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STABLE RANGE FOR TENSOR PRODUCTS OF EXTENSIONS  
OF  $K$  BY  $C(X)$

by

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INTRODUCTION

In [4] G. Nagy studied the stable rank of the algebras  $\mathcal{T}_n$  of  $n$ -dimensional Toeplitz operators. For  $n$  even his results are exact, while for odd  $n$  the stable rank is only determined up to 1. In this paper we obtain general results for the stable rank of tensor products of Brown-Douglas-Fillmore-extensions [9] of  $K$  (the algebra of compact operators) by commutative  $C^*$ -algebras. Our results are sharp under some additional hypothesis which are fulfilled if the spectra of the commutative  $C^*$ -algebras involved are manifolds. In particular we complete Nagy's results by obtaining the exact value of the stable rank of  $C(X) \otimes \mathcal{T}_n$  if  $X$  is a compact manifold. We obtain also some estimates for M.A. Rieffel's connected stable rank  $\text{csr}(A)$  and introduce an absolute connected stable rank,  $\text{acsr}(A)$ , which turns out to be quite useful and show that it equals the stable rank of  $A \otimes C([0,1])$ .

The first section deals with preliminary material due to

M.A. Rieffel and G. Nagy. The second section contains some general results concerning the stable rank, the connected stable rank and the absolute connected stable rank. In the third section we generalize G.Nagy's estimates. The last section contains some technical lemmas and the proof of the main theorem.

I would like to express my gratitude to Professor Dan Voiculescu for suggesting me this problem.

# I

We begin by recalling some definitions and results from [5].

For a unital  $C^*$ -algebra  $A$  and a natural number  $n$  we consider  $Lg_n(A)$ , the set of  $n$ -tuples of elements of  $A$  which generate  $A$  as a left ideal: the topological stable rank of  $A$  is the least integer  $n$  (if it does not exist it will be taken by definition to be  $\infty$ ) such that  $Lg_n(A)$  is dense in  $A^n$ . According to [6] this number coincides with the usual Bass stable rank of  $A$ , denoted  $sr(A)$  which is the least integer  $n \geq 1$  such that for any  $(a_1, \dots, a_n, a_{n+1}) \in Lg_{n+1}(A)$  there exists  $b_1, \dots, b_n \in A$  such that  $(a_1 + b_1 a_{n+1}, \dots, a_n + b_n a_{n+1}) \in Lg_n(A)$  (if no such  $n$  exists we take  $sr(A)$  to be  $\infty$ ). We denote by  $GL(n, A)$  the group of invertible elements of  $M_n(A)$  and by  $GL^\circ(n, A)$  the connected component of 1 in this group. We shall denote by  $csr(A)$  the connected stable rank of  $A$  which is the least integer  $n \geq 1$  such that the action of  $GL^\circ(m, A)$  by left multiplication on  $Lg_m(A)$  is transitive for any  $m \geq n$  (if no such integer exists we take  $csr(A) = \infty$ ). According to corollary 8.5 of [5]  $csr(A)$  is also the least integer  $n$  such that  $Lg_m(A)$  is connec-



ted for any  $m \geq n$ . If  $A$  is not unital we take  $sr(A)$  ( $csr(A)$ ) to be  $sr(\tilde{A})$  ( $csr(\tilde{A})$ ) where  $\tilde{A}$  is the algebra obtained from  $A$  by adjoining a unit.

We shall use the following results proved in [5], here  $J$  is a twosided ideal in the  $C^*$ -algebra  $A$ :

$$(1.1) \quad sr(J) \leq sr(A) \quad (\text{thm. 4.4})$$

$$(1.2) \quad sr(A/J) \leq sr(A) \quad (\text{thm. 4.3})$$

$$(1.3) \quad sr(A) \leq \max \{ sr(J), sr(A/J), csr(A/J) \} \quad (\text{thm. 4.11})$$

$$(1.4) \quad sr(A \otimes K) = \begin{cases} 1 & \text{if } sr(A) = 1 \\ 2 & \text{if } sr(A) \geq 2 \end{cases} \quad (\text{thm. 6.4})$$

Let  $A = \lim_{c \in I} A_c$  be an inductive limit of  $C^*$ -algebras over the directed set  $I$ , the following results are proved analogously as theorem 5.1 of [5] :

$$(1.5) \quad sr(A) \leq \liminf_{c \in I} sr(A_c)$$

$$(1.6) \quad csr(A) \leq \liminf_{c \in I} csr(A_c)$$

The following results are simple consequences of definitions ( $A, B$  are  $C^*$ -algebras) :

$$(1.7) \quad sr(A \oplus B) = \max \{ sr(A), sr(B) \}$$

$$(1.8) \quad csr(A \oplus B) = \max \{ csr(A), csr(B) \}$$

Lemma 2 of [4] reads :

$$(1.9) \quad csr(A) \leq \max \{ csr(J), csr(A/J) \}$$

We shall use also the following consequence of a classical theorem in dimension theory (see [5] thm. 1.1 and proposition 1.7)

$$(1.10) \quad \text{sr}(C(X)) = [\dim(X)/2] + 1.$$

here  $X$  is a compact space.

It is a simple observation that  $x = (x_1, \dots, x_s) \in \text{Lg}_s(A)$  if and only if there exists  $\varepsilon > 0$  such that  $x_1^* x_1 + \dots + x_s^* x_s \geq \varepsilon 1$  if and only if  $\varphi(x_g^* x_g) > 0$  for any  $g \in \{1, \dots, s\}$  and any pure state  $\varphi$  of the  $C^*$ -algebra  $A$ , this also shows that  $f \in C(X, A^s)$  belongs to  $\text{Lg}_s(C(X, A))$  if and only if  $f(x) \in \text{Lg}_s(A)$  for any point  $x$  in the compact space  $X$ .

If  $A$  is a  $C^*$ -algebra and  $x, y \in A^s$  we shall denote by  $\|x - y\| = \max_g \|x_g - y_g\|$  for  $x = (x_1, \dots, x_s)$ ,  $y = (y_1, \dots, y_s)$ .

## II

In this section we prove some general results concerning the stable rank and the connected stable rank for  $C^*$ -algebras. We also introduce the notion of absolute connected stable rank of  $A$  and prove that it is equal to  $\text{sr}(C([0, 1]) \otimes A)$ .

(2.1) LEMMA. Let  $J$  be a twosided ideal in the  $C^*$ -algebra  $A$  and  $\pi: A \rightarrow A/J$  the quotient map, suppose that  $s \geq \text{sr}(A)$ . Let  $\varepsilon > 0$  and  $x = (x_1, \dots, x_s) \in A^s$  be such that  $\pi(x) = (\pi(x_1), \dots, \pi(x_s)) \in \text{Lg}_s(A/J)$ , then there exists  $x' \in \text{Lg}_s(A)$  such that  $\|x' - x\| < \varepsilon$  and  $\pi(x') = \pi(x)$ . Moreover, if  $y = (y_1, \dots, y_s) \in A^s$  is such that  $\pi(y_1)\pi(x_1) + \dots + \pi(y_n)\pi(x_n) = 1$  then there exists  $y' \in A^s$  such that  $y'_1 x'_1 + \dots + y'_s x'_s = 1$  and  $\pi(y'_g) = \pi(y_g)$  for  $1 \leq g \leq s$ .



PROOF. Denote also by  $\pi$  the map  $M_n(A) \rightarrow M_n(A/J)$ . The set  $V_\delta = \{\exp(\pi(a))\pi(x) \mid \|a\| < \delta, a \in M_S(A)\}$  is a neighbourhood of  $\pi(x)$  in  $Lg_S(A/J)$ . Choose  $x'' \in \pi^{-1}(V) \cap Lg_S(A)$  such that  $\|x'' - x\| < \varepsilon/2$ . Then  $\pi(x'') = \exp(\pi(a))\pi(x)$  for a suitable  $a \in M_S(A)$  with  $\|a\| < \delta$ . Let  $x' = \exp(-a)x'' \in Lg_S(A)$  then  $\|x' - x\| \leq \|x' - x''\| + \|x'' - x\| \leq \|\exp(-a) - 1\| \|x''\| + \varepsilon/2 < (e^\delta - 1)(\|x\| + \varepsilon/2) + \varepsilon/2 < \varepsilon$  if  $\delta$  is small enough.

Since  $\pi(y_1 x_1 + \dots + y_s x_s) = 1$  we get  $y_1 x_1 + \dots + y_s x_s = 1 - a$  with  $a \in J$ . Let  $\{u_i\}$  be an approximate unit for  $J$  and  $y_1'', \dots, y_s'' \in A^S$  such that  $y_1'' x_1' + \dots + y_s'' x_s' = 1$ . Let  $y_g' = (1 - u_c) y_g + u_c y_g''$  for  $1 \leq g \leq s$  then  $\pi(y_g') = \pi(y_g)$  for  $1 \leq j \leq s$  and  $y_1' x_1' + \dots + y_s' x_s' = 1 - (1 - u_c)a$ . Choosing  $c$  big enough we obtain the desired conclusion.

Let  $B_r = \{x = (x_1, \dots, x_r) \in \mathbb{R}^r \mid x_1^2 + \dots + x_r^2 \leq 1\}$ ,  $S^{r-1} = \partial B_r$ . The last lemma yields (for  $J = C(B_r) \otimes B$  and  $A = C(B_r) \otimes B$ ) the following corollary

(2.2) COROLLARY. Let  $B$  be a  $C^*$ -algebra,  $s \geq sr(C(B_r) \otimes B)$  and  $f \in C(B_r, B^S)$  such that  $f(x) \in Lg_S(B)$  for any  $x \in S^{r-1}$  then for any  $\varepsilon > 0$  there exists  $g \in C(B_r, Lg_S(B))$  such that  $f|_{S^{r-1}} = g|_{S^{r-1}}$  and  $\|g - f\| < \varepsilon$ .

(2.3) DEFINITION. Let  $A$  be unital a  $C^*$ -algebra,  $n \geq 1$  an integer, then we say that  $n$  is in the absolute connected stable range of  $A$  if for any nonempty connected open set  $V \subset A^n$ ,  $V \cap Lg_n(A)$  is nonempty and connected. We shall denote by  $acsr(A)$  the least integer  $n$  such for any  $m \geq n$   $m$  is in the absolute connected stable range of  $A$ , if no such integer exists we let  $acsr(A) = \infty$ .

(2.4) LEMMA. Let  $n$  be an integer, then  $n$  is in the absolute connected stable range of the  $C^*$ -algebra  $A$  if and only if  $n \geq \text{sr}(C(I) \otimes A)$  and, consequently,  $\text{acsr}(A) = \text{sr}(C(I) \otimes A)$  ( $I = [0, 1]$ ).

PROOF.  $A$  is a quotient of  $C(I) \otimes A$  and thus (1.2) shows that  $a = \text{sr}(C(I) \otimes A) \geq \text{sr}(A)$ . Let  $n \geq a$ ,  $V \subset A^n$  be an open connected nonempty set. We shall show that  $V \cap \text{Lg}_n(A)$  is connected and conclude that  $n$  is in the absolute connected stable range of  $A$ . Let  $x_0, x_1 \in V \cap \text{Lg}_n(A)$ , there exists a continuous function  $f: I \rightarrow V$  such that  $f(0) = x_0$ ,  $f(1) = x_1$ . Let  $\varepsilon > 0$  be less than the distance between the compact set  $f([0, 1])$  and the closed set  $A^n \setminus V$ , by the last corollary there exists a continuous function  $g: I \rightarrow \text{Lg}_n(A)$  such that  $g(0) = f(0) = x_0$ ,  $g(1) = f(1) = x_1$  and  $\|g - f\| < \varepsilon$ , then  $g$  takes its values in  $\{y \mid d(y, f([0, 1])) < \varepsilon\} \subset V$ .

Conversely, let  $n$  be in the absolute connected stable range of  $A$ , it is then obvious that  $n \geq \text{sr}(A)$ . Let  $\varphi \in C(I, A^n)$ ,  $\varepsilon > 0$ , choose  $m$  big enough such that  $\|\varphi(t_1) - \varphi(t_2)\| < \varepsilon/3$  for any  $t_1, t_2 \in I$ ,  $|t_1 - t_2| < 2/m$ . Since  $n \geq \text{sr}(A)$  we can choose  $y_g \in \text{Lg}_n(A)$  such that  $\|y_g - \varphi(g/m)\| < \varepsilon/3$  for  $0 \leq g \leq m$ . By assumption each of the sets  $V_g = \{y \in \text{Lg}_n(A) \mid \|y - \varphi(g/m)\| < \varepsilon/3\}$  is open and connected containing  $y_{g-1}$  and  $y_g$  for  $1 \leq g \leq m$  thus we may find  $\psi \in C(I, \text{Lg}_n(A))$ ,  $\psi(g/m) = y_g$  and  $\psi(t) \in V_j$  for  $t \in [(j-1)/m, j/m]$ , then  $\|\psi(t) - \varphi(t)\| \leq \|\psi(t) - \varphi(j/m)\| + \|\varphi(j/m) - \varphi(t)\| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon$  proving that  $n \geq \text{sr}(C(I) \otimes A)$ .

(2.5) COROLLARY. 1°  $\text{csr}(A \otimes K) \leq 2$  for  $A$  a  $C^*$ -algebra and  $K$  the algebra of compact operators on a separable Hilbert space.

2°  $\text{csr}(C(X)) \leq [(m+1)/2] + 1$  if  $X$  is a compact space of dimension  $m$ .



Let  $A, A_1, \dots, A_n$  be unital  $C^*$ -algebras,  $\pi_g: A_g \rightarrow A$  morphisms of unital  $C^*$ -algebras, consider the following  $C^*$ -subalgebra of the  $C^*$ -algebra  $A_1 \oplus \dots \oplus A_n$ .  $B = \{x = (x_1, \dots, x_n) \in A_1 \oplus \dots \oplus A_n \mid \pi_1(x_1) = \pi_2(x_2) = \dots = \pi_n(x_n)\}$  and denote it by  $\prod_A A_g$  (it is the usual fiberproduct), denote  $\bigcap_{g=1}^n A_g$  by  $A'$ .

(2.6) PROPOSITION. Suppose that  $s \geq sr(A_g)$  for  $1 \leq g \leq n$  and  $s \geq sr(\bigcap_{g=1}^n \pi_g(A_g))$  then  $s \geq sr(\prod_A A_g)$ .

PROOF. Denote  $\prod_A A_j$  by  $B$ . Let  $x = (x_1, \dots, x_s) \in B^s$ ,  $\varepsilon > 0$ ,  $x_j = (x_{j1}, \dots, x_{jn}) \in A_1 \oplus \dots \oplus A_n$ ,  $1 \leq j \leq s$ , such that  $\pi_1(x_{g1}) = \dots = \pi_n(x_{gn}) = t_g$ . Let  $t = (t_1, \dots, t_s) \in (\bigcap_{j=1}^n \pi_j(A_j))^s = A'^s$ . Let  $t' \in Lg_s(A')$  such that  $\|t' - t\| < \varepsilon$ , then there exists  $\|x'_{gk} - x_{gk}\| < \varepsilon$

such that  $\pi_g(x'_{gk}) = t'_g$  and  $(x'_{1k}, \dots, x'_{sk}) \in Lg_s(A_k)$ . An obvious application of lemma (2.1) shows that  $x' = (x'_1, \dots, x'_s) \in Lg_s(B)$  if  $x'_j = (x'_{j1}, \dots, x'_{jn})$ .

(2.7) COROLLARY. Let  $J_1, \dots, J_n \subset A$  be closed two-sided ideals in the  $C^*$ -algebra  $A$ , then  $sr(A/J) \leq \max_{1 \leq k \leq n} \{sr(A/J_k)\}$  if  $J = J_1 \cap \dots \cap J_n$ .

PROOF. Using induction on  $n$  we may suppose that  $n=2$ . Let  $\pi_j: A/J_j \rightarrow A/(J_1 + J_2) = B$ , then  $A/(J_1 \cap J_2)$  is isomorphic with  $\prod_B A/J_j$ .

Let  $A$  and  $B$  be two unital  $C^*$ -algebras,  $\varphi, \psi \in \text{End}(A, B)$  we say that  $\varphi$  and  $\psi$  are homotopic if there exists  $\eta: I \rightarrow \text{End}(A, B)$  such that  $\eta(0) = \varphi$ ,  $\eta(1) = \psi$  and the map  $\eta_x: I \rightarrow B$   $\eta_x(t) = \eta(t, x)$  is continuous for any  $x \in A$ . We say that

B homotopically dominates A if there exists  $\varphi \in \text{End}(A, B)$ ,  $\psi \in \text{End}(B, A)$  such that  $\psi \circ \varphi$  and  $\text{id}_A$  are homotopic.

(2.8) LEMMA. Let A and B be unital C\*-algebras such that B homotopically dominates A, then  $\text{csr}(A) \leq \text{csr}(B)$ .

PROOF. Let  $\varphi$  and  $\psi$  as above,  $x = (x_1, \dots, x_s) \in \text{Lg}_s(A)$ , suppose that  $s \geq \text{csr}(B)$ , then X can be joined to  $\psi(\varphi(x))$  by an arc in  $\text{Lg}_s(A)$  and  $\varphi(x)$  can be joined by an arc in  $\text{Lg}_s(B)$  to  $(1, 0, \dots, 0)$ , this shows that x can be joined to  $(1, 0, \dots, 0)$  by an arc in  $\text{Lg}_s(A)$  proving that  $s \geq \text{csr}(A)$  and hence that  $\text{csr}(A) \leq \text{csr}(B)$ .

(2.9) COROLLARY.  $\text{csr}(C(B_r) \otimes A) = \text{csr}(A)$ .

(2.10) PROPOSITION.  $\text{csr}(M_m(A)) \leq \{(\text{csr}(A)-1)/m\} + 1$  (here  $\{x\}$  denotes the least integer greater than x and  $M_{m,n}(A)$  denotes the set of matrices with entries in A with m lines and n rows,  $M_{m,m}(A)$  is also denoted by  $M_m(A)$ ).

PROOF. Let  $x \in M_m(A)^s$ , we may view x as an element of  $M_{ms,m}(A)$ :  $x = (a_{ig})$   $1 \leq i \leq ms$ ,  $1 \leq g \leq m$ .  $x \in \text{Lg}_s(M_m(A))$  means that the equation (1)  $\sum_{i=1}^{ms} b_{ki} a_{ij} = \delta_{kj}$  has a solution. This shows that the first row is in  $\text{Lg}_{ms}(A)$ . If  $ms \geq \text{csr}(A)$  there exists  $B_1 \in \text{GL}^\circ(ms, A)$  such that

$$B_1 x = \begin{pmatrix} 1 & x_1 \\ 0 & x_2 \\ \vdots & \\ 0 & \end{pmatrix}$$

where  $x_1$  stands for  $((Bx)_{12}, (Bx)_{13}, \dots, (Bx)_{1m})$



a straightforward computation shows that the first row in the matrix  $x_2$  is in  $Lg_{ms-1}(A)$ . By induction we get that there exists  $B \in GL^o(ms, A)$  such that

$$Bx = \begin{pmatrix} 1 & & * \\ 0 & 1 & \\ \vdots & 0 & 1 \\ \vdots & & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

whenever  $ms-m+1 \geq csr(A)$  and this matrix can be joined with the matrix  $(\delta_{ij})_{1 \leq i \leq ms, 1 \leq j \leq m}$  in  $Lg_s(M_m(A))$ .

(2.11) COROLLARY. Let  $A$  be an AF-C\*-algebra and  $B$  a C\*-algebra, then  $sr(A \otimes B) \leq sr(B)$ ,  $csr(A \otimes B) \leq csr(B)$ . Equality holds if  $A$  is commutative.

PROOF. By theorem 6.1 of [5] we know that  $sr(M_m(A)) = \{(sr(A)-1)/m\} + 1 \leq sr(A)$ , the last proposition shows that  $csr(M_m(A)) \leq csr(A)$ . Using (1.5), (1.6), (1.7) and (1.8) we get the desired inequalities. The rest is obvious.

(2.12) COROLLARY. Let  $A$  be a C\*-algebra, then  $csr(A)=1$  implies  $csr(A \otimes K)=1$ .

### III

In this section we obtain some preliminary results using mainly inequality (1.3).

Let us fix some notations to be used from now on. We shall denote by  $X(X_0, \dots, X_n)$  a compact space of dimension

$m(m_0, \dots, m_n)$ ,  $X = X_0 \times \dots \times X_n$ .

Let  $A_1, \dots, A_n$  be BDF-extensions of  $K$  by  $C(X_g)$ , they satisfy an exact sequence:

$$(3.1) \quad 0 \rightarrow K \rightarrow A_j \rightarrow C(X_j) \rightarrow 0$$

Let  $A = C(X_0) \otimes A_1 \otimes \dots \otimes A_n$ ,  $B_j = C(X_0) \otimes \dots \otimes C(X_j) \otimes A_{j+1} \otimes \dots \otimes A_n$ , for  $0 \leq j \leq n$  (thus  $B_0 = A$ ,  $B_n = C(X)$ ),  $C_1 = C(X_0) \otimes \dots \otimes C(X_{j-1}) \otimes A_{j+1} \otimes \dots \otimes A_n$ , then from (3.1) we get the following exact sequence:

$$(3.2) \quad 0 \rightarrow K \otimes C_{j-1} \rightarrow B_{j-1} \rightarrow B_j \rightarrow 0 \quad 1 \leq j \leq n$$

Let us denote by  $c_j = \text{csr}(B_j)$ ,  $s_j = \text{sr}(B_j)$ ,  $0 \leq j \leq n$ , from (1.3) and (1.9) we get

$$(3.3) \quad c_{j-1} \leq \max\{c_j, \text{csr}(K \otimes C_{j-1})\} \leq \max\{c_j, \text{csr}(C_{j-1})\}$$

$$s_j \leq s_{j-1} \leq \max\{c_j, s_j, \text{sr}(K \otimes C_{j-1})\}.$$

So we obtain  $c_0 \leq \max_{1 \leq j \leq n} \{c_n, \text{csr}(K \otimes C_{j-1})\}$ ,  $s_n \leq s_0 \leq \max_{1 \leq j \leq n} \{s_n, c_n, \text{sr}(K \otimes C_{j-1}), \text{csr}(K \otimes C_{j-1})\}$ , but  $S_n = \lfloor m/2 \rfloor + 1$ ,  $c_n \leq \lfloor (m+1)/2 \rfloor + 1$ . If  $m=0$  we can prove by induction on  $n$  (using (3.3), (2.5) and (2.12)) that  $\text{csr}(K \otimes C_{j-1}) = 1$ ,  $\text{sr}(K \otimes C_{j-1}) = 1$  and thus  $S_0 = C_0 = 1$ . (This also follows by a theorem of L.G. Brown which implies that  $A$  is an AF-C\*-algebra).

(3.4) PROPOSITION. Let  $m \neq 1$  then



$$\lfloor m/2 \rfloor + 1 \leq \text{sr}(A) \leq \max \{ \lfloor m/2 \rfloor + 1, \text{csr}(C(X)) \}$$

$$\text{csr}(A) \leq \text{csr}(C(X))$$

For  $m=1$   $\text{sr}(A)$ ,  $\text{csr}(A) \in \{1, 2\}$ .

PROOF. We have proved everything except  $\text{csr}(A) \leq \text{csr}(C(X))$  but this follows from (3.3) by induction on  $n$  since  $C_{j-1}$  has at most  $n-1$  extension terms in the tensor product.

$$(3.5) \text{ COROLLARY. } \lfloor m/2 \rfloor + 1 \leq \text{sr}(A) \leq \lfloor (m+1)/2 \rfloor + 1$$

(3.6) REMARK. This corollary is the analogue of theorem 6 of [4]. Its proof is inspired by that in [4].

To improve the last corollary we shall make from now on the following assumptions:  $X_j$  is the inverse limit of  $m_j$ -dimensional finite CW-complexes,  $\dim(X_j) = m_j$  for  $0 \leq j \leq n$  and  $\dim(X) = \dim(X_0) + \dim(X_1) + \dots + \dim(X_n) = m$ .

$$(3.7) \text{ LEMMA. Suppose that } m_0 > 0, \text{ then } \text{sr}(A) = \lfloor m/2 \rfloor + 1.$$

PROOF. Using (1.5) we may suppose that  $X_0$  is a finite CW-complex of dimension  $m_0 > 0$ . Let  $X_0 = X'_0 \cup U_1 \cup \dots \cup U_k$  where  $X'_0$  is a CW complex of dimension less than  $m_0$  and  $U_j$  are disjoint open set such that there exists functions  $\varphi_j: B_{m_0} \rightarrow \bar{U}_j$  such that

$$\varphi_j(\partial B_{m_0}) \subset X'_0 \text{ and } \varphi_j|_{\overset{\circ}{B}} \text{ is a homeomorphism of } \overset{\circ}{B}_{m_0} \text{ onto}$$

$$U_j \quad (1 \leq j \leq k) \quad \text{and } X'_0 \cap U_j = \emptyset. \text{ Then } \text{sr}(C(X'_0) \otimes A_1 \otimes \dots \otimes A_n) \leq$$

$$\lfloor m/2 \rfloor + 1 = s \text{ by corollary (3.5). Let } f \in C(X_0, (A_1 \otimes A_2 \otimes \dots \otimes A_n)^S),$$

$$\varepsilon > 0. \text{ There exists } g \in C(X'_0, (A_1 \otimes \dots \otimes A_n)^S), \quad g(x) \in$$

$$Lg_S(A_1 \otimes \dots \otimes A_n), \quad \|g - f|_{X'_0}\| < \varepsilon/2. \text{ Extend } g \text{ to } X_0 \text{ such that}$$

the last inequality be fulfilled. Using the functions  $\varphi_j$  we pullback  $g$  to a function  $g_j = g \circ \varphi_j \in C(B_{m_0}, (A_1 \otimes \dots \otimes A_n)^S)$ . But  $\text{csr}(C(B_{m_0}) \otimes A_1 \otimes \dots \otimes A_n) = \text{csr}(A_1 \otimes \dots \otimes A_n)$  by corollary (2.9), thus  $\text{csr}(C(B_{m_0}) \otimes A_1 \otimes \dots \otimes A_n) \leq [m/2] + 1 = s$  by corollary (2.5). By corollary (2.2) there exists  $h_j \in C(B_{m_0}, \text{Lg}_S(A_1 \otimes \dots \otimes A_n))$ ,  $\|h_j - g_j\| < \varepsilon/2$ ,  $h_j|_{s^{m_0-1}} = g_j|_{s^{m_0-1}}$ . Then the function  $h: X_0 \rightarrow \text{Lg}_S(A_1 \otimes \dots \otimes A_n)$  given by  $h(x) = g(x)$  for  $x \in X_0^*$  and  $h(x) = h_j(\varphi_j^{-1}(x))$ ,  $x \in U_j$  satisfies  $\|h - f\| < \varepsilon$  and  $h|_{X_0^*}$ ,  $h \circ \varphi_j$  are continuous, this shows that  $h$  is continuous.

#### IV

This section contains some results concerning the irreducible representations of  $A$  and the proof of the main theorem.

We know that there exists a unique class of irreducible representations of  $K$  and hence a unique class of irreducible representations of  $A_j$  not vanishing on  $K$  (see [1]), denote by  $\rho_j$  an element of this class. Let  $\pi_j: A_j \rightarrow C(X_j)$  the quotient map and  $\sigma_j = \pi_j \otimes 1$  here 1 stands for the identity map on  $A_1 \otimes \dots \otimes A_{j-1} \otimes A_{j+1} \otimes \dots \otimes A_n$ .

(4.1) LEMMA. Let  $\pi$  be an irreducible representation of  $A = A_1 \otimes \dots \otimes A_n$ , then one of the following statements is true:

- 1°  $\pi$  is unitarily equivalent to  $\rho_1 \otimes \dots \otimes \rho_n = \rho$
- 2°  $\pi$  factors through  $\sigma_j$  (i.e. there exists  $j \in \{1, \dots, n\}$  and  $\pi_0$  such that  $\pi = \pi_0 \circ \sigma_j$ ).



PROOF. Let  $J_j = \ker \sigma_j$ , there are exactly two possibilities (see [1]) - 1°  $\pi|_{J_1 J_2 \dots J_n}$  is irreducible, and hence unitarily equivalent to  $\rho_1 \otimes \dots \otimes \rho_n / J_1 \dots J_n$  since  $J_1 \dots J_n \cong K$ , this shows ([1]) that  $\pi$  and  $\rho$  are unitarily equivalent; 2° there exists  $j$ ,  $1 \leq j \leq n$  such that  $\pi|_{J_j} = 0$  but this means that factors through  $\sigma_j$ .

Let us observe that  $\rho(J) = K$  ( $J = J_1 J_2 \dots J_n$ ).

The following lemma shows why the case  $m=1$  is an exceptional one, let  $\rho = \rho_1 \otimes \dots \otimes \rho_n$ ,  $H = H_\rho$  (see [1] for notation).

(4.2) LEMMA. Let  $\varepsilon > 0$ ,  $x, y \in A = A_1 \otimes \dots \otimes A_n$ , then there exists  $x_1, y_1 \in A$  such that  $\|x - x_1\| < \varepsilon$ ,  $\|y - y_1\| < \varepsilon$  and  $\ker \rho(x_1) \cap \ker \rho(x_2) = \{0\}$ .

PROOF. We shall denote by  $[M]$  the orthogonal projection onto the closed subspace  $M$  of a Hilbert space  $\mathcal{H}$ .

Suppose first that at least one of the operators  $\rho(x)$  and  $\rho(y)$ , say  $\rho(x)$ , has the property that  $\dim H / \rho(x)H = \infty$ . Then there exists a compact operator  $T$ ,  $\|T\| < \varepsilon/2$ , such that  $\dim H / (\overline{\rho(x) + T})H = \infty$ , (see [3]) there exists also a compact operator  $T_1$ ,  $\|T_1\| < \varepsilon/2$  with right support  $r(T_1) = \ker(\rho(x) + T) \cap \ker \rho(y)$  and left support  $\ell(T_1) \leq 1 - [\text{Ran}(\rho(x) + T)]$ . If we let  $y_1 = y$  and  $x_1$  such that  $\|x_1 - x\| < \varepsilon$  and  $\rho(x_1 - x) = T + T_1$  we obtain the desired conclusion.

Suppose now that  $H / \rho(x)H$  and  $H / \rho(y)H$  are finite dimensional then there exists an  $\delta > 0$  such that  $(0, \delta) \cap \sigma(\rho(x)\rho(x)^*) = \emptyset$ , let  $f(0) = 1$  and  $f(t) = 0$  for  $t \geq \delta$ , then  $f(\rho(x)\rho(x)^*) = 1 - [\rho(x)H]$  and  $f(\rho(x)^*\rho(x)) = [\ker \rho(x)]$ . By assumption there exists  $a \in J$  such

that  $\rho(a) = \rho(f(xx^*))$ , since  $A/(J_1 + \dots + J_n)$  is commutative we get that  $f(xx^*) - f(x^*x) \in J_1 + \dots + J_n$  thus obtaining that there exists  $b \in J_1 + \dots + J_n$  such that  $\rho(b) = [\ker \rho(x)]$ . We want to show that  $\text{Ran } \rho(y) / \ker \rho(x) = \text{Ran } \rho(yb)$  has infinite codimension to conclude the proof as above. Suppose the contrary then the same will be true for any  $T$  close enough to  $\rho(xb)$  and we may find such a  $T$  of the following particular type  $T = \sum_{j=1}^n (\sum_{k=1}^{m_j} A_{jk} \otimes F_{jk} \otimes B_{jk})$  with  $F_{jk}$  a finite rank operator on  $H_{\rho_j}$  and  $A_{jk}$  and  $B_{jk}$  bounded operators. Let  $\xi_j \perp \text{Ran } F_{jk}$  ( $1 \leq j \leq n$ ,  $1 \leq k \leq m_j$ ) then  $\xi = \xi_1 \otimes \dots \otimes \xi_n$  is orthogonal to  $\text{Ran } T$  and these vectors span an infinite dimensional vector space, this is a contradiction.

(4.3) OBSERVATION. Suppose that  $m_n = \dim(X_n) = 1$  then  $\text{sr}(A_n) \in \{1, 2\}$  by proposition (3.4). Let  $x \in A_n$ ,  $\varepsilon > 0$ , there exists  $x_1 \in A_n$  such that  $\|x_1 - x\| < \varepsilon/2$  and  $\pi_n(x_1) \in \text{Lg}_1(C(X_n)) = \text{GL}(1, C(X_n))$  this shows that  $\rho_n(x_1)$  is a Fredholm operator which will be left-invertible if and only if  $\ker \rho_n(x_1) = 0$ . If  $\text{ind}(\rho_n(x_1)) = \dim \ker(\rho_n(x_1)) - \dim \ker(\rho_n(x_1)^*) \leq 0$  we can find a finite rank operator  $F$  such that  $\|F\| < \varepsilon/2$  and  $\ker(\rho_n(x_1) + F) = \{0\}$  thus finding  $x_2 = x_1 + a(\rho_n(a) = F)$  such that  $\rho_n(x_2)$  is left invertible. Lemma (4.1) shows that  $x_2 \in \text{Lg}_1(A_n)$ . We obtained that  $\text{sr}(A_n) = 1$  if and only if any Fredholm operator in  $A_n$  has index 0, if and only if the composed map  $H^1(X_n) \rightarrow K_1(C(X_n)) \rightarrow K_0(K) = \mathbb{Z}$  is trivial (see [7]).

Suppose that  $X, X_0, \dots, X_n$  satisfy the assumptions preceding lemma (3.7),  $A = C(X_0) \otimes A_1 \otimes \dots \otimes A_n$ ,  $m_j = \dim(X_j)$ ,  $0 \leq j \leq n$



(4.4) THEOREM. Let  $m \neq 1$  then  $\text{sr}(A) = \lfloor m/2 \rfloor + 1$ . If  $m=1$  let  $m_1 \leq \dots \leq m_n$ , then  $\text{sr}(A) = \text{sr}(A_n)$ .

PROOF. Let  $s = \lfloor m/2 \rfloor + 1$ . For  $m=0$  the theorem has already been proved. For  $m=1$  there are two possibilities: 1°  $\dim(X_0)=1$ ,  $\dim(X_1)=\dots=\dim(X_n)=0$ , then  $\text{sr}(A)=1=\text{sr}(A_n)$  by lemma (3.7); 2°  $\dim(X_0)=\dots=\dim(X_{n-1})=0$ ,  $\dim(X_n)=1$ , then  $C(X_0) \otimes A_1 \otimes \dots \otimes A_{n-1}$  is an AF-C\*-algebra (see [2]) and thus, by corollary (2.11)  $\text{sr}(A) \leq \text{sr}(A_n)$ , the reverse inequality follows from the fact that  $A_n$  is a quotient of  $A$ .

Suppose now that  $m \geq 2$ . We shall use induction on  $n$ , for  $n=0$  there is nothing to prove. Let  $x=(x_1, \dots, x_s) \in A^s$ ,  $\varepsilon > 0$ . If  $m_0 > 0$  the theorem follows from lemma (3.7), if  $m_0=0$   $C(X_0)$  is an AF-C\*-algebra and thus  $\text{sr}(A) = \text{sr}(A_1 \otimes \dots \otimes A_n)$  by corollary (2.11), we may suppose then that  $A = A_1 \otimes \dots \otimes A_n$ . By the induction hypothesis there exists  $x'=(x'_1, \dots, x'_s) \in A^s$  such that  $\|x-x'\| < \varepsilon/2$  and  $\sigma_j(x') \in \text{Lg}_s(A_1 \otimes \dots \otimes A_{j-1} \otimes C(X_j) \otimes \dots \otimes A_n)$  we may also find  $x''$  such that the last condition be fulfilled,  $\|x''-x'\| < \varepsilon/2$  and  $\ker \rho(x''_1) \cap \ker \rho(x''_2) \neq \{0\}$  then by lemma (4.1)  $x'' \in \text{Lg}_s(A)$ .

(4.5) REMARK. It is obvious that if  $A$  and  $B$  are C\*-algebras as in the theorem (4.4) then  $\text{sr}(A \otimes B) \leq \text{sr}(A) + \text{sr}(B)$  (this answers a question from [5] in a particular case).

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