INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

SOME REMARKS ON ANTICANONICAL MODELS AND NORMAL SURFACES WITH VANISHING PLURIGENERA

by

Lucian BADESCU

PREPRINT SERIES IN MATHEMATICS

No. 1/1986

BUCURESTI

SOME REMARKS ON ANTICANONICAL MODELS AND NORMAL SURFACES WITH VANISHING PLURIGENERA

by

Lucian BADESCU*)

January 1986

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania

SOME REMARKS ON ANTICANONICAL MODELS AND NORMAL SURFACES WITH VANISHING PLURIGENERA

To Professor G. Galbură on his 70th birthday

by Lucian Bădescu

Introduction

This paper is motivated by our previous study of normal projective degenerations of rational and ruled surfaces [4].

The first section is a variation on a fundamental result of Zariski (see [19]) concerning the finite generatedness of the graded k-algebra $R(X,D)$ associated to a smooth projective surface X together with a divisor D such that the Iitaka dimension $\kappa(X,D)$ is 2, and is greatly infuenced by Sakai's theory of normal varieties (see [12] and [13]) as well as by [3].

In section 2 we relate the class of normal rational surfaces with at most rational singularities and vanishing plurigenera (which naturally occur as degenerations of rational surfaces) with the class of smooth open surfaces with vanishing logarithmic plurigenera.

Throughout the paper we shall fix an algebraically closed field k of arbitrary characteristic. Sometimes k will be assumed to be the field $\mathbb{C}$ of complex numbers. The terminology and notations (when not explained) are the standard ones.

§1. Some remarks on a result of Zariski

Let (X,D) be a pair consisting of a smooth projective surface over k, and D a divisor on X. Let us denote by $\kappa(X,D)$ the Iitaka dimension (or the D-dimension) of X (see [7]), and by $R(X,D)$ the graded k-algebra

$\bigoplus_{n\geq 0} H^0(X,O_X(nD))$. If we assume that $\kappa(X,D)\geq 0$ (i.e. $|nD|\neq\emptyset$ for some $n>0$), then by [19] there is a unique decomposition (the Zariski decomposition of D) $D = P + N$ such that: P is a numerically effective $\mathbb{Q}$-divisor (nef divisor) (i.e a divisor with rational coefficients such that $P.C \geq o$ for every curve C on X), $N \geq o$, and if $N \neq o$ then the intersection matrix of $\mathrm{Supp}(N)$ is negative definite and $P.C = o$ for every irreducible component C of $\mathrm{Supp}(N)$. The divisor P (resp. N) will be referred to as the semi-positive (resp. negative) part of D.

Recall the following fundamental result proved in [19].

Theorem A (Zariski). i) If $K(X,D) \leq 1$ then $R(X,D)$ is a finitely generated k-akebra.

ii) If $K(X,D) = 2$ then $R(X,D)$ is a finitely generated k-algebra iff there is a positive integer m such that mP is an integral divisor and the linear system $|mP|$ has no fixed components (or, equivalently, by another result of Zariski [19], if there is a positive integer n such that nP is an integral divisor and the linear system $|nP|$ has no base points).

Inspired by the work of Sakai (especially [12] and [13]) as well as by [3], we are going to make some elementary remarks concerning part ii) of the above theorem, trying to interpret it in terms of the concept of the model of (X,D). Let us first explain what the model of a pair (X,D) with $K(X,D) = 2$ is (see [13]).

First of all, $K(X,D) = 2$ iff $P^2 > o$ (this is an easy consequence of Riemann-Roch theorem, see e.g. [15]). Let A be the set of all irreducible curves C on X such that $P.C = o$. Since $P^2 > o$ the Hodge index theorem implies that this set is finite and the intersection matrix of A is negative definite. Thus one can apply a well known criterion of contractibility due to Grauert and Artin to deduce that there is a unique birational morphism $u: X \longrightarrow Y$, with Y a complete normal 2-dimensional algebraic space over k, such that u blows down the connected components of A to points and yields an isomorphism between $X-A$ and $Y-u(A)$. Set $D' = u_*(D)$. Then D' is a Weil divisor on Y, and the pair (Y,D') is by definition the model of (X,D).

Before stating the first two results which show the use of this concept, we need Mumford's definition of the intersection number of two Weil divisors on a complete normal 2-dimensional algebraic space Z (see [11], or also [13]). Let $f: Z' \longrightarrow Z$ be a resolution of the singularities of Z and D a $\mathbb{Q}$ - Weil divisor on Z. Let $E_1, \ldots, E_n$ be the irreducible components of all exceptional fibres of f. Then Mumford defines first the inverse image $f^*(D)$ of D by $f^*(D) = D' + \sum_{j=1}^{n} a_j E_j$, where D' is the strict transform of D by f, and the rational numbers a_j are uniquely determined by the following linear equations $D'.E_h + \sum_{j=1}^{n} a_j E_j.E_h = o$, $h = 1, \ldots, n$ (recall that the intersection matrix $\| E_i.E_j \|$ is negative definite).

Note that in general $f^*(D)$ is only a $\mathbb{Q}$-divisor even if D is integral. However, if D is an integral Cartier divisor then $f^*(D)$ is also integral and coincides with the usual inverse image of D. Now, Mumford defines the intersection number $D_1.D_2$ of two $\mathbb{Q}$-Weil divisors D_1 and D_2 by the formula

$$D_1.D_2 = f^*(D_1).f^*(D_2).$$

It turns out that $D_1.D_2$ is a rational number which is independent of the resolution f. Finally, let $F = \sum_i a_i F_i$ be a $\mathbb{Q}$-Weil divisor on a complete normal 2-dimensional algebraic space Z, where F_i is irreducible and reduced and $F_i \neq F_j$ for $i \neq j$. Let $[F]$ denote the divisor $[F] = \sum_i [a_i] F_i$, where if r is a real number, $[r]$ denotes the greatest integer $\leq r$. Then one defines the associated sheaf $O_Z(F)$ of the $\mathbb{Q}$-Weil divisor F by $O_Z(F) = O_Z([F])$, which is a reflexive rank one sheaf on Z. This definition allows one to define the graded k-algebra $R(Z,F)$ of the pair (Z,F) and the F-dimension of Z, $\kappa(Z,F)$, in the same way as at the beginning. The one has the following result, essentially due to Sakai ([12] and [13]):

Proposition 1. i) *Let (Y,D') be the model of the pair (X,D), with X a smooth projective surface and D a $\mathbb{Q}$-divisor such that $\kappa(X,D) = 2$. Then D' is numerically ample (i.e. $D'^2 > 0$ and $D'.C' > 0$ for every curve C' on Y) and $u^*(D') = P$ (the semi-positive part of D).*

ii) $u_*(O_X(nD)) \cong O_Y(nD')$ *for every $n \geq 0$, and in particular, $R(X,D) \cong R(Y,D')$.*

iii) *Conversely, let Y be a complete normal 2-dimensional algebraic space, D' a numerically ample $\mathbb{Q}$-Weil divisor on Y, and $u: X \longrightarrow Y$ the minimal resolution of Y. Then for every $\mathbb{Q}$-divisor D of the form $D = u^*(D') + N$, with $N \geq 0$ and Supp(N) contained in the exceptional fibres of u, one has $\kappa(X,D) = 2$ and (Y,D') is the model of (X,D).*

Proof. i) Since $D' = u_*(D)$ one has $D = u^*(D') + N'$, where N' is a $\mathbb{Q}$-divisor whose support is contained in the exceptional set A of u. Let $E_1, \ldots, E_n$ be all irreducible components of A. Since both P and $u^*(D')$ are perpendicular on each E_i, we get $N.E_i = N'.E_i$ for every $i = 1, \ldots, n$, and since the intersection matrix of A is negative definite, it follows that $N = N'$ and $u^*(D') = P$. Therefore $D'^2 = P^2 > 0$ since $\kappa(X,D) = 2$. On the other hand, if C' is an arbitrary irreducible curve on Y and C is its strict transform by u, we have $u^*(C') = C + N''$, with $\mathrm{Supp}(N'') \subseteq A$ and $N''.E_i + C.E_i = 0$ for every $i = 1, \ldots, n$. Consequently, $D'.C' = u^*(D').u^*(C') = P.C + P.N'' = P.C > 0$ because C is not a component of A.

ii) Everything follows from i) and from [13], especially theorem (6.2).

iii) The proof is obvious because $D = u^*(D') + N$ is nothing but the Zariski decomposition of D. Q.E.D.

The next result interprets part ii) of theorem A in the setting of models.

Proposition 2. Let (X,D) be a pair consisting of a smooth projective surface X and a $\mathbb{Q}$-divisor D such that $\kappa(X,D) = 2$. Then $R(X,D)$ is a finitely generated k-algebra iff D' is a $\mathbb{Q}$-Cartier divisor on the model (Y,D') of (X,D), i.e. if there is a positive integer n such that nD' is an (integral) Cartier divisor. In particular, if $R(X,D)$ is a finitely generated k-algebra then Y is a projective surface.

Proof. If nD' is a Cartier divisor for some $n > o$ then by proposition 1, i) and Nakai-Moishezon criterion of ampleness (for the variant of Nakai-Moishezon criterion in the case of algebraic spaces over $\mathbb{C}$, see [5]) we infer that nD' is actually an ample Cartier divisor. In particular, Y is a projective surface. Using [6], proposition (3.3) we get that $R(Y,D')$ is a finitely generated k-algebra, and by proposition 1, ii), $R(X,D)$ (which is isomorphic to $R(Y,D')$) is also finitely generated. Alternatively, if D were an integral divisor, the linear system $|mnD'|$ has no base points for some $m \gg o$, and therefore $u^*(mnD') = mnP$ (proposition 1) is integral and has no base points. By theorem A, $R(X,D)$ is finitely generated.

Conversely, assume $R(X,D)$ finitely generated; by theorem A, the linear system $|mP|$ has no base points for some $m > o$ such that mP is integral. Let $\varphi : X \longrightarrow \mathbb{P} = \mathbb{P}^d$ be the associated morphism such that $\varphi^*(O_{\mathbb{P}}(1)) \cong O_X(mP)$. Since $P.E_i = o$ for every $i = 1,\dots,n$ (we are keeping the notations of the proof of proposition 1), we infer that $\varphi(E_i)$ is a point for every i, and by the definition of Y, the morphism φ factors as $X \xrightarrow{u} Y \xrightarrow{\psi} \mathbb{P}$. If $L = \psi^*(O_{\mathbb{P}}(1))$, we have $u^*(L) \cong O_X(mP)$, and hence by projection formula, $L \cong u_*(O_X(mP))$. On the other hand, since $u^*(mD') = mP$, using theorem (6.2) of [13] we get $u_*(O_X(mP)) \cong O_Y(mD')$. Thus $L \cong O_Y(mD')$, and since L is invertible, mD' is a Cartier divisor. Q.E.D.

Remark. Proposition 2 is inspired from [3], where the particular case $D = -K_X$ and $\kappa^{-1}(X) := \kappa(X,-K_X) = 2$ (whith K_X a canonical divisor of X) was treated. It was also shown in [3] that the "anticanonical" model Y is always a projective surface. On the other hand, there are examples of ruled non-rational surfaces X over $\mathbb{C}$ with $\kappa^{-1}(X) = 2$ such that $R^{-1}(X) := R(X,-K_X)$ is not finitely generated (see [17]). Thus we see that in general Y projective does not imply $R(X,D)$ finitely generated.

Keeping the notations and assumptions of proposition 1, the morphism $u : X \longrightarrow$

is nothing but a resolution of the singularities of Y. Let $u_o : X_o \longrightarrow Y$ be the minimal resolution of Y (in the sense that there are no exceptional curves of the first kind in the exceptional fibres of u). It is a general fact that u (uniquely dominates u_o, i.e. there is a unique birational morphism $v: X \longrightarrow X_o$ such that $u_o \circ v = u$. Since $\kappa(X,D) = 2$, D may be assumed effective, and since the effectiveness is preserved by direct images, $D_o := v_*(D)$ is also effective. Thus D_o has a Zariski decomposition (say) $D_o = P_o + N_o$, with P_o (resp. N_o) the semi-positive (resp. negative) part of D_o.

We claim that $P_o = v_*(P)$, $P = v^*(P_o)$ and $N_o = v_*(N)$. To prove this claim, we may assume that v is the blowing down morphism of an exceptional curve of the first kind E such that $P.E = o$ (in Sakai's terminology, such a curve is called D-redundand, see [12] for the case $D = -K_X$). Indeed, v is a composition of a finite number of such blowing downs. Set $P_1 = v_*(P)$ and $N_1 = v_*(N)$. Since P is nef and since this property is preserved by direct images, P_1 is also nef. Since $P.E = o$ we get $P = v^*(P_1)$. Since N is effective, N_1 is also effective. Let C_1 be an irreducible component of N_1 and C the strict transform of C_1 by v. Then C is a component of N and we have $P_1.C_1 = v^*(P_1).v^*(C_1) = P.(C + (C.E)E) = P.C = o$. Since $P_1^2 = P^2 > o$ and $P_1.C = o$, by Hodge index theorem we infer that the intersection matrix of $\mathrm{Supp}(N_1)$ is negative definite (unless $N_1 = o$). Therefore $D_o = P_1 + N_1$ is a Zariski decomposition of D_o and the claim follows from the uniqueness of Zariski decomposition.

The claim shows in particular that $\kappa(X_o,D_o) = 2$ and that (Y,D') is also the model of (X_o,D_o). Now, from proposition 1 we deduce that $P = u^*(D')$ and $P_o = u_o^*(D')$. Since N and N_o are effective we can apply theorem (6.2) of [13] to deduce that $u_*(O_X(nD)) \cong O_Y(nD') \cong u_{o*}(O_{X_o}(nD_o))$ for every $n \geq o$. In particular, $R(X,D) \cong R(X_o,D_o)$. These considerations prove the following result (essentially due to Sakai, at least in case $D = -K_X$, although his proof works along different lines, see [12]):

<u>Proposition 3.</u> <u>Let (X,D) be as in proposition 1, i). Then there is a unique birational morphism</u> $v: X \longrightarrow X_o$ <u>such that</u> X_o <u>is a smooth projective surface with no</u> D_o<u>-redundand exceptional curves of the first kind,</u> $R(X,D) \cong R(X_o,D_o)$ <u>and the pairs</u> (X,D) <u>and</u> (X_o,D_o) (<u>with</u> $D_o = v_*(D)$) <u>have the same model.</u>

When we are dealing with a problem involving the graded ring R(X,D) (e.g. the finite generatedness of of R(X,D)), proposition 3 shows that there is no loss of generality in assuming that X contains no D-redundand exceptional curves of the first kind. For example, this reduction is essential in order to define the effective $\mathbb{Q}$-divisor Δ on X which carries a lot of information about the singularities of the model (Y,D') of (X,D). Recall that Δ is by definition the unique $\mathbb{Q}$-divisor on X satisfying the following two conditions:

a) $\mathrm{Supp}(\Delta) \subseteq A$, where A is the exceptional set of u, and

b) $\Delta . E = -K_X . E$ for every irreducible component E of A, with K_X a canonical divisor of X.

Since X contains no D-redundand exceptional curves of the first kind, $K_X . E \geqslant o$, and hence $\Delta . E \leqslant o$ for every component E of A. As one can easily see, the support of Δ is precisely the union of the exceptional fibres of u over all singular points of Y which are not rational double points. Alternatively, one could define Δ as the unique $\mathbb{Q}$-divisor on X such that $u^*(K_Y) = K_X + \Delta$, where K_Y is a canonical (Weil) divisor on Y.

Let r be the smalles positive integer such that $\tilde{\Delta} = r\Delta$ is an integral divisor. For example, Y is Gorenstein iff either $\Delta = o$ (in which case Y has only rational double points as singularities), or r = 1 and the dualizing sheaf of Δ is isomorphic to O_Δ (see [2], theorem (4.2)). The following simple (but useful generalization of this fact was noticed by Sakai in [13], theorem (4.2): if Z is an integral Weil divisor on Y, then Z is a Cartier divisor iff $u^*(Z)$ is integral and $O_X(u^*(Z)) \otimes O_{\tilde{\Delta}} \cong O_{\tilde{\Delta}}$. The proof of this generalization is practically the same as in case $Z = K_Y$. Now applying this fact to the divisor mD', and using propositions 1 and 2 we get:

<u>Proposition 4.</u> <u>Let (X,D) be a pair as in proposition 1, i). Suppose that X contains no D-redundand exceptional curves of the first kind (by proposition 3 we can always assume this), and let $\tilde{\Delta}$ be the $\mathbb{Q}$-divisor defined above. Then the following statements are equivalent:</u>

i) <u>R(X,D) is a finitely generated k-algebra.</u>

ii) $O_X(mP) \otimes O_{\tilde{\Delta}} \cong O_{\tilde{\Delta}}$ <u>for some positive integer m such that mP is integral.</u>

<u>Remark.</u> If m is a positive integer such that mP is integral, then $mP.E = o$ for every irreducible component E of $\tilde{\Delta}$, or else (in Lipman's notations [9]), $O_X(mP) \otimes O_{\tilde{\Delta}} \in \mathrm{Pic}^\circ(\tilde{\Delta})$. Thus, condition ii) of proposition 4 means that a certain element of $\mathrm{Pic}^\circ(\tilde{\Delta})$ is a torsion. Since the latter condition is always fulfilled if either Y has at most rational singularities, or if k is the algebraic closure

of a finite field (see [1]; [9]) we get:

<u>Corollary. If Y has at most rational singularities, or if k is the algebraic closure of a finite field, then R(X,D) is a finitely generated k-algebra.</u>

<u>Examples.</u> 1) <u>The anticanonical model of a surface.</u> Let X be a smooth projective surface of anticanonical dimension $\kappa^{-1}(X) = 2$, and (Y,D') the anticanonical model of X (i.e. the model of the pair $(X,-K_X)$). Then obviously $D' = -K_Y$ and one has two possibilities: either X is a rational surface, or X is ruled non-rational. The case where X is rational is studied in detail by Sakai in [12], where it is proved that Y is projective with at most rational singularities (and hence the anticanonical ring $R^{-1}(X)$ is finitely generated). The second case is studied in [3], where it is proved that Y is still projective, with precisely one non-rational singularity, and such that $R^{-1}(X)$ is finitely generated iff Y is $\mathbb{Q}$-Gorenstein i.e. there is a positive integer m such that mK_Y is a Cartier divisor. Moreover, a quite precise description of the singularities of Y is given. Part iii) of proposition 1 above applied to $D = -K_X$ slightly improves the converse part of the main result of [3].

2) <u>The canonical ring of a normal surface.</u> Let Y be a complete normal 2-dimensional algebraic space and $u:X \longrightarrow Y$ the minimal resolution of Y. Consider the canonical ring $R(Y) = R(Y,K_Y)$ of Y. Since $u^*(K_Y) = K_X + \Delta$, applying theorem (6.2 of [13] we get that $R(Y) \cong R(X,K_X+\Delta)$. We are interested in finding a characterization of the fact that R(Y) is finitely generated. Since the only problem is when $\kappa(Y) = \kappa(Y,K_Y) = 2$, we shall assume this from now on. If Y is smooth (or more generally, Gorenstein) it is well known that R(Y) is finitely generated (Mumford, Appendix to [19], in the smooth case, and Sakai [14], in the Gorenstein case). We shall need the following:

<u>Lemma. In the above notations and assumptions, let</u> (Y_o,D') <u>be the model of the pair</u> $(X,K_X+\Delta)$. <u>Then</u> $D' = K_{Y_o}$.

<u>Proof.</u> This lemma follows from [13], where Y_o is called the canonical model of Y (in fact the structure morphism $X \longrightarrow Y_o$ factors uniquely as $X \xrightarrow{u} Y \longrightarrow Y_o$, so Y dominates Y_o).

Nevertheless, we shall give another proof of this lemma. Let $K_X + \Delta = P + N$ be the Zariski decomposition of $D = K_X + \Delta$. Let $v:X \longrightarrow X_o$ be the birational morphism given by proposition 3, and put $\Delta_o = v_*(\Delta)$. We have $D_o = v_*(K_X) + v_*(\Delta)$ $= K_o + \Delta_o$ (where K_o denotes K_{X_o}). If $D_o = P_o + N_o$ is the Zariski decomposition of D_o, we have seen in the proof of proposition 3 that $N_o = v_*(N)$, $P_o = v_*(P)$ and $P = v^*(P_o)$. We note that if E is a component of Δ, we have $P.E = o$. Indeed, by the definition of Δ, $o = D.E = P.E + N.E$, and if $P.E \neq o$, then $N.E < o$ (recall that P is nef), and hence E would be a component of N (since $N \geqslant o$), a contradiction with $P.E \neq o$.

Let E_o be an arbitrary component of Δ_o ; then $v^*(E_o) = E + H$, where E is the strict transform of E by v (which is a component of Δ) and H is a $\mathbb{Q}$-divisor on X whose components are all perpendicular on P. Then $P_o.E_o = v^*(P_o).v^*(E_o) = P.(E+H) = P.E = o$. Therefore by Hodge index theorem we get that the intersection matrix of $\mathrm{Supp}(\Delta_o)$ is negative definite.

If Z is a positive $\mathbb{Q}$-divisor such that $\Delta_o - Z$ and $N_o - Z$ are both effective, then $K_o + (\Delta_o - Z) = P_o + (N_o - Z)$ is the Zariski decomposition of $K_o + (\Delta_o - Z)$, and hence $R(X,D) \cong R(X_o,D_o) \cong R(X_o,K_o+\Delta_o) \cong R(X_o,K_o+(\Delta_o - Z))$. Therefore, by changing Δ_o a little bit (in such a way that $R(X_o,D_o)$ remains unaffected), we may assume that Δ_o and N_o have no common components. Then we claim that $N_o = o$, i.e. $K_o + \Delta_o = P_o$ is nef. Indeed, otherwise it would exist an irreducible curve C on X_o such that $o > D_o.C = P_o.C + N_o.C$. As above, this immediately implies that C should be a component of N_o, and in particular, $P_o.C = o$ and $C^2 < o$. Since C is not a component of Δ_o and Δ_o is effective, $\Delta_o.C \geqslant o$, and recalling that $D_o.C < o$, we get $K_o.C < o$. In other words, C is a D_o-redundand exceptional curve of the first kind, contradicting the definition of X_o. Thus $K_o + \Delta_o$ is nef.

Now we examine a little the irreducible curves C on X_o such that $P_o.C = K_o.C + \Delta_o.C = o$. Since $K_o.C \geqslant o$ we get that either $\Delta_o.C < o$ (if $K_o.C > o$), or that $K_o.C = \Delta_o.C = o$. In the first case C is a component of Δ_o, while in the second, C is a non-singular rational curve such that $C^2 = -2$, and either C does not meet $\mathrm{Supp}(\Delta_o)$, or C is a component of Δ_o. It follows that the model of the pair (X_o,D_o) (which is the same with the model of (X,D)) has finitely many singular points such that $\mathrm{Supp}(\Delta_o)$ is the union of the exceptional fibres of the morphism $f_o: X_o \longrightarrow Y_o$ over all singularities of Y_o which are not rational double points. One also has that $f_o^*(K_{Y_o}) = K_o + \Delta_o$. Indeed, this amounts to showing that $(K_o + \Delta_o).C = o$ for every component C of Δ_o ; and since $K_o + \Delta_o = P_o$, these equalities were already checked before. Thus $f_o^*(D_o) = K_{Y_o}$, Q.E.D.

Combining the above lemma with proposition 2 we get:

Proposition 5. Let Y be a complete normal 2-dimensional algebraic space such that $K(Y) = 2$, and $u: X \longrightarrow Y$ the minimal resolution of Y. Then $R(Y)$ is a finitely generated k-algebra iff the model of the pair $(X, u^*(K_Y))$ is $\mathbb{Q}$-Gorenstein.

Remark. Another proof of proposition 5 was given recently by Sakai in [16].

Using proposition 5, one can easily construct an example of a normal projective surface Y whose canonical ring $R(Y)$ is not finitely generated. Let B be a smooth projective curve of genus $g \geqslant 2$, L a line bundle on B of degree e such that $o < e < 2g-2$, and Y the projective cone over the polarized curve (B,L). Then the minimal resolution $u: X \longrightarrow Y$ of Y is the geometrically ruled surface $X = \mathbb{P}(O_B \oplus L^{-1})$ such that $E = u^{-1}(y)_{red}$ is a section of the canonical projection $\pi: X \longrightarrow B$ $(E^2 = -e)$. A simple calculation shows that $\Delta = (2g-2+e)/e.E$, and if F is a fibre of π, $K_X + \Delta = (2g-2-e)/e.E + (2g-2-e)F$ (up to numerical equivalence). In particular, $K_X + \Delta$ is nef, $(K_X + \Delta)^2 > o$, and hence $K(Y) = K(X, K_X + \Delta) = 2$. In the notations of the proof of the lemma above we have $Y = Y_o$ because the base number of Y is one. By proposition 5, $R(Y)$ is finitely generated iff Y is $\mathbb{Q}$-Gorenstein. On the other hand, it is well known (and easy to see) that Y is $\mathbb{Q}$-Gorenstein iff the line bundle (of degree o) $M = O_B(eK_B) \otimes L^{\otimes 2-2g}$ is a torsion in $\mathrm{Pic}^o(B)$. If $k = \mathbb{C}$, one can obviously choose L such that M is not a torsion, and therefore for such a cone Y, $R(Y)$ is not finitely generated.

§2. Vanishing plurigenera and logarithmic plurigenera

First of all recall that if Y is a normal projective surface one can define the n-genus $p_n(Y)$ of Y by $p_n(Y) = h^o(Y, O_Y(nK_Y))$ for every $n \geqslant 1$ (see [4]). Recall also that if Z is a smooth (not necessarily complete) surface one can define the logarithmic n-genus $\bar{p}_n(Z)$ of Z in the following way: choose a smooth "compactification" $\bar{Z}$ of Z (i.e. a smooth projective surface $\bar{Z}$ containing Z as a Zariski open subset and such that $D = \bar{Z} - Z$ is a (reduced) divisor with normal crossings) and put $\bar{p}_n(Z) = h^o(\bar{Z}, O_{\bar{Z}}(n(K_{\bar{Z}}+D))$ for every $n \geqslant 1$. This definition is due to Iitaka and turns out to be independent of the compactification of Z. The case we shall be interested in here is the one when Z is the smooth locus Y_o of a normal projective surface Y such that a compactification $\bar{Z}$ will be just the minimal resolution $u: X \longrightarrow Y$ (more precisely, $\bar{Z} = X$ and $Z = Y_o$ is embedded in X as an open subset via u^{-1}).

In connection with the problem of classifying all normal projective degenerations of rational and ruled surfaces (see [4]) we were led to classify all normal surfaces with vanishing plurigenera. The result is the following (see [4]):

Theorem B ([4]). Let Y be a normal projective surface over $\mathbb{C}$ such that $p_n(Y) = 0$ for every $n \geq 1$, and $u: X \longrightarrow Y$ the minimal resolution of Y. Then Y belongs to one of the following classes of surfaces:

(A) The normal projective surfaces Y such that X is a rational surface and Y has at most rational singularities.

(B) The normal projective surfaces Y with the following properties: X is a ruled surface of irregularity $q > 0$, Y has precisely one non-rational singularity y, the geometric genus of (Y,y) is q, the irreducible components of $u^{-1}(y)$ are: a section of the canonical ruled fibration $\pi : X \longrightarrow B$ (with B a smooth projective curve of genus q) plus (possibly) some components of the degenerated fibres of π, while the exceptional fibre of u over any rational singularity of Y is contained in a degenerated fibre of π.

(C) The normal projective surfaces Y with X a ruled surface of irregularity $q > 0$ such that Y has at most rational singularities, and the exceptional fibre of u over any singularity of Y is contained in a degenerated fibre of the canonical ruled fibration $\pi : X \longrightarrow B$.

Remarks. 1) If Y is a normal projective surface such that $p_n(Y) = 0$ for every $n \geq 1$, then $q(Y) > 0$ iff Y belongs to class (C), and conversely, every surface Y from class (C) has $p_n(Y) = 0$ for every $n \geq 1$ and $q(Y) > 0$ (see [4]). Here $q(Y) = h^1(Y, O_Y)$

2) Not all surfaces from class (A) or class (B) have $p_n(Y) = 0$ for every $n \geq 1$. Examples of such surfaces from class (B) were already given in [4]. At the end of the section we shall give examples of surfaces Y from class (A) with $p_n(Y) \neq 0$ for some $n \geq 1$.

3) Let Y be a surface belonging to class (B), Y_0 the smooth locus of Y, and $E_0, E_1, \ldots, E_n$ the components of all the exceptional fibres of u, with E_0 the section of π and $E_1, \ldots, E_n$ contained in the fibres of π. Then $E_0, \ldots, E_n$ are all smooth and $D = E_0 + E_1 + \ldots + E_n$ is a divisor with normal crossings. Let $b_1, \ldots, b_m \in B$ be points such that $E_1, \ldots, E_n$ are contained in the union of the fibres $F_j = \pi^{-1}(b_j)$ $(j = 1, \ldots, m)$ and set $B' = B - \{b_1, \ldots, b_m\}$. Then $Y_0 = X - D \supseteq X - (E_0 + F_1 + \ldots + F_m) = \mathbb{A}^1 \times B'$, or else, using the terminology of Miyanishi, Y_0 contains a cylinderlike open subset (see [10]). Since $K_X = -2E_0 + D'$, with D' a divisor whose support is contained in the fibres of π, then $K_X + D = -E_0 + D''$, with D'' a divisor whose support is contained in the fibres if π. Thus $|n(K_X + D)| = \emptyset$ for every

$n \geq 1$, or else, $\bar{p}_n(Y_o) = o$ for every $n \geq 1$.

4) Similarly, if Y is a surface belonging to class (C) then Y_o contains a Zariski open subset isomorphic to $\mathbb{P}^1 \times B'$ $(B' \subseteq B)$ and $\bar{p}_n(Y_o) = o$ for every $n \geq 1$.

Proposition 6. Let Y be a surface belonging to class (A) and Y_o the smooth locus of Y. Then $p_n(Y) = o$ for every $n \geq 1$ implies that $\bar{p}_n(Y_o) = o$ for every $n \geq 1$. If Y has at most quotient singularities the converse is also true.

Proof. Since the singularities of Y are all rational, there is a positive integer a such that aK_Y is a Cartier divisor (see [9]). Then

$$(1) \quad u^*(aK_Y) = aK_X + a\Delta \text{, with } \Delta \text{ the divisor introduced in } \S 1.$$

Since aK_Y is Cartier, $D_a = a\Delta$ is an integral divisor. The hypothesis that $p_n(Y) = o$ for every $n \geq 1$ translates into:

$$(2) \quad |n(aK_X + D_a)| = o \quad \text{for every } n \geq 1.$$

Let $Z > o$ be an arbitrary positive integral divisor on X with support contained in the exceptional fibres of u, and consider the following exact sequence

$$o \longrightarrow O_X(naK_X+nD_a) \longrightarrow O_X(naK_X+nD_a+Z) \longrightarrow L \longrightarrow o,$$

with $L = O_Z(naK_X+nD_a+Z)$. Taking into account of (1) and the fact that aK_Y is a Cartier divisor, we infer that $L \cong O_Z(Z)$. Thus we have $H^o(Z,L) \cong H^o(Z,O_Z(Z) \cong$ $\cong H^1(Z,\omega_Z \otimes O_Z(-Z)) \cong H^1(Z,O_Z(K_X))$, where $\omega_Z = O_Z(Z+K_X)$ and the middle isomorphism comes from duality on the curve Z. Since $u:X \longrightarrow Y$ is the minimal resolution of Y, we have $\deg_E(O_Z(K_X)/E) = K_X.E \geq o$ for every component of Z. From proposition (11.1) of [9] we infer that $H^1(Z,O_Z(K_X)) = o$ and hence $H^o(Z,L) = o$. Recalling the above exact sequence we infer that for every $Z \geq o$ with support in the exceptional fibres of u one has $H^o(X,O_X(naK_X+nD_a+Z)) \cong H^o(X,O_X(naK_X+nD_a)) = o$ for every $n \geq 1$. In particular, for every $Z' > o$ with support contained in the exceptional fibres of u, we have

$$(3) \quad |na(K_X+Z')| = \emptyset \quad \text{for every } n \geq 1.$$

Indeed, for every $n \geq 1$ it is sufficient to take $Z > naZ'$. In particular, (3) holds for the reduced curve D with support the union of all exceptional fibres of u. Since Y has only rational singularities, the curve D has normal crossings, and therefore we get $\bar{p}_n(Y_o) = o$ for every $n \geq 1$.

Conversely, if Y has only quotient singularities, by a result of Watanabe [18] the coefficients of Δ are all <1, and in particular, $D \geqslant \Delta$. Thus, our hypothesis $|n(K_X+D)| = \emptyset$ for every $n \geqslant 1$ implies $|n(K_X+\Delta)| = \emptyset$, or else, $p_n(Y) = 0$ for every $n \geqslant 1$. Q.E.D.

Proposition 7. Let Y be a surface over $\mathbb{C}$ belonging to class (A) such that $p_n(Y) = 0$ for every $n \geqslant 1$, and $y \in Y$ an arbitrary singularity of Y. Then there exists a normal projective surface Y', with only one singular point $y' \in Y'$, and a resolution $v: X' \longrightarrow Y'$ (not necessarily the minimal one) such that:

i) The local rings $O_{Y,y}$ and $O_{Y',y'}$ are isomorphic,

ii) There is a surjective morphism $\pi : X' \longrightarrow \mathbb{P}^1$ whose general fibre is $\mathbb{P}^1$, and

iii) Either $v^{-1}(y')$ is contained in a degenerated fibre of π, or the irreducible components of $v^{-1}(y')$ are: a section of π plus (possibly) some components of the degenerated fibres of π.

Proof. First take $u_1: Y_1 \longrightarrow Y$ the minimal resolution of all singularities of Y but y, and denote by $y_1 = u_1^{-1}(y)$. Clearly we have $O_{Y_1,y_1} \cong O_{Y,y}$. If there is an exceptional curve E of the first kind of Y_1 not passing through y_1, take the blowing down morphism $f: Y_1 \longrightarrow Y_2$ of E to a point and set $y_2 = f(y_1)$. Then Y_2 is smooth outside y_2 and $O_{Y_2,y_2} \cong O_{Y_1,y_1} \cong O_{Y,y}$. If Y_2 has an exceptional curve of the first kind not passing through y_2, repeat the procedure. After a finite number of such steps one obtains a pair (Y',y'), with Y' a normal projective surface, smooth outside the singularity y', and such that $O_{Y',y'} \cong O_{Y,y}$. It is a general fact that $p_n(Y) \geqslant p_n(Y_1)$ for every $n \geqslant 1$. Thus, our hypothesis implies that $p_n(Y_1) = 0$ for every $n \geqslant 1$. We claim that $p_n(Y') = 0$ for every $n \geqslant 1$. To see this it is sufficient to check that $p_n(Y_2) = 0$ for every $n \geqslant 1$, and this is a consequence of $K_{Y_1} = f^*(K_{Y_2}) + E$.

Thus we may assume that Y is smooth outside y and there are no exceptional curves of the first kind not passing through y. From our hypothesis and proposition 6 we get $\overline{p}_n(Y_o) = 0$ for every $n \geqslant 1$. A smooth compactification of Y_o is the minimal resolution $u: X \longrightarrow Y$ of (Y,y). The curve $D = u^{-1}(y)_{red}$ is connected (with normal crossings) and $Y_o \cong X-D$ does not contain any exceptional curve of the first kind. Therefore we are in position to apply theorem I 3.13 of [10] to deduce that Y_o contains a cylinderlike open subset $U = \mathbb{A}^1 \times C$ (with $C \subseteq \mathbb{P}^1$). In particular, we get a birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ inducing a (biregular) isomorphism between $\mathbb{A}^1 \times C$ and $\varphi(\mathbb{A}^1 \times C)$. Eliminating the indeterminacies of φ we get a commutative diagram

$$\begin{array}{ccc} & X' & \\ \chi \swarrow & & \searrow \psi \\ \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & X \end{array}$$

with χ a minimal composition of blowing points (outside $\varphi(\mathbb{A}^1 \times C)$) such that ψ is a morphism. Put $v = u \circ \psi$ and $\pi = p_2 \circ \chi$, where p_2 is the second projection of $\mathbb{P}^1 \times \mathbb{P}^1$. Then the resolution $v: X' \longrightarrow Y$ satisfies the requirements of the proposition. Q.E.D.

Remark. By the work of Kumar and Murthy [8] the structure of surfaces Y belonding to class (A) and such that $p_n(Y) = o$ for every $n \geqslant 1$ is well understood if Y has only one singular point y such that $D = u^{-1}(y)_{red}$ is a rational smooth curve (with self-intersection $D^2 = -m, m \geqslant 2$). Actually, a very fine analysis of these surfaces according to the logarithmic Kodaira dimension is carried out in [8]. In particular, it is proved that $p_n(Y) = o$ for every $n \geqslant 1 \Longleftrightarrow \bar{p}_n(Y_o) = o$ for every $n \geqslant 1 \Longleftrightarrow O_{Y,y}$ is isomorphic to $k[T^m, T^{m-1}U, \ldots, TU^{m-1}, U^m]_{(o)}$.

Let us consider the following example taken from [8]. Let $P_1, \ldots, P_{10}$ be ten points in general position in $\mathbb{P}^2$ and C a sextic curve passing doubly through each P_i (a simple counting constants shows that such a sextic exists, and moreover, C is a rational curve having ten ordinary double points at $P_1, \ldots, P_{10}$). Consider the surface X obtained by blowing up $\mathbb{P}^2$ at $P_1, \ldots, P_{10}$ and denote by D the strict transform of C in X. Then $D = P^1$ and $D^2 = -4$, so that there is a birational morphism $u: X \longrightarrow Y$ such that u(D) is a normal point y (hence of the above type) and u is an isomorphism between X-D and $Y_o = Y - \{y\}$. In particular, Y belongs to class (A). However, it is proved in [8] that $\bar{p}_n(Y_o)$ are not all zero (in fact, $\bar{p}_n(Y_o)$ may be o or 1, and there are integers n o such that $\bar{p}_n(Y_o) = 1$, i.e. the logarithmic Kodaira dimension of Y_o is zero). On the other hand, since (Y,y) is a quotient singularity, by proposition 6, not all $p_n(Y)$ are zero. Therefore there are surfaces belonging to class (A) with very simple rational singularities such that not all plurigenera vanish.

R e f e r e n c e s

1. M. Artin, Some numerical criteria of contractibility of curves on algebraic surfaces, Amer. J. Math. 84 (1962) 485-496.

2. L. Bădescu, Applications of the Grothendieck duality theory to the study of normal isolated singularities, Rev. Roum. Math. Pur. Appl. 24 (1979)673-689

3. L. Bădescu, Anticanonical models of ruled surfaces, Ann. Univ. Ferrara 29 (1983 165-177.

4. L. Bădescu, Normal projective degenerations of rational and ruled surfaces, to appear in Journ. reine angew. Mathematik (Crelle).

5. C. Bănică, K. Ueno, On the Hilbert-Samuel polynomial in complex analytic geometry, J. Math. Kyoto Univ. 2o (198o) 381-389.

6. M. Demazure, Anneaux normaux gradués, Séminaire sur les surfaces, Paris (1979).

7. S. Iitaka, On D-dimension of algebraic varieties, J. Math. Soc. Japan 23 (1971) 356-373.

8. N.M. Kumar, M.P. Murthy, Curves with negative self-intersection on rational surfaces, J. Math. Kyoto Univ. 22 (1983) 767-777.

9. J. Lipman, Rational singularities with applications to algebraic surfaces and unique factorization, Publ. Math. IHES 36 (1969) 195-297.

1o. M. Miyanishi, Non-complete algebraic surfaces, Springer Lect. Notes Math. Vol. 857 (1981).

11. D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. IHES 9 (1961) 229-246.

12. F. Sakai, Anticanonical models of rational surfaces, Math. Ann. 269 (1984) 389-41o.

13. F. Sakai, Weil divisors on normal surfaces, Duke Math. J. 51 (1984) 877-888.

14. F. Sakai, Enriques classification of normal Gorenstein surfaces, Amer. J. Math 1o4 (1981) 1233-1241.

15. F. Sakai, D-dimension of algebraic surfaces and numerically effective divisors Compos. Math. 48 (1983) 1o1-118.

16. F. Sakai, Classification of normal surfaces, Preprint.

17. F. Sakai, Anti-Kodaira dimension of ruled surfaces, Sci. Reports Saitama Univ. Ser. A 1o (1982) 1-7.

18. K. Watanabe, On plurigenera of normal isolated singularities I, Math. Ann. 253 (198o) 241-262.

19. O. Zariski, The theorem of Riemann-Roch for high multiple of an effective divisor on an algebraic surface, Annals Math. 76 (1962) 56o-615.

INCREST Bucharest, Dept. of Mathematics
B-dul Păcii 22o, 79622 Bucharest, Romania