + h_1\,\eta(\sqrt{h})$$

Denote $M_2(h_1) = M_2'(h_1) + M_2''(h_1)$ and using (16)-(20) in (15) we get

21) $$x^h(2h_1) = x_0 + \int_{h_1}^{2h_1} \tilde{f}(t,y(t))dt + \sum_{j=1}^{2} \int_0^h u_{ij}(t)[g_1,b_j](t,y(t))dt +$$

$$\int_0^{2h_1} \tilde{\sigma}(t,y(t))dw(t) + 2h_1\,\eta(\sqrt{h}) + M_1(h_1) + M_2(h_1)$$

where $M_1(h_1)$ is defined in (14), and

22) $$E\,|M_1(h_1) + M_2(h_1)|^2 = E\,|M_1(h_1)|^2 + E\,|M_2(h_1)|^2 \le 2h_1\,\eta(h)$$

Finally, for $t = h$, we get $M_i(h_1)$, $i = 1, \ldots, m\tilde{m}$, such that

23) $$x^h(h) = x^h(m\tilde{m}h_1) = x_0 + \int_0^h \tilde{f}(t,y(t))dt + \sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}} \int_0^h u_{ij}(t)[g_i,b_j](t,y(t))dt +$$

$$+ \int_0^h \tilde{\sigma}(t,y(t))dw(t) + h\,\eta(\sqrt{h}) + M_1(h) = y(h) + h\,\eta(\sqrt{h}) + M_1(h)$$

where

24) $$M_1(h) = \sum_{i=1}^{m\tilde{m}} M_i(h_1) = \int_0^h [\tilde{\sigma}(t,x^h(t)) - \tilde{\sigma}(t,y(t))]dw(t) +$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}} (m\tilde{m}/\sqrt{h_1}) \int_0^h [\tilde{p}_1^0(t,h)(\partial)/(\partial x)(u_{ij}g_i)\tilde{\sigma}(t,x^h(t)) +$$

$$+ \tilde{p}_2^0(t,h)(\partial)/(\partial x)(b_j)\tilde{\sigma}(t,x^h(t))]dw(t)$$

and $E\,|M_1(h)|^2 \le h\,\eta(h)$

Lemma was proved for $t'' = h$, $t' = 0$.

For the next interval $[h,2h]$ we have to repeat the computations done on $[0,h]$. Using (23) we get

25) $$(E \max_{t\in[h,2h]} |x^h(t)-y(t)|^2)^{\frac{1}{2}} \le [(E\,|x^h(h) - y(h)|^2)^{\frac{1}{2}} + \eta(\sqrt{h})](1-Ch)^{-1} \le$$

$$\le \eta(\sqrt{h})(1+h+\sqrt{h})(1-Ch)^{-1}$$

where $C > 0$ is the Lipschitz constant for $\tilde{f}$ and $\tilde{\sigma}$.

For $t = 2h$ we get a similar represntation as in $t = h$. Namely

26) $$x^h(2h) = x^h(h) + \int_h^{2h} \tilde{f}(t,y(t))dt + \sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}} \int_h^{2h} u_{ij}(t)[g_i,b_j](t,y(t))dt +$$

$$+ \int_h^{2h} \sigma(t,y(t))dw(t) + h\,\eta(\sqrt{h}) + M_2(h) = y(2h) + 2h\,\eta(\sqrt{h}) + M_1(h) + M_2(h)$$

where $M_1(h)$ is defined in (24) and

27) $$M_2(h) = \int_h^{2h} [\sigma(t,x^h(t)) - \sigma(t,y(t))]dw(t) +$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} (m\tilde{m})/(\sqrt{h_1}) \int_h^{2h} [\tilde{p}_1^1(t,h)(\partial)/(\partial x)(u_{ij}g_i)\sigma(t,x^h(t)) +$$

$$+ \tilde{p}_2^1(t,h)(\partial b_j)/(\partial x)\sigma(t,x^h)(t))]dw(t)$$

fulfil

28) $$E\,|M_2(h)|^2 = h\,\eta(h),\ E\,|M_1(h) + M_2(h)|^2 = E\,|M_1(h)|^2 +$$

$$+ E\,|M_2(h)|^2 = 2h\,\eta(h)$$

By using an induction argument we get (see (25), (26))

29) $$(E \max_{t\in[kh,(k+1)h]} |x^h(t) - y(t)|^2)^{\frac{1}{2}} \le \eta(\sqrt{h})(1+kh+\sqrt{kh})(1-Ch)^{-1} = \eta(\sqrt{h})$$

and $M_1(h), \ldots, M_k(h)$ such that

30) $$x^h(kh) = y(kh) + kh\,\eta(\sqrt{h}) + \sum_{i=1}^{k} M_i(h) = y(kh) + kh\,\eta(\sqrt{h}) +$$

$$+ M^h(kh),\ k=0,1,\ldots,N-1,$$

where $M^h(t)$ is defined by

31) $$M^h(t) = \int_0^t [\sigma(s,x^h(s)) - \sigma(s,y(s))]dw(s) + \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} (m\tilde{m})/(\sqrt{h_1})$$

$$\int_0^t [\tilde{p}_1(s,h)(\partial)/(\partial x)(u_{ij}g_i)\sigma(s,x^h(s)) + \tilde{p}_2(s,h)(\partial b_j)/(\partial x)\sigma(s,x^h(s))]dw(s))$$

and fulfil

31') $$E\,|M^h(k''h) - M^h(k'h)|^2 = E\,\Big|\sum_{i=k'}^{k''} M_i(h)\Big|^2 = \sum_{i=k'}^{k''} E\,|M_i(h)|^2 =$$

$$= (h''-k')h\,\eta(h),\ k' < k'',\ k',k'' \in \{0,1,\ldots,N\}.$$

From (30-(31') we get the conclusion. The proof is complete.

The approximation equation (4) has some coefficients $u_I(t,h)$ depending on h being unbounded with respect to h. These functions $u_I(t,h)$ are of class C^1 in $t\in[0,T]$, and with respect to h they fulfil the following condition

$$h\,u_I(t,h) = \eta(\sqrt{h})v_I(t,h),\ h^2(\partial u_I)/(\partial t)(t,h) = \eta(\sqrt{h})v_I^1(t,h)$$

where $v_I(\cdot,h)$, $v_I^1(\cdot,h)$ are uniformly bounded with respect to h.

These properties are essential in order to make the next step of reducing the order of a system which has unbounded coefficients with respect to the parameter h.

In order to get an equation of zero order we need to known how to reduce the order of an equation of the type (4) but with unbounded coefficients with respect to h, such that the statement in Lemma 1 is still true. Now we consider the following stochastic equation

S) $$dy = [f(t,y) + \sum_{i=1}^{m} u_i(t,h)g_i(t,y) + \sum_{|I|=2}^{L+1} u_I(t,h)g_I(t,y)]dt +$$

$$+ \sigma(t,y)dw(t), y(0) = x_0,\ t\in[0,T],$$

where x_0, f, g_i, σ are as in (1) and $u_i(\cdot,h)\in C([0,T];R)$, $u_I(\cdot,h)\in C^1([0,T];R)$.

With respect to the parameter h we assume that there exist $0 < r(h)$ $(r(h) = \eta(h))$ and a partition π_r of [0,T] with intervals of the lenght r such that

a) $$r u_I(t,h) = \eta(\sqrt{h})v_I(t,h),\ 1 \le |I| \le L+1,\ t\in[0,T]$$

b) $$r^2(\partial u_I)/(\partial t)(t,h) = \eta(\sqrt{h})v_I^1(t,h),\ t\in[0,T],\ 2\le |I| \le L+1$$

where $v_I(\cdot)$, $v_I^1(\cdot)$ are uniformly bounded with respect to h

Definition

A system S of order L for which there exists $r(h) > 0$ such that $u_I(\cdot)$, $1\le |I| \le L+1$, fulfil (a) and (b) is called of index (L,r).

Lemma 2

Assume (1.1) and (1.2) fulfilled for f, g_i, σ. Let (S) be a system of index (L,r) where $r = \eta(h)$. Then there exists a sistem (S_1) of index $(L-1, r_1)$ with $r_1 = T/MK^4$, $M = \text{card}\{I: |I| = L+1\}$, $K = T/r$, such that the corresponding solution $y(\cdot)$ in (S) and $y_1(\cdot)$ in (S_1) with $y(0) = y_1(0) = x_0$ fulfil

c_1) $$y_1(t'') - y_1(t') = y(t'') - y(t') + (t''-t')\eta(\sqrt{h}) + M^h(t'') - M^h(t')$$

$$t' < t'',\quad t',\quad t''\in\{0,\tilde r_1, 2\tilde r_1, \ldots, K^3\tilde r_1, \ldots, 2K^3\tilde r_1, \ldots, K^4\tilde r_1 = T\},$$

$$\tilde r_1 = T/K^4 = Mr_1$$

c_2) $M^h(t)$ is a martingale and $E|M^h(t'')-M^h(t')|^2 = (t'' - t')\eta(h)$

(see (62))

C_3) the coefficients $\tilde{u}_I(\cdot)$ in (S_1) fulfil (a) and (b) with r replaced by r_1

Proof.

We shall use the same general scheme as in Lemma 1.

The system (S) is rewritten as

32) $$dy = [f^h(t,y) + \sum_{i=1}^{m}\sum_{j=1}^{m} u_{ij}(t,h)[g_i, b_j](t,y)]dt + \sigma(t,y)dw(t),$$

$$y(0) = x_0,\ t \in [0,T]$$

where $\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}} v_{ij}(t,h)[g_i, b_j](t,y) = \sum_{|I| = L+1} u_I(t,h)g_I(t,y)$ and

$$f^h(t,y) = f(t,y) + \sum_{i=1}^{m} u_i(t,h)g_i(t,y) + \sum_{|I|=2}^{L} u_I(t,h)g_I(t,y).$$

Let K be the natural number such that $Kh = T$ and define $r_1 = T/m\tilde{m}K^4$, $\tilde{r}_1 = T/K^4$. Let $p_1(\cdot)$, $p_2(\cdot)$ $p^0[0,1]$ and A^k_{ij} $i = 1, \ldots, m$, $j = 1, \ldots, \tilde{m}$, be a partition of $[k\tilde{r}_1, (k+1)\tilde{r}_1]$ with $|A^k_{ij}| = r_1$, such that

33) $$p_i(0) = p_i(1) = 0,\ (dp_i)/(dt)(0) = (dp_i)/(dt)(1),\ i = 1,2,$$

$$\int_0^1 p_2(t)\left(\int_0^t p_1(s)ds\right)dt = 1.$$

Define $p^k_i(t,h) : [k\tilde{r}_1, (k+1)\tilde{r}_1] \to R$ as in the system (4), considering $h_1 = r$, and $h = \tilde{r}_1$.

Let $y_1(\cdot)$ be the solution of the following differential equation

S_1) $$dy = \Big\{ f^h(t,y) + \sum_{k=0}^{K^4-1} \sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}} (m\tilde{m}/\sqrt{r_1})[p^k_1(t,h)u_{ij}(t,h)g_i(t,y) +$$
$$p^k_2(t,h)bj(t,y)]\Big\} dt + \sigma(t,y)dw(t),\ y(0) = x_0,\ t \in [0,T],$$

where f^h, g_i, b_j, u_{ij} are as in (32).

We have

34) $$y_1(r_1) = x_0 + \int_0^{r_1} f^h(t,y_1(t))dt + (m\tilde{m}/\sqrt{r_1}) \int_0^{r_1} [p^0_1(t,h)u_{11}(t,h)g_1(t,y_2(t)) +$$
$$+ p^0_2(t,h)b_1(t,y_1(t)]dt + \int_0^{r_1} \sigma(t,y_1(t))dw(t) = x_0 + T_1 + T_2 + T_3$$

Denote $\tilde{y}(s) = y(s\,r_1)$, $\tilde{y}_1(s) = y_1(sr_1)$, $w(s) = (1/\sqrt{h_1})w(sr_1)$, $s \in [0,1]$. It follows

35) $$\begin{cases} d\tilde{y} = r_1[f^h(sr_1,\tilde{y}) + \sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}} u_{ij}(sr_1,h)[g_i,b_j](sr_1,\tilde{y})]ds + \\ \quad + \sqrt{r_1}\,\sigma(sr_1,\tilde{y})d\tilde{w}(s) \\ d\tilde{y}_1 = \Big\{ r_1 f^h(sr_1,\tilde{y}_1) + \sqrt{r_1}\, m\tilde{m}[p_1(s)u_{11}(sr_1,h)g_1(sr_1,\tilde{y}_1) + \\ \quad + p_2(s)b_1(sr_1,\tilde{y}_1)]\Big\} ds + \sqrt{r_1}\,\sigma(sr_1,\tilde{y}_1)d\tilde{w}(s),\ \tilde{y}_1(0) = y_1(0) = x_0,\ s \in [0,1] \end{cases}$$

By hypothesis (see (a)) we have

36) $rf^h(t,y) = \eta(\sqrt{h})\ \tilde{f}^h(t,y),\ \ |\tilde{f}^h(t,y'') - \tilde{f}^h(t,y')| \leq C\ |y'' - y'|\ (\forall) t \in [0,T]$ for some $C > 0$, $r^2 u_{ij}(t,h) = r\ \eta(\sqrt{h})\, v_{ij}(t,h)$, $v_{ij}(\cdot,h)$ uniformly bounded with respect to h. Using (36) in (35) we get

37) $E\ |y(t) - y_1(t)|^2 \leq \sqrt{r_1}\ \eta(h),\ (\forall)\ t \in [0,r_1],$

and similarly we obtain

38) $E\ |y(t) - y_1(t)|^2 \leq \sqrt{r_1}\ \eta(h)\ (\forall) t \in [0,\tilde{r}_1]$

Using (37) it follows

39) $T_1 = \int_0^{r_1} f^h(t,y(t))dt + \int_0^{r_1}[f^h(t,y(t)) - f^h(t,y_1(t))]dt =$

$\int_0^{r_1} f^h(t,y(t))dt + r_1\eta(h)$

40) $T_3 = \int_0^{r_1}\sigma(t,y(t))dw(t) + \int_0^{r_1}[\sigma(t,y_1(t)) - \sigma(t,y(t))]dw(t) =$

$\int_0^{r_1}\sigma(t,y(t))dt + M_1'(r_1)$ where

41) $E\ |M_1'(r_1)|^2 = r_1\eta(h)$

Denote

$$u_{ij}^h(t) = u_{ij}(t,h),\ \ \mathcal{L}u(t,x) = (\partial/\partial t) + \sum_{i=1}^{n} f_i^h(t,x)(\partial/\partial x_i) +$$

$$+ 1/2\sum_{i,j=1}^{n} a_{ij}(t,x)(\partial^2/\partial x_i\,\partial x_j))u(t,x),\ \text{where } a=\sigma\sigma^*$$

Computation shows

42) $T_2 = m\tilde{m}\sqrt{r_1}\ \int_0^1[p_1(s)u_{11}^h(sr_1)g_1(sr_1,\tilde{y}_1(s)) + p_2(s)b_1(sr_1,\tilde{y}_1(s))]ds =$

$= m\tilde{m}r_1^{3/2}[\int_0^1 p_1(s)ds\int_0^s \mathcal{L}(u_{11}^h\ g_1)(s_1r_1,\tilde{y}_1(s_1))ds_1 +$

$+\ {}_0^1p_2(s)ds\int_0^s(\mathcal{L}\,b_1(s_1 r_1,\tilde{y}_1(s_1))ds_1] +$

$+\ m\tilde{m}r_1[\int_0^1 p_1(s)ds\int_0^s(p_1(s_1)(\partial/\partial x)(u_{11}^h g_1)(u_{11}^h g_1)(s_1 r_1,\tilde{y}_1(s_1)) +$

$$+ p_2(s_1)(\partial)/(\partial x)(u_{11}^h g_1) b_1(s_1 r_1, \tilde{y}_1(s_1)))) ds_1 +$$

$$+ \int_0^1 p_2(s) ds \int_0^s p_1(s_1)(\partial b_1)/(\partial x)(u_{11}^h g_1)(s_1 r_1, \tilde{y}_1(s_1)) +$$

$$+ p_2(s_1)(\partial b_1)/(\partial x) b_1(s_1 r_1, \tilde{y}_1(s_1)) ds_1] +$$

$$(m\tilde{m})/(\sqrt{r_1}) \int_0^{r_1} [\tilde{p}_1^0(t,h)(\partial/(\partial x)(u_{11}^h g_1)\sigma(t, y_1(t)) + \tilde{p}_2^0(t,h)(\partial b_1)/(\partial x)\sigma(t, y_1(t))] dw(t) =$$

$T_2' + T_2'' + M_1''(r_1)$, where

$$\tilde{p}_i^k(t,h) = \int_0^t p_i^k(s,h) ds, \ i = 1,2.$$

Using (b) (see the hypotheses) it follows

43) $\quad r^2 |\mathcal{L}(u_{11}^h g_1)(t,x)| \leq \eta(h), \ r^2 |\mathcal{L} b_1(t,x)| \leq \eta(h), \ t \in [0,T], \ x \in R^n,$

Since $\sqrt{r_1} = \eta(r) = \eta(h)$, using (43) in (42) we get

44) $\quad T_2' = r_1 \eta(h)$

By hypothesis (see (1.2) and (a)) we have

45) $\quad |(\partial(u_{11}^h g_1)/\partial x)\sigma(t,y)|^2 \leq C, \ |\partial b_1/\partial x)\sigma(t,y)|^2 \leq C r^{-2}\eta(h)$

$t \in [0,T], \ y \in R^n$, for some constant $C > 0$.

Using (45) in (42) we get

46) $\quad E|M_1''(r_1)|^2 \leq r_1^2 \int_0^1 [(\tilde{p}_1(s))^2 |(\partial(u_{11}^h g_1))/(\partial x)\sigma(sr_1, \tilde{y}_1(s))|^2 +$

$$+ (\tilde{p}_2(s))^2 |(\partial b_1/(\partial x)\sigma(sr_1, \tilde{y}_1(s_1))|^2] ds \leq r_1 \eta(h)$$

and since $\int_0^1 p_i(s)(\tilde{p}_i(s))^l ds = 0$, $1 = \int_0^1 p_2(s)\tilde{p}_1(s) ds = - \int_0^1 p_1(s)\tilde{p}_2(s) ds$, for $l = 1,2,\ldots$, we get

47) $\quad T_2'' = m\tilde{m} r_1 u_{11}^h(0)[g_1, b_1](0, x_0) + r_1 \eta(\sqrt{h}) =$

$$= \int_0^{\tilde{r}_1} u_{11}(t,h)[g_1, b_1](t, y(t)) dt + r_1 \eta(\sqrt{h})$$

Since $T_2 = T_2' + T_2'' + M_1''(r_1)$, and T_1, T_3 are estimated in (39) and (40) the equation (34) beomes

48) $\quad y_1(r_1) = x_0 + \int_0^{r_1} f^h(t, y(t)) dt + \int_0^{\tilde{r}_1} u_{11}(t,h)[g_1, b_1](t, y(t)) dt +$

$$+ \int_0^{r_1} \sigma(t, y(t_1)) dw(t) + r_1 \eta(\sqrt{h}) + M_1(r_1)$$

where

49) $M_1(r_1) = M'_1\ (r_1) + M''_1\ (r_1) = \int_0^{r_1}[\sigma(t,y_1(t)) - \sigma(t,y(t))]dw(t) +$

$+ (m\tilde{m})/(\sqrt{r_1}) \int_0^{r_1}[\tilde{p}_1^0(t,h)(\partial)/(\partial x)(u_{11}^h g_1)\sigma(t,y_1(t)) +$

$+ \tilde{p}_2^0(t,h)(\partial b_1)/(\partial x)\sigma(t,y_1(t))]dw(t),\ E\,|M_1(r_1)|^2 = r_1\eta(h)$

On the interval $[r_1,2r_1]$ we repeat the computation done on $[0_1, r_1]$ and we get

50) $(E\,|y(t) - y_1(t)|^2)^{1/2} \leq (E\,|y(r_1) - y_1(r_1)|^2)^{1/2} + r\eta(\sqrt{h}) \leq$

$\leq 2r\eta(\sqrt{h}) = \eta(h), t \in [r_1,2r_1]$

51) $y_1(2r_1) = y_1(r_1) + \int_{r_1}^{2r_1} f^h(t,y_1(t))dt +$

$+ (m\tilde{m})/(\sqrt{r_1}) \int_{r_1}^{2r_1}[p_1^0(t,h)u_{12}(t,h)g_1(t,y_1(t)) + p_2^0(t,h)b_2(t,y_1(t))]dt +$

$+ \int_{r_1}^{2r_1}\sigma(t,y_1(t))dw(t) = y_1(r_1) + \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3$

Using (36) and (50) we obtain

52) $\tilde{T}_1 = \int_{r_1}^{2r_1} f^h(t,y(t))dt + \int_{r_1}^{2r_1}[f^h(t,y_1(t)) - f^h(t,y(t))]dt =$

$= \int_{r_1}^{2r_1} f^h(t,y(t))dt + r_1\eta(h)$

and

53) $\tilde{T}_3 = \int_{r_1}^{2r_1}\sigma(t,y(t))dw(t) + M'_2(r_1),$

where

$M'_2 = \int_{r_1}^{2r_1}[\sigma(t,y_1(t)) - \sigma(t,y(t))]dw(t)$, fulfils $E\,|M'_2\ (r_1)|^2 = r_1\eta(h)$.

Also, with the same computations as in (42) we get

54) $\tilde{T}_2 = \tilde{T}'_2 + \tilde{T}''_2 + M''_2\ (r_1)$, where $\tilde{T}'_2 = r_1\eta(h)$

55) $M''_2\ (r_1) = (m\tilde{m})/(\sqrt{r_1}) \int_{r_1}^{2r_1}[\tilde{p}_1^0(t,h)(\partial)/(\partial x)(u_{12}^h g_1)\sigma(t,y_1(t)) +$

$+ \tilde{p}_2^0(t,h)(\partial b_2)/(\partial x)\sigma(t,y_1(t))]dw(t),\ E|M''_2\ (r_1)|^2 = r_1\eta(h)$

56) $\tilde{T}''_2 = m\tilde{m}r_1 u_{12}(r_1,h)[g_1,b_2](r,y_1(r_1)) + r_1\eta(\sqrt{h})$

and using (b) in the hypothesis we obtain

57) $\tilde{T}''_2 = m\tilde{m}r_1 u_{12}(0,h)[g_1,b_2](0,x_0) + r_1\eta(\sqrt{h}) = \int_0^{r_1} u_{12}(t)h)[g_1,b_2](t,y(t))dt +$

$+ r_1\eta(\sqrt{h})$

Denote $M_2(r_1) = M'_2(r_1) + M''_2(r_1)$. Using (52) - (57) in (51) we get

58) $$y_1(2r_1) = x_0 + \int_0^{2r_1} f^h(t,y(t))dt + \sum_{j=1}^{2} \int_0^{\tilde{r}_1} u_{ij}(t,h)[g_1,b_j](t,y(t))dt +$$
$$+ \int_0^{2r_1} \sigma(t,y(t))dw(t) + 2r_1\, \eta(\sqrt{h}) + M_1(r_1) + M_2(r_1)$$

where

$$E\,|M_1(r_1) + M_2(r_1)|^2 = E\,|M_1(r_1)|^2 + E\,|M_2(r_1)|^2 = 2r_1\, \eta(h)$$

Finally for $t = m\tilde{m}r_1 = \tilde{r}_1$ we get

59) $$y_1(\tilde{r}_1) = x_0 + \int_0^{\tilde{r}_1} f^h(t,y(t))dt + \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} \int_0^{\tilde{r}_1} u_{ij}(t,h)[g_i,b_j](t,y(t))dt +$$
$$+ \int_0^{\tilde{r}_1} \sigma(t,y(t))dw(t) + \tilde{r}_1\, \eta(\sqrt{h}) + \sum_{i=1}^{m\tilde{m}} M_i(r_1) = y(\tilde{r}_1) + \tilde{r}_1\, \eta(\sqrt{h}) + M_1(\tilde{r}_1)$$

where

60) $$M_1(\tilde{r}_1) = \int_0^{\tilde{r}_1} [\sigma(t,y_1(t)) - \sigma(t,y(t))]dw(t) +$$
$$+ \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} (m\tilde{m})/(\sqrt{r_1}) \int_0^{\tilde{r}_1} [\tilde{p}_1^0(t,h)(\partial)/(\partial x)(u_{ij}^h b_j)\sigma(t,y_1(t)) +$$
$$+ \tilde{p}_2^0(t,h)(\partial b_j)/(\partial x)\sigma(t,y_1(t))]dw(t),\ E\,|M_1(\tilde{r}_1)|^2 = \tilde{r}_1\, \eta(h)$$

Generally, we get

61) $$y_1(k\tilde{r}_1) = y(k\tilde{r}_1) + k\tilde{r}_1\, \eta(\sqrt{h}) + M^h(k\tilde{r}_1)$$
$k = 0,1,2, \ldots ,K^3, \ldots ,2K^3, \ldots ,K^4.$

where

62) $$M^h(t) = \int_0^t [\sigma(s,y_1(s)) - \sigma(s,y(s))]dw(s) +$$
$$+ \sum_{i=1}^{m} \sum_{j=1}^{\tilde{m}} (m\tilde{m})/(\sqrt{r_1}) \int_0^{r_1} [\tilde{p}_1(s,h)(\partial)/(\partial x)(u_{ij} g_i)\sigma(s,y_1(s)) +$$
$$+ \tilde{p}_i(s,h)(\partial b_j)/(\partial x)\sigma(s,y_1(s))]dw(s),$$

$E\,|M^h(k''r_1) - M^h(k'r_1)|^2 = (k'' - k')\tilde{r}_1\, \eta(h)$ if $k' < k''$,

$k',k'' \in \{0,1,2, \ldots ,K^3, \ldots ,2K^3, \ldots ,K^4\}$ and

$p_i(t,h) = p_i^k(t,h)$, $t \in [k\tilde{r}_1,(k+1)\tilde{r}_1]$, $k = 0,1, \ldots ,K^3, \ldots ,2K^3, \ldots ,K^4-1$,

$\tilde{p}_i(t,h) = \int_0^t p_i(s,h)ds$, $i = 1,2$.

Using (61) and (62) we get the conclusion (C_1) and (C_2) in the statement.

It remains to prove (a) and (b) for S_1 using $r_1(h)$ in the place of $r(h)$. Since $f^h(t,y)$ in S_1 is defined by using the previous functions $u_I(t,h), 1 \leq |I| \leq L$, and $r_1 \cong r^4$ we have to prove (a) and (b) only for $(1)/(\sqrt{r_1})p_1^k(t,h)u_{ij}(t,h)$ and $(1)/(\sqrt{r_1})p_2^k(t,h)$. In this respect, multiplying these functions by r_1 and taking into account that $ru_{ij}(t,h) = \eta(\sqrt{h})v_{ij}(t,h)$, and $p_1^k(\cdot)$, $p_2^k(\cdot)$ are uniformly bounded with respect to h, we obtain (a) for S_1.

By definition $(\partial)/(\partial t)p_j^k(t,h) = (1)/(r_1)\bar{p}_j^k(t,h), j = 1,2$, where $\bar{p}_j^k(\cdot)$ are uniformly bounded with respect to h. It shows that multiplying by r_1^2 the derivatives with respect to t of the indicated functions we get (b) for S_1.

The proof is complete.

Now we are in position to prove Theorem.

Proof of Theorem

By hypothesis, the conditions in Lemma 1 are fulfilled. Therefore we can associate to (1) a system of index $(L-1,h_1)$ defined in (4), with $h_1 = h/M$, $M = \text{card}\{I: |I| = L+1\}$, such that the solutions in (1) and (4) fulfil the statement in Lemma 1. Denote $x^h(\cdot)$, $M^h(\cdot)$ in Lemma 1 by $y_1(\cdot)$ and $M_1^h(\cdot)$ respectively. We have

63) $$y_1(t'') - y_1(t') = y(t'') - y(t') + (t''-t')\eta(\sqrt{h}) + M_1^h(t'') - M_1^h(t'),$$

$t' < t''$, $t',t'' \in \{0,h,2h,\ldots,Nh = T\}$, where

$$E|M_1^h(t'') - M_1^h(t')|^2 = (t''-t')\eta(h)$$

The equation (4) fulfils the hypotheses in Lemma 2 and using Lemma 2 (L-1) times we get the equations S_k of the index $(L-k, r_k)$, $k = 2,\ldots,L$, such that the solution $y_k(\cdot)$ in S_k fulfils (C_1) - (C_3) in Lemma 2.

Namely

64) $$y_k(t'') - y_k(t') = y_{k-1}(t'') - y_{k-1}(t') + (t''-t')\eta(\sqrt{h}) + M_k^h(t'') - M_k^h(t')$$

for any $t' < t''$, $t',t'' \in \{0,\tilde{r}_k,2\tilde{r}_k,\ldots,N_k\tilde{r}_k = T\}$, $N_k = (N_{k-1})^4 = (T/r_{k-1})^4$, $\tilde{r}_k = T/N_k$, $r_k = \tilde{r}_k/M_k$, $M_k = \text{card}\{I: |I| = L+1-k\}$.

Since $\tilde{r}_k$ divides r_{k-1} $(r_{k-1} = (N_{k-1})^3\tilde{r}_k)$ and r_{k-1} divides $\tilde{r}_{k-1}$, it follows that $\tilde{r}_k$ divides $\tilde{r}_{k-1}$, $k = 2,\ldots,L$. Let $t \in [0,T]$, be fixed. Then there exist natural numbers $m_1,\ldots,m_L$, such that

$$t \in [m_L\tilde{r}_L, m_L\tilde{r}_L + \tilde{r}_L) \quad \text{and} \quad m_k\tilde{r}_k \in [m_{k-1}\tilde{r}_{k-1}, m_{k-1}\tilde{r}_{k-1} + \tilde{r}_{k-1}),$$

$k = 2,\ldots,L.$

Using (C_3) in Lemma 2 we get

$(E \max\limits_{s\in I_k} |y_k(s) - y_k(m_k \tilde{r}_k)|^2)^{1/2} \leq \eta(\sqrt{h})$, where

$I_k = [m_k \tilde{r}_k, m_k \tilde{r}_k + \tilde{r}_k]$.

By definition $y_L(.)$ is the solution in system of the index $(0, r_L)$ and denote it by $x^h(.)$. If follows

$$65) \quad x^h(t) - y(t) = y_L(m_L\tilde{r}_L) - y(m_L\tilde{r}_L) + y_L(t) - y_L(m_L\tilde{r}_L) - y(t) + y(m_L\tilde{r}_L) = y_L(m_L\tilde{r}_L) - y(m_L\tilde{r}_L) + 2\eta(\sqrt{h}) =$$

$$= y_L(m_L\tilde{r}_L) - y_{L-1}(m_L\tilde{r}_L) + \eta(\sqrt{h}) + y_{L-1}(m_{L-1}\tilde{r}_{L-1}) - y_{L-2}(m_{L-1}\tilde{r}_{L-1}) + \eta(\sqrt{h}) + \ldots + y_2(m_2\tilde{r}_2) - y_1(m_2\tilde{r}_2) + \eta(\sqrt{h}) + y_1(m_1\tilde{r}_1) - y(m_1\tilde{r}_1) + 2\eta(\sqrt{h})$$

Using (64) in (65) we deduce

$$66) \quad x^h(t) - y(t) = (L+1)\eta(\sqrt{h}) + \sum_{i=1}^{L} m_i\tilde{r}_i \, \eta_i(\sqrt{h}) + \sum_{i=1}^{L} M_i^h(m_i\tilde{r}_i)$$

and

$$67) \quad \max_{t\in[0,T]} E\,|x^h(t) - y(t)|^2 \leq C\,[(1+L(T+1))\,\eta(h) + LT\,\eta(h)] \leq C_1 h$$

for some constant $C_1 > 0$ and the proof is complete using the metric d_2. Now suppose $L = 1$. By definition we have

$$x^h(t) - y(t) = \int_0^t [f(s,x^h(s)) - f(s,y(s))]\,ds + \int_0^t [\sigma(s,x^h(s)) - \sigma(s,y(s))]\,dw(s) + \sum_{i=1}^{m} \int_0^t v_i^h(s) g_i(s,x^h(s))\,ds - \sum_{|I|=2} \int_0^t u_I(s) g_I(s,y(s))\,ds$$

and using Lemma 1 we get

$$\sum_{i=1}^{m} \int_0^t v_i^h(s) g_i(s,x^h(s))\,ds = \sum_{i=1}^{m} \int_0^{m_1\tilde{r}_1} v_i^h(s) g_i(s,x^h(s))\,ds + \sum_{i=1}^{m} \int_{m_1\tilde{r}_1}^{t} v_i^h(s) g_i(s,x^h(s))\,ds =$$

$$= x^h(m_1\tilde{r}_1) - x_0 - \int_0^{m_1\tilde{r}_1} f(s,x^h(s))\,ds - \int_0^{m_1\tilde{r}_1} \sigma(s,x^h(s))\,dw(s) +$$

$$+\eta(\sqrt{h})=y(m_1\tilde{r}_1)-x_0+m_1\tilde{r}_1\,\eta(\sqrt{h})-\int_0^{m_1\tilde{r}_1} f(s,x^h(s))ds-$$

$$-\int_0^{m_1\tilde{r}_1}\sigma(s,x^h(s))dw(s)+\eta(\sqrt{h}) = \int_0^{m_1\tilde{r}_1}[f(s,y(s))-$$

$$-f(s,x^h(s))]\,ds+\int_0^{m_1\tilde{r}_1}[\sigma(s,y(s))-\sigma(s,x^h(s))]\,dw(s)+$$

$$+\sum_{|I|=2}\int_0^t u_I(s)\,g_I(s,y(s))ds+4\,\eta(\sqrt{h})$$

Since

$$E\left|\int_0^{m_1\tilde{r}_1}[\sigma(s,y(s))-\sigma(s,x^h(s))]\,dw(s)\right|^2\le E\max_{t\le\tau}$$

$$\left|\int_0^t[\sigma(s,y(s))-\sigma(s,x^h(s))]\,dw(s)\right|^2\le$$

$$\int_0^\tau E\,|\sigma(s,y(s))-\sigma(s,x^h(s))|^2\,ds$$

and

$$E\left|\int_0^{m_1\tilde{r}_1}[f(s,y(s))-f(s,x^h(s))]\,ds\right|^2\le E\max_{t\le\tau}$$

$$\left|\int_0^t[f(s,y(s))-f(s,x^h(s))]\,ds\right|^2\le\tau\int_0^\tau E\,|f(s,y(s))-$$

$$-f(s,x^h(s))|^2ds$$

it follows

$$E\max_{t\le\tau}|x^h(t)-y(t)|^2\le C\int_0^{\tau} E(\max_{s\le t}|x^h(s)-y(s)|^2)dt+4\,\eta(\sqrt{h})$$

and $d_1(x^h(.),\ y(.))\le C_1\sqrt{h}$ for some constant $C_1>0$.

The proof is complete.

REMARK 4.

Using the conclusion $d_1(x^h(.),\ y(.))\le C_1\sqrt{h}$, for $L=1$, and since the reduced system of $x^h(.)$ for each h fixed fulfils the same conditions as the original system of $y(.)$ it follows that there exist a sequence $\{x_\delta(.)\}_{\delta>0}$ of solutions in (1) such that $\lim\ E\max_{t\in[0,T]}|x_\delta(t)-y(t)|^2=0$ uniformly with respect to $x_0, U_i(.)$ and $U_I(.)$ in bounded sets.

From the last statement it follows that there exists a sequence $\{x_\delta(.)\}_{\delta>0}$ of solutions in (1) such that with probability one $\lim_{\delta\to 0} \max_{t\in[0,T]} |x_\delta(t)-y(t)| = 0$, uniformly with respect to x_0, $u_i(.)$, $U_I(.)$ in bounded sets.

If we relax the hypotheses in theorem by neglecting (1.2) and replacing $g_j \in C_b^{1,L+1}([0,T]\, x R^n)$ in (1.1) with $\partial g_j/\partial x_k \in C_b^{1,L}([0,T]\, x R^n)$ then using a standard argument of truncation it follows that there exists a sequence $\{x_\delta(.)\}_{\delta>0}$ of solutions in (1) such that with probability one $\lim_{\delta\to 0} \max_{t\in[0,T]} |x_\delta(t)-y(t)| = 0$ uniformly with respect to x_0, $u_i(.)$ and $U_I(.)$ in bounded sets.

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