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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS

No.20/1986

BUCURESTI

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April 1986

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Abstract

Let G be a noncomutative finite group and let p be the smallest prime dividing its order. It is shown that if $\alpha \in Aut(G)$, then at most $1/p$ of the elements of G can satisfy $\alpha(x) = x^3$(here $p \neq 2$). The structure of the nonabelian groups of odd order having an automorphism cubing exactly $1/p$ of the elements of the group is given. These groups are closed to be abelian, in the sense that either have nilpotency class 2 or they have an abelian normal subgroup of index p.

1 INTRODUCTION

Let G be a group and let n be a fixed integer. An <u>n-automorphism</u> of G is an automorphism which maps every element of G to its n-th power. If G has an n-automorphism for n =-1,2 or 3, then G is abelian, a result which is long known. On the other hand, it is shown in [7] that for every other nonzero value of n there exist nonabelian groups admitting a non-trivial n-automorphism.

For n = -1,2 and 3 and for nonabelian finite group G it remains to determine how large a proportion of elements of G can be mapped by an automorphism of G to their n-th power. This problem was solved in [4] - [6] for n = -1,2; in this case, those nonabelian finite groups with maximum possible number of elements x mapped by an automorphism to x^n were classified.

These groups are in some sense closed to be abelian (they either have an abelian normal subgroup of minimum index, or they are nilpotent of small nilpotency class).

The aim of this paper is to classify those finite nonabelian groups of odd order that admit an automorphism cubing as large a proportion of group elements as possible. This case (groups of odd order) is a very lucky one since the method of proof of [6] works as well, the results obtained being different from those of [6] only in some details.

If p is the smallest prime dividing $|G|$, then not more than $1/p$ of the elements of G can be cubed by an automorphism. We then classify all finite nonabelian groups G of odd order having an automorphism which cubes exactly $|G|/p$ elements. The classification is given for fixed-point-free automorphisms and for automorphisms with nontrivial fixed subgroup and may be summarized as follows:

THEOREM. Let G be a nonabelian group of odd order, let p be the smallest prime divisor of $|G|$ and let $\alpha \in Aut(G)$ cubing $|G|/p$ elements of G. Let T_α the set of elements of G cubed by α and let F_α the fixed subgroup of α.

(i) If $F_\alpha = 1$, then G is nilpotent of class 2, $|G'| = p$, $G^p \cap G' = 1$ and $p \geq 5$.

(ii) If $F_\alpha \neq 1$, then $T_\alpha \lhd G$, $|G:T_\alpha| = p$, T_α is abelian and there exists $f \in F_\alpha \setminus T_\alpha$ of order p.

2 NOTATIONS AND PRELIMINARY REMARKS

In what follows, if the contrary is not assumed, G will be a finite group of odd order. Any notation not explicitly defined is standard and conforms to that of [2].

$\mathcal{G}_p$	the set of all groups with order divisible by the prime p but by no smaller prime
α	an automorphism of the group G
T_α	the set $\{g \in G \,/\, \alpha(g) = g^3\}$
A_α	a subgroup of G maximal in T_α
F_α	the set $\{g \in G \,/\, \alpha(g) = g\}$; F_α is a subgroup of G .
$Z = Z(G)$	the centre of G

Since $2 \nmid |G|$, $T_\alpha \cap F_\alpha = 1$. It is immediate that $\alpha(T_\alpha) = T_\alpha$, that $\alpha(A_\alpha) = A_\alpha$ and that A_α is abelian (the last since the restriction of α to A_α is a 3-automorphism of A_α). Moreover, T_α and A_α contain no elements of order 3.

3 PRELIMINARY RESULTS

Since our problem is closely related to 3-automorphisms, we first give a proof of the following well-known result just for the convenience of the reader:

3.1 LEMMA If G is an arbitrary group (not necessary finite) which has a 3 - automorphism, then G is abelian.

Proof. Let α as stated and let $x, y \in G$. Since $\alpha(xy) = (xy)^3 = x^3y^3$, it follows that $(yx)^2 = x^2y^2$ (1)

Then $(yx)^4 = ((yx)^2)^2 \overset{(1)}{=} (x^2y^2)^2 \overset{(1)}{=} y^4x^4$ and $y^4x^4 = (yx)^4 =$ $= (yx)(yx)^3$. It follows that $x(yx)^3 = y^3xx^3$, whence $x\alpha(yx) = \alpha(y)x\alpha(x)$. But $\alpha \in \mathrm{Aut}(G)$ and $x\alpha(y)\alpha(x) = \alpha(y)x\alpha(x)$, which yields $x\alpha(y) =$ $= \alpha(y)x$. Now G must be abelian since x,y were arbitrary in G and α is bijective.

The following trivial result will be used:

3.2 LEMMA If $F_\alpha = 1$, the map $\beta : G \longrightarrow G$, $\beta(x) = x^{-1}\alpha(x)$ is bijective.

The main result of this section is the following

3.3 THEOREM Let $G \in \mathcal{G}_p$ and let $\alpha \in \mathrm{Aut}(G)$ such that $p\,|T_\alpha| > |G|$. Then G is abelian (and $T_\alpha = G$).

3.4 COROLLARY Let $G \in \mathcal{G}_p$ be nonabelian and let $\alpha \in \mathrm{Aut}(G)$. Then

$$p|T_\alpha| \leq |G|.$$

We shall prove 3.3 in a sequence of lemmas.

3.5 LEMMA Let $f \in F_\alpha$. Then $T_\alpha f^r \cap T_\alpha f^s \neq \Phi$ only if $f^{r-s} = 1$.

Proof. Let $t_1, t_2 \in T$ such that $t_1 f^r = t_2 f^s$ (1)

Applying α one gets $t_1^3 f^r = t_2^3 f^s$ (2)

From (1) and (2) we obtain $t_1^2 = t_2^2$; since $2 \nmid |G|$, $t_1 = t_2$ and this shows that $f^{r-s} = 1$.

3.6 <u>LEMMA</u> (i) Let $g \in G \setminus A_\alpha$ such that $A_\alpha g \cap T_\alpha \neq \Phi$. Then

$$|A_\alpha g \cap T_\alpha| = |C_A(g)| \leq |A| / p .$$

(ii) Let $g \in G \setminus A_\alpha$ with $g^{-1}A_\alpha g = A_\alpha$. Then $A_\alpha g \cap T_\alpha = \Phi$.

(iii) Let $p|T_\alpha| > |G|$. Then $A_\alpha = N_G(A_\alpha)$.

Proof. (i) Set $A = A_\alpha$ and let $t \in Ag \cap T_\alpha$, so that $t = a_1 g$, $a_1 \in A$ and $Ag = At$.

If $a \in A$, $t \in T_\alpha$ and $at \in T_\alpha$, then $\alpha(at) = (at)^3 = a^3t^3$, whence $(ta)^2 = a^2t^2$. By repeating the argument of 3.1 one obtains $\alpha(at) \in C_G(a)$. It is easy to prove that $\alpha(C_G(a)) = C_G(a)$. Indeed, let $x \in C_G(a)$; then $\alpha(x)\alpha(a) = \alpha(a)\alpha(x)$, i.e. $\alpha(x)a^3 = a^3\alpha(x)$ and $\alpha(x) \in C_G(a^3)$. But $3 \nmid |A|$, forcing $\alpha(x) \in C_G(a)$.

Thus $\alpha(at) \in C_G(a)$ implies $at \in C_G(a)$, so $aat = ata$, whence $at = ta$. We have then shown that for $a \in A$, $t \in T_\alpha$ we have $at \in T_\alpha$ if and only if $at = ta$. Therefore $Ag \cap T_\alpha = At \cap T_\alpha = (C_A(t))t$ and since $C_A(t) = C_A(g)$ one obtains $Ag \cap T_\alpha = C_A(g)t$, whence $|Ag \cap T_\alpha| = |C_A(g)t| = |C_A(g)|$.

Suppose now that $g \in G \setminus A$. Then $C_A(g) < A$, for if the contrary, $C_A(g) = A$ implies $A < \langle C_A(g), t \rangle < T_\alpha$, contradicting the maximality of A in T_α. Since $G \in \mathcal{G}_p$, $|A:C_A(g)| \geq p$, so $|C_A(g)| \leq |A|/p$, proving (i).

(ii) We claim that if $t \in T_\alpha \cap N_G(A)$, then $t \in C_G(A)$.
Let $a \in A$ such that $t^{-1}at \in A$. Then $t^{-3}a^3t^3 = \alpha(t^{-1}at) = (t^{-1}at)^3 = t^{-1}a^3t$, whence $t^{-2}a^3t^2 = a^3$ and t^2 commutes with a^3. Since $3 \nmid |A|$, t^2 commutes with a and since $2 \nmid |G|$, t commutes with a, proving the claim.

Let now $g \in (G \setminus A) \cap N_G(A)$ and suppose there exists $a_1 \in A$ such that $a_1 g \in T_\alpha$. Then $a_1 g \in T_\alpha \cap N_G(A)$ and by the above claim $a_1 g \in C_G(A)$. Since A is abelian, $C_A(g) = A$, contradiction with (i). Thus we have proved that $A_\alpha g \cap T = \Phi$.

(iii) Suppose there exists $g \in (G \setminus A) \cap N_G(A)$. Then $g^r \in N_G(A) \setminus A$ for $r = \overline{1,p-1}$. By (ii), the set $A \cup Ag \cup \ldots \cup Ag^{p-1}$ has exactly $1/p$ of its elements in A (those of A). It follows from (i) that $p|T_\alpha| \leq |G|$. Consequently, if $p|T_\alpha| > |G|$, then $A = N_G(A)$.

3.7 <u>LEMMA</u> If $F_\alpha = 1$, then for every $t \in T_\alpha$ we have $g \in C_G(t)$ if and only if $g^{-1}\alpha(g) \in C_G(t)$.

Proof. If $[g,t] = 1$, then $\alpha([g,t]) = [\alpha(g),t^3] = 1$; since T does not contain elements of order 3 it follows that $[\alpha(g),t] = 1$. Thus $g \in C_G(t)$ implies $g^{-1}\alpha(g) \in C_G(t)$. The lemma follows now by 3.2.

3.8 <u>LEMMA</u> If $F_\alpha = 1$, then T_α contains at most one element from every conjugacy class of G.

Proof. Let $t \in T_\alpha$ and $g \in G$ such that $gtg^{-1} \in T_\alpha$. Then $gt^3g^{-1} = \alpha(gtg^{-1}) = \alpha(g)t^3\alpha(g^{-1})$, whence $g^{-1}\alpha(g) \in C_G(t^3) = C_G(t)$. By 3.7, $g \in C_G(t)$, so $gtg^{-1} = t$.

<u>THE PROOF OF THEOREM 3.3</u> If $|G| = p$, the result is clear. Let G be a minimal counterexample to our theorem, so that $G \in \mathcal{S}_p$ is nonabelian and $p|T_\alpha| > |G|$.

We may suppose that $F_\alpha = 1$. For suppose $F_\alpha \neq 1$ and choose $f \neq 1$ in F_α. By 3.5, none of the classes Af^i, $i = \overline{1,p-1}$ contains elements of T_α. The other cosets of G with respect to $A = A_\alpha$ (excepting A itself) have at most $1/p$ of their elements in T_α by 3.6 (ii). Thus $F_\alpha \neq 1$ implies $p|T_\alpha| \leq |G|$, which is not the case.

We next show that $Z = Z(G) \neq 1$. If $Z = 1$, each nontrivial conjugacy class of G contains at least p elements since $G \in \mathcal{S}_p$. By 3.8, G has at least $|T_\alpha|$ conjugacy classes, so $|G| \geq (|T_\alpha|-1)p + 1$. Since $|T_\alpha| > |G|/p$, one obtains $|T_\alpha| \geq 1 + |G|/p$, whence the contradiction $|G| \geq |G| + 1$.

Thus it remains to consider the case $Z \neq 1$ and $F_\alpha = 1$.

Let $Z^* = Z \cap T_\alpha$. If $Z^* < Z$, then at must $1/p$ of the elements of Z lie in T_α; by 3.8, at must $1/p$ of the elements of $G \setminus Z$ lie in T_α. Therefore $p|T_\alpha| \leq |G|$, contradicting $p|T_\alpha| > |G|$. It follows that $Z^* = Z$, so $Z \subseteq T_\alpha$.

Since $p|T_\alpha| > |G|$, G has by 3.6 (iii) a proper subgroup $A = A_\alpha$ such that $A = N_G(A)$; thus G is not nilpotent and G/Z cannot be abelian. But induces on G/Z the automorphism $\bar{\alpha}$ defined by $\bar{\alpha}(Zg) = Z\alpha(g)$. Since $|G:Z| < |G|$ and $G/Z \in \mathcal{G}_q$ $(q \geqslant p)$ we can suppose $q|T_{\bar{\alpha}}| \leqslant |G/Z|$ (otherwise G/Z would be abelian and G nilpotent, which is not the case).
Now $|T_{\bar{\alpha}}| \leqslant |T_\alpha||Z|$, so $p|T_\alpha| \leqslant |G|$, a contradiction which proves the result.

Corollary 3.4 raises the question of the structure of the nonabelian groups $G \in \mathcal{G}_p$ $(p > 2)$ with an automorphism α such that $p|T_\alpha| = |G|$.

4 GROUPS IN WHICH $p|T_\alpha| = |G|$ AND $F_\alpha = 1$

Let observe first that in this case T_α cannot be a subgroup of G. For if contrary, since $G \in \mathcal{G}_p$, we have $T_\alpha \lhd G$ is abelian. But G being nonabelian, there exixtx $x \in G \setminus A$ and $a \in A$ such that $a \neq a^x$, contradicting 3.8.

We prove next that $Z \not\subseteq T_\alpha$. Suppose the contrary and argue by contradiction. We have here two cases:

Case I. Every conjugacy class of G contains elements of T_α.

In this case, by 3.8, there are $|T_\alpha| = |G|/p$ conjugacy classes in G. By a result of Joseph (see for example [6]) we obtain that

$$1/p \leq 1/p^2 (1 + (p^2 - 1)/|G'|)$$

whence $|G'| \leq p + 1$. Since $p \neq 2$, $|G'| \neq p + 1$, forcing $|G'| = p$ and hence every noncentral conjugacy class of elements of G has exactly p elements. The class equation now furnishes a contradiction.

Case II. There exist conjugacy classes in G without elements in T_α. The proof of this case is much more involved, but it runs along the same lines as in [6].

We are now in the following situation : $F_\alpha = 1$, $p|T_\alpha| = |G|$, $|Z:Z^*| = p$ where $Z^* = Z \cap T_\alpha$, every noncentral conjugacy class of elements has exactly p elements and exactly one of these lies in T_α.

Choose $z \in Z \setminus Z^*$. Then the sets $T_\alpha z^i$, $i = \overline{0,p-1}$ are pairwise disjoint and since $p|T_\alpha| = |G|$ it follows that $G = T_\alpha Z$. This shows (by 3.1) that G/Z is abelian. Let $t_1, t_2 \in T_\alpha$ such that $c = [t_1, t_2] \neq 1$. Of course $c \in Z$ and consequently (see e.g. lemma 2.2, p. 19 of [2])

$\alpha(c) = \alpha([t_1, t_2]) = [t_1^3, t_2^3] = [t_1, t_2]^9 = c^9$. Now $c \notin Z^* = Z \cap T_\alpha$ since otherwise $\alpha(c) = c^3 = c^9$ and $c^6 = 1$, contradicting the fact that T_α does not contain elements of order 3.

Thus $Z = Z^* \times G'$, where $G' = \langle c \rangle$ and $|c| = p$. Since $\alpha(c) = c^9$, it follows that $p \neq 3$, otherwise we would obtain a contradiction because is injective.

Since $t^p \in Z \cap T_\alpha = Z^*$ for all $t \in T_\alpha$ and since $p \neq 2$, it follows that $(t_i t_j)^p = t_i^p t_j^p$ for all $t_i, t_j \in T_\alpha$, thus $G^p < Z^*$ and consequently $G^p \cap G' = 1$.

The structure result in the case $F_\alpha = 1$ is the following

4.1 THEOREM A necessary and sufficient condition that a nonabelian group $G \in \mathcal{G}_p$ $(p > 2)$ have an automorphism α such that $F_\alpha = 1$ and $p|T_\alpha| = |G|$ is that

(i) G is nilpotent of class 2 with $|G'| = p$

(ii) $G^p \cap G' = 1$

(iii) $p \geqslant 5$

Proof. The necessity has been established above. Suppose now that G

satisfies (i) - (iii) . Then G/Z is an elementary abelian p-group and $Z = Z^{*} \times G'$, where Z^{*} is a subgroup of G such that $G^{p} < Z^{*} < G$. Thus $G/Z = \langle Za_1, \ldots, Za_k, Zx_1, \ldots, Zx_k \rangle$, where

$[x_i, x_j] = [a_i, a_j] = 1$, for all $i,j = \overline{1,k}$

$[a_i, x_j] = 1$ $(i \neq j)$

$[a_i, x_i] = c$ $i = \overline{1,k}$ and

$G' = \langle c \rangle$, $|c| = p$. If $A = \langle a_1, \ldots, a_k, Z^{*} \rangle$, then every element $g \in G$ is uniquely expressible as

$$g = ac^{s}x_1^{q_1} \ldots x_k^{q_k}, \text{ where } a \in A,\ s = \overline{0,p-1},\ q_i = \overline{0,p-1},\ i = \overline{1,k}.$$

Define $\alpha \in \mathrm{Aut}(G)$ by

$$\alpha(g) = \alpha(ac^{s}x_1^{q_1} \ldots x_k^{q_k}) = a^{3}c^{9s}x_1^{3q_1} \ldots x_k^{3q_k}.$$

Then $p|T_\alpha| = |G|$, for given $a \in A$ and $q_i \in \mathbb{Z}$, $i = \overline{1,k}$, there is exactly one $0 \leq s \leq p-1$ with $\alpha(g) = g^{3}$. Of course, $F_\alpha = 1$ and the theorem is proved.

5 GROUPS IN WHICH $p|T_\alpha| = |G|$ AND $F_\alpha \neq 1$

Up to the end of this section we shall work in the following conditions : $G \in \mathcal{G}_p$ $(p>2)$ is a nonabelian group, $\alpha \in \mathrm{Aut}(G)$, $F_\alpha \neq 1$ and $p|T_\alpha| = |G|$.

By 3.5, $|F_\alpha| = p$. Let $F_\alpha = \langle f \rangle$; then we have the relation

5.2 $$G = T_\alpha \cup T_\alpha f \cup \ldots \cup T_\alpha f^{p-1}$$

We need some technical lemmas.

5.3 LEMMA The conjugacy class containing f has no elements in T_α.

Proof. Deny the lemma and suppose that there exists $g \in G$ such that $gfg^{-1} \in T_\alpha$. By 5.2 there exist $t \in T_\alpha$ and integer r such that

$g = tf^r$. Thus $(tf^r)f^3(tf^r)^{-1} = ((tf^r)f(tf^r)^{-1})^3 = \alpha((tf^r)f(tf^r)^{-1}) =$

$= \alpha(tft^{-1}) = t^3ft^{-3}$, whence $tf^3t^{-1} = t^3ft^{-3}$, so $f^3 = t^2ft^{-2}$ (1)

Applying α to (1) we obtain $f^3 = t^6ft^{-6}$, which together with (1) gives $f = t^4ft^{-4}$, i.e. $t^4 \in C_G(f)$. Since $2 \nmid |G|$, $t \in C_G(f)$ and

$$tft^{-1} = f \qquad (2)$$

From (2) it follows $f^3 = tf^3t^{-1}$ and by (1) we have $tf^3t^{-1} = t^2ft^{-2}$ which yields $f^3 = tft^{-1} = f$, contradicting $2 \nmid |G|$.

5.4 <u>LEMMA</u> Let $G_1 = AF_\alpha < G$ such that

(i) $A < G_1$, $|G_1 : A| = p$

(ii) $\alpha(A) = A$

Then $A \subseteq T_\alpha$ (thus A is abelian).

Proof. Let $a \in A$. By 5.2, exactly one of the elements af^i , $i = \overline{0,p-1}$ belongs to T_α . Suppose this element is af^j. Then $(af^j)^3 = \alpha(af^j) = \alpha(a)f^j$, which yields $(f^ja)^2 = a^{-1}\alpha(a) \in A$. Since $2 \nmid |G|$, $f^ja \in A$, so $f^j \in A$. But it is clear that $A \cap F_\alpha = 1$, so $j = 0$ and $a \in T_\alpha$.

The case $T_\alpha < G$ is settled by

5.5 <u>THEOREM</u> A nonabelian group $G \in \mathcal{G}_p (p > 2)$ has an automorphism α such that $F_\alpha \neq 1$, $T_\alpha < G$, $|G : T_\alpha| = p$ if and only if G has an abelian (normal) subgroup A of index p and an element $f \in G \setminus A$ of order p .

Proof. If $T_\alpha < G$, then $F_\alpha \neq 1$ and the generator f of F_α is of order p . Conversely, given A and f as in the statement, the map $\alpha(af^r) = a^3f^r$, $a \in A$, $r = \overline{0,p-1}$, defines an automorphism of G with $T_\alpha = A$.

We now proceed to prove the following result, which together with 4.1 and 5.5 establishes the final characterization theorem stated in the first section.

5.6 THEOREM If $G \in \mathcal{G}_p$ $(p>2)$ is nonabelian and has an automorphism α such that $F_\alpha \neq 1$ and $p|T_\alpha| = |G|$, then $T_\alpha < G$. Consequently, the only groups in the case $F_\alpha \neq 1$ are those of 5.5.

We shall assume from now on that T_α is not a subgroup of G, that $|F_\alpha| = p$ and that $p|T_\alpha| = |G|$ and eventually derive a contradiction.

5.7 LEMMA Let $A = A_\alpha$ be a subgroup of maximal in T_α $(A \neq T_\alpha)$. Then there exists a coset decomposition

$$G = A \cup Af \cup \ldots \cup Af^{p-1} \cup Ag_1 \cup \ldots \cup Ag_n$$

such that

(i) $Af^j \cap T_\alpha = \Phi$, $j = \overline{1,p-1}$

(ii) $|Ag_i \cap T_\alpha| = |C_A(g_i)| \geqslant |A|/p$, $i = \overline{1,n}$.

Proof. (i) follows from 5.2.
Exactly $1/p$ of the elements of $A \cup Af \cup \ldots \cup Af^{p-1}$ belong to T_α and by 3.6 every other coset must have exactly $1/p$ of its elements in T_α according to the condition $p|T_\alpha| = |G|$. This proves (ii) .

We try now to obtain the desired contradiction in order to prove 5.6.
Case I. $Z \not\subset T_\alpha$.

Then $Z^* = Z \cap T_\alpha < Z$. In order to have $p|T_\alpha| = |G|$, two conditions must be satisfied: $|Z:Z^*| = p$ and every noncentral conjugacy class of G has exactly $1/p$ of its elements in T_α. By 5.3 $f \in Z$ and $Z = \langle Z^*, f\rangle$; by 5.2, $G = T_\alpha F_\alpha$, so G/F_α is abelian by 3.1. Thus $G' \leq F_\alpha$ and since $|F_\alpha| = p$, $G' = F_\alpha$. Hence $G = \langle T_\alpha\rangle$ and there exist $t_1, t_2 \in T_\alpha$ such that $f = [t_1, t_2] \neq 1$. Of course $f \in Z$ and applying α one gets
$f = \alpha(f) = \alpha([t_1, t_2]) = [t_1^3, t_2^3] = [t_1, t_2]^9 = f^9$, contradicting $2 \nmid |G|$.

Case II. $Z \subset T_\alpha$.

This case is much more complicated; however, using the technical lemmas obtained above, it is not difficult to derive a contradiction exactly as in [6]. The situation is different from those of [6] : if the automorphism α squares many elements instead of cubing them, then the corresponding theorem for 5.6 is not true, another class of groups which satisfy all the conditions being obtained, with G/Z nonabelian of order 27.

In the cubing case, this situation is ruled out, thus obtaining a contradiction which ends the proof of 5.6.

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