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by

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1. Introduction.

Let $g:\mathbb{C}^n \longrightarrow \mathbb{C}$ and $h:\mathbb{C}^m \longrightarrow \mathbb{C}$ be polynomial maps. Our mai aim is the study of the polynomial maps $f:\mathbb{C}^n\times\mathbb{C}^m \longrightarrow \mathbb{C}$; $f(x,y)=g(x)+h(y)$ and $f(x,y)=g(x)\cdot h(y)$. In fact, the methods which are presented in this note, can be used also in the more general case $f(x,y)=P(g(x),h(y))$, where $P:\mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ is a polynomial map with two variables. But in this general case, the formulation of the results becomes more complicated depending on the complexity of the topology of affine curve $P=0$.

The main results are the followings :

- the generic fiber of the map $f(x,y)=g(x)+h(y)$ is homotopic equivalent with the join space of the generic fibe of the maps g and h.
- the determination of the topology of generic fiber in the product case : $f(x,y)=g(x)\cdot h(y)$.
- the monodromy in both cases.

The results of this note are natural continuations of the results of M.Sebastiani and R.Thom [Se-T] (the local case of isolated singularities), K.Sakamoto [Sa_1] , [Sa_2] (the local case), and M.Oka [O] (the case of weighted homogeneous polynomials.).

2. The sum case : $P(c,d)=c+d$

Let $g:\mathbb{C}^n \longrightarrow \mathbb{C}$ and $h:\mathbb{C}^m \longrightarrow \mathbb{C}$ be polynomial maps. Then there exists a finite set $\Lambda_g=\{c_1,\dots,c_t\}$ (respectively $\Lambda_h=\{d_1,\dots,d_s\}$) such that $g:\mathbb{C}^n-g^{-1}(\Lambda_g) \longrightarrow \mathbb{C}-\Lambda_g$ (respectively $h:\mathbb{C}^m-h^{-1}(\Lambda_h) \longrightarrow \mathbb{C}-\Lambda_h$) is a C^∞ locally trivial fibration.

We define $f:\mathbb{C}^n\times\mathbb{C}^m \longrightarrow \mathbb{C}$ by $f(x,y)=g(x)+h(y)$ and $\Lambda_f=\Lambda_g+\Lambda_h = \{c_i+d_j \mid c_i\in\Lambda_g, d_j\in\Lambda_h\}$. If we introduce the set $L_e=\{(c,d)\in\mathbb{C}\times\mathbb{C} \mid c+d=e\}$ for all $e\in\mathbb{C}$, and the map $u=g\times h:\mathbb{C}^n\times\mathbb{C}^m \longrightarrow \mathbb{C}\times\mathbb{C}$, $u(x,y)=(g(x),h(y))$, then $f^{-1}(e)=u^{-1}(L_e)$. Hence, the study of the polynomial f is in strong connection with the study of the map u and the mutual position of the line L_e and the set $\Lambda=(\mathbb{C}\times\Lambda_h)\cup(\Lambda_g\times\mathbb{C})$. If $e\notin\Lambda_f$, then let $\{C_i\}_{i=\overline{1,t}} = \{(c_i, e-c_i)\}_{i=\overline{1,t}} = L_e\cap(\Lambda_g\times\mathbb{C})$ and $\{D_j\}_{j=\overline{1,s}} = \{(e-d_j, d_j)\}_{j=\overline{1,s}} = L_e\cap(\mathbb{C}\times\Lambda_h)$.

Lemma 2.1. There exists a C^∞ diffeomorphism $v:\mathbb{R}^2\longrightarrow L_e$, such that $v^{-1}(C_i)\subset\mathbb{R}\times(0,\infty)$ $(i=1,\dots,t)$ and $v^{-1}(D_j)\subset\mathbb{R}\times(-\infty,0)$ $(j=1,\dots,s)$.

This is the consequence of the following

Homogeneity Lemma [Mi1]

Let n_i and n_i' $(i=1,\dots,l)$ be arbitrary points of the smooth, connected manifold N. Then there exists a diffeomorphism $v: N\longrightarrow N$ which (is smoothly isotopic to the identity and) carries n_i into n_i'. (In addition v can be chosen such that the set $\{x\in N : v(x)\neq x\}$ is compact).

We define the following sets in L_e: $v(\mathbb{R}\times[0,\infty))=\mathcal{C}$, $v(\mathbb{R}\times(-\infty,0])=\mathcal{D}$, $v(\mathbb{R}\times\{0\})=\gamma$ and $v((0,0))=\Gamma$. Hence $C_i\in\mathcal{C}$ $(i=1,\dots,t)$ and $D_j\in\mathcal{D}$ $(j=1,\dots,s)$.

First result is the following

Theorem 2.2. The restricted map $f:\mathbb{C}^n\times\mathbb{C}^m-f^{-1}(\Lambda_f)\longrightarrow\mathbb{C}-\Lambda_f$ is a C^∞ locally trivial fibration.

Proof. If we denote $\mathcal{C}_\varepsilon=v(\mathbb{R}\times[\varepsilon,\infty))$ and $\mathcal{D}_\varepsilon=v(\mathbb{R}\times(-\infty,-\varepsilon])$, then there exists a sufficiently small $\varepsilon>0$, such that $C_i\in\mathcal{C}_\varepsilon$ $(i=1,\dots,t)$ and $D_j\in\mathcal{D}_\varepsilon$ $(j=1,\dots,s)$. We define a C^∞ function $\varphi:L_e\longrightarrow[0,1]$ by $\varphi=\varphi'\circ v^{-1}$, where $\varphi'(r_1, r_2) = \varphi''(r_2)$ is a C^∞ function with

$$\varphi''(r_2) = \begin{cases} 1 & r_2 \leq -\frac{\varepsilon}{2} \\ 0 & r_2 \geq \frac{\varepsilon}{2} \end{cases}$$

Let $e \notin \Lambda_f$. Because g is trivial over $pr_1(L_e - \mathcal{C}_\varepsilon) = P$ (where pr_1 is the first projection $pr_1 : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$), there exists a diffeomorphism $\psi = (\psi_1, \psi_2) : g^{-1}(P) \longrightarrow P\times G$ such that $\psi_1 = g$. It can be shown, that there exists $\varepsilon' > 0$, such that the map $\tilde{\phi} : B_{\varepsilon'}\times P\times G \longrightarrow B_{\varepsilon'}\times P\times G$, $\tilde{\phi}(a,c,x) = (a, c+\varphi(c,e-c)\cdot(a-e), x)$ is bijective and local diffeomorphism in any point (a,c,x) , hence is diffeomorphism (here $B_{\varepsilon'} = \{a \in \mathbb{C} : |a-e| < \varepsilon'\}$).

We define the diffeomorphism

$\phi^g = (1, \phi_2^g) : B_{\varepsilon'}\times g^{-1}(P) \longrightarrow B_{\varepsilon'}\times g^{-1}(P)$ by the diagram :

$$\begin{array}{ccc} B_{\varepsilon'}\times g^{-1}(P) & \xrightarrow{(1,\phi_2^g)} & B_{\varepsilon'}\times g^{-1}(P) \\ \wr\downarrow (1,\psi) & & \wr\downarrow (1,\psi) \\ B_{\varepsilon'}\times P\times G & \xrightarrow{\tilde{\phi}} & B_{\varepsilon'}\times P\times G \end{array}$$

Therefore $g(\phi_2^g(a,x)) = g(x) + \varphi(g(x), e-g(x))\cdot(a-e)$ $\quad (*)$

The map $(1,\phi_2^g)$ can be extended by the identity, hence we have constructed a diffeomorphism $(1,\phi_2^g) : B_{\varepsilon'}\times\mathbb{C}^n \longrightarrow B_{\varepsilon'}\times\mathbb{C}^n$, such that $(*)$ holds.

In a similar way we obtain a diffeomorphism $(1,\phi_2^h) : B_{\varepsilon'}\times\mathbb{C}^m \longrightarrow B_{\varepsilon'}\times\mathbb{C}^m$, such that

$$h(\phi_2^h(a,y)) = h(y) + (1-\varphi)(e-h(y), h(y))\cdot(a-e) .$$

We define $\phi : B_{\varepsilon'}\times f^{-1}(e) \longrightarrow f^{-1}(B_{\varepsilon'})$ by $\phi(a,x,y) = (\phi_2^g(a,x), \phi_2^h(a,y))$. Then ϕ is diffeomorphism and $f(\phi_2^g(a,x), \phi_2^h(a,y)) = a$.

Because $f^{-1}(e) = u^{-1}(L_e)$, it is important to study the restricted map $u : u^{-1}(L_e) \longrightarrow L_e$.

Lemma 2.3. Let $e \notin \Lambda_f$. Then

a.) $u : u^{-1}(L_e - \Lambda) \longrightarrow L_e - \Lambda$ is a C^∞ locally trivial fibration.

b.) $u^{-1}(\mathcal{D}) \approx G \times h^{-1}(pr_2 \mathcal{D}) \approx G \times \mathbb{C}^m$

$u^{-1}(\mathcal{C}) \approx g^{-1}(pr_1 \mathcal{C}) \times H \approx \mathbb{C}^n \times H$

$u^{-1}(\gamma) \approx u^{-1}(\Gamma) \times \mathbb{R} \approx G \times H \times \mathbb{R}$

(where G and H are the generic fibres of the polynomials g and h and " $\approx$ " means "diffeomorphic").

Proof: a.) Let $(c,d) \in L_e - \Lambda$ .Then there exists a neighbourhood U of (c,d) such that g (respectively h) is trivial over $pr_1 U$ ($pr_2 U$). Hence there exist the diffeomorphisms:

$\psi^g = (g, \psi_2^g) : g^{-1}(pr_1 U) \longrightarrow pr_1 U \times G$ and

$\psi^h = (h, \psi_2^h) : h^{-1}(pr_2 U) \longrightarrow pr_2 U \times H$.

Then $\psi : u^{-1}(U) \longrightarrow U \times G \times H$ by

$\psi(x,y) = ((g(x),h(y)), \psi_2^g(x), \psi_2^h(y))$ is a trivialisation of u over U.

b.) The map $(\psi_2^g, p_2) : u^{-1}(\mathcal{D}) \longrightarrow G \times h^{-1}(pr_2 \mathcal{D})$,

$(\psi_2^g, p_2)(x,y) = (\psi_2^g(x), y)$ is a diffeomorphism.

Because $\Lambda_h \subset pr_2 \mathcal{D}$, we have $h^{-1}(pr_2 \mathcal{D}) \approx h^{-1}(\mathbb{C})$. ∎

With this preparations we can prove the following

Theorem 2.4. Let f be a polynomial in $\mathbb{C}^n \times \mathbb{C}^m$ such that $f(x,y) = g(x) + h(y)$. Let $F = f^{-1}(e)$ $(e \notin \Lambda_f)$, $G = g^{-1}(c)$ $(c \notin \Lambda_g)$ and $H = h^{-1}(d)$ $(d \notin \Lambda_h)$. Then there is a homotopy equivalence between F and $G * H$ (where $G * H$ is the join of G and H).

Proof. From Lemma 2.3. , there is a homotopy equivalence between $F = u^{-1}(L_e)$ and $u^{-1}(L_e - (\gamma - \Gamma))$, which is homotopic equivalent with $\mathbb{C}^n \times H \bigcup_{G \times H} G \times \mathbb{C}^m$ (in the disjoint union of the spaces $\mathbb{C}^n \times H$ and $G \times \mathbb{C}^m$ we identify the subspace $G \times H \subset \mathbb{C}^n \times H$, $G \times H \subset G \times \mathbb{C}^m$).

We have the following homotopy equivalences of pairs of spaces:

$$(\mathbb{C}^n \times H, G\times H) \sim (\mathbb{C}^n \times H \bigsqcup_{G\times H\times\{0\}} G\times H\times[0,1] \ , \ G\times H\times\{1\}) \sim ((\mathrm{Con}\ G)\times H, G\times H)$$

(where Con G is the cone over G).

If we define $X_1 = \{[x,t,y] \in G*H : t \leq \frac{1}{2}\}$ and

$X_2 = \{[x,t,y] \in G*H : t \geq \frac{1}{2}\}$, then $(X_1, X_1\cap X_2) \sim (G\times(\mathrm{Con}H), G\times H)$

and $(X_2, X_1\cap X_2) \sim ((\mathrm{Con}\ G)\times H, G\times H)$

Therefore $F \sim X_1 \bigsqcup_{X_1\cap X_2} X_2 = G * H$. ■

Corollary 2.5.

a.) $\tilde{H}_q(F) = \bigoplus_{p+r=q-1} \tilde{H}_p(G)\otimes\tilde{H}_r(H) \oplus \bigoplus_{p+r=q-2} \mathrm{Tor}(\tilde{H}_p(G), \tilde{H}_r(H))$

(for the proof, see [Mi_2]).

b.) If G is n_1-connected and H is n_2-connected ,

then F is n_1+n_2+2-connected. In particular F is connected.

c.) $\pi_1(F)$ = the free group of rank $(a-1)\cdot(b-1)$, where

$H_o(G) = \mathbb{Z}^a$, $H_o(H) = \mathbb{Z}^b$.

Remark 2.6. The homology groups can be calculated using the Mayer-Vietoris exact sequence :

$$\ldots \to \tilde{H}_q(u^{-1}(\gamma)) \longrightarrow \tilde{H}_q(u^{-1}(\mathcal{C}))\oplus\tilde{H}_q(u^{-1}(\mathcal{D})) \longrightarrow \tilde{H}_q(F) \xrightarrow{\partial_*} \tilde{H}_{q-1}(u^{-1}(\gamma)$$

If we use Lemma 2.4. we obtain the following exact sequence:

$$0 \longrightarrow \tilde{H}_q(F) \xrightarrow{r_*\circ\partial_*} \tilde{H}_{q-1}(G\times H) \longrightarrow \tilde{H}_{q-1}(\mathbb{C}^n\times H)\oplus\tilde{H}_{q-1}(G\times\mathbb{C}^m) \longrightarrow 0$$

which is equivalent with Corollary 2.5.a.

(Here the isomorphism r_* is induced by the retraction $r : u^{-1}(\gamma) \longrightarrow u^{-1}(\Gamma)$).

Now, we describe the monodromy of f.

If $e \in \Lambda_f$, then $e = c_{i_l} + d_{j_l}$, $l=1,\ldots,r$, $i_l \in \{1,\ldots,t\}$, $j_l \in \{1,\ldots,s\}$. If we take $e(t) = e + \rho\cdot\exp(2\pi i t)$ with $\rho > 0$, sufficiently small and $t \in [0,1]$, then

$C_i(t) = (c_i, e - c_i + \rho\exp(2\pi i t))$ and $D_j(t) = (e - d_j + \rho\cdot\exp(2\pi i t), d_j)$.

Hence the points C_{i_1} and D_{j_1} turn around each other.

If we take $v_t : \mathbb{R}^2 \longrightarrow L_{e(t)}$ $(t \in [0,1])$ such that v_t is a C^∞ diffeomorphism with $v_t^{-1}(D_j(t)) \subset \mathbb{R}\times(-\infty,0)$ and $v_t^{-1}(C_i(t)) \subset \mathbb{R}\times(0,\infty)$, then we denote $v_1(\mathbb{R}\times\{0\}) = m_e(\gamma)$.

We can consider that the point $\Gamma = v_t((0,0))$ is invariable. Then it is easy to prove the following

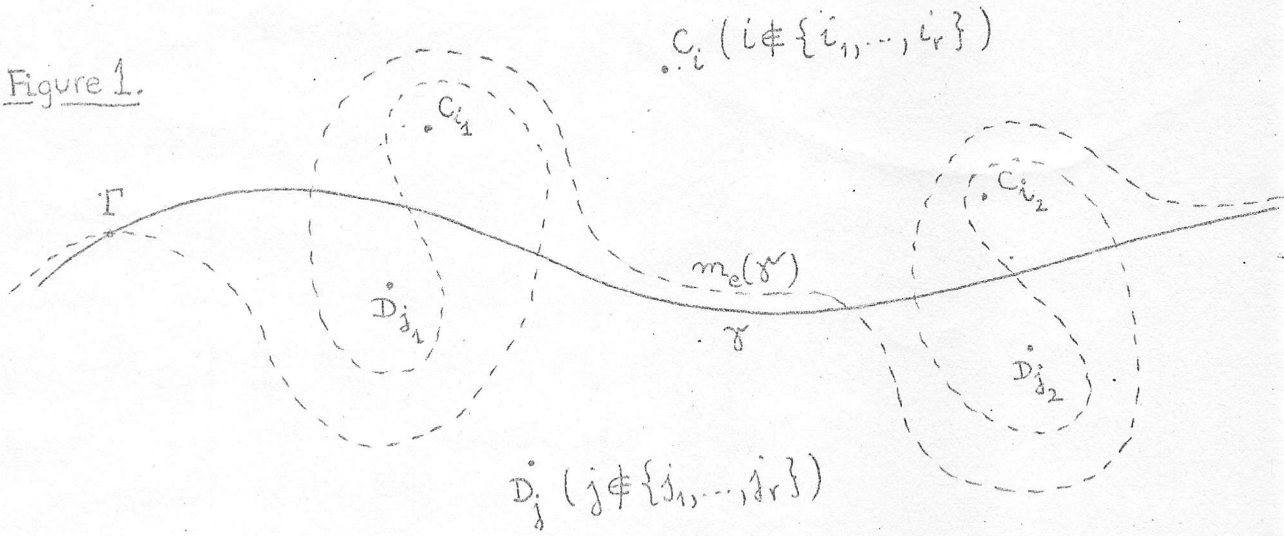
<u>Theorem 2.7.</u> If $m_e^q : \widetilde{H}_q(F) \longrightarrow \widetilde{H}_q(F)$ denote the monodromy action induced by the path $t \longmapsto e(t)$, then we have the following commutative diagram :

$$\begin{array}{ccccc} \widetilde{H}_q(F) & \xrightarrow{\partial_*} & \widetilde{H}_{q-1}(u^{-1}(\gamma)) & \xrightarrow[\sim]{(r_1)_*} & \widetilde{H}_{q-1}(G\times H) \\ \downarrow m_e^q & & & & \| \text{ identity} \\ \widetilde{H}_q(F) & \xrightarrow{\partial_*} & \widetilde{H}_{q-1}(u^{-1}(m_e(\gamma))) & \xrightarrow[\sim]{(r_2)_*} & \widetilde{H}_{q-1}(G\times H) \end{array}$$

where $(r_1)_*$ and $(r_2)_*$ are induced by the retractions $r_1 : u^{-1}(\gamma) \longrightarrow u^{-1}(\Gamma)$ and $r_2 : u^{-1}(m_e(\gamma)) \longrightarrow u^{-1}(\Gamma)$.

The following drawing illustrate the change $\gamma \longmapsto m_e(\gamma)$

<u>Figure 1.</u>



<u>Corollary 2.8.</u> If $\Lambda_g = \{c\}$, $\Lambda_h = \{d\}$ and if m_g and m_h denote the monodromy actions of the maps g and h around the points c and d, then the monodromy action of the f around the point $c+d$ is $m_g * m_h$.

<u>Proof.</u> Let B_D and B_C be open disks centred at $v_o^{-1}(D)$ respectively at $v_o^{-1}(C)$ with sufficiently small radius and let $\overline{DC}$ be the segment with endpoints at $v_o^{-1}(D)$ and $v_o^{-1}(C)$. Then there is a homotopy equivalence between $u^{-1}(v_o(B_D \cup \overline{DC} \cup B_C))$ and $u^{-1}(L_e) = F$. If we choose v_t such that $m_e(\gamma) \cap v_o(\overline{DC}) = \gamma \cap v_o(\overline{DC}) = I$, then we have the following figure :

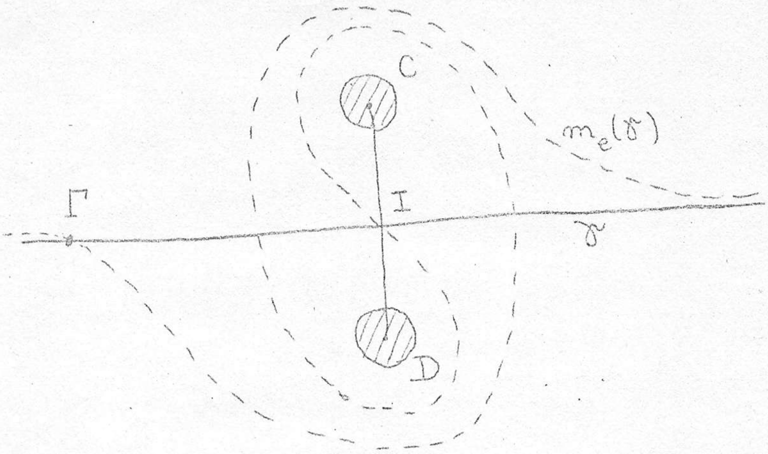


Figure 2.

Let $(\Gamma I)_i$ be the path $(\Gamma I)_i(t)=v_i(t\cdot v_i^{-1}(I))$ $t\in[0,1]$ (i=0,1)

and
$$(\Gamma I)_o^{-1}\circ(\Gamma I)_1(t)=\begin{cases}(\Gamma I)_1(2t) & t\in[0,\frac{1}{2}]\\ (\Gamma I)_o(2-2t) & t\in[\frac{1}{2},1]\end{cases}$$

Then we have the following commutative diagram

$$\begin{array}{ccccc}\tilde{H}_q(F) & \xrightarrow{\partial_*} & \tilde{H}_{q-1}(u^{-1}(m_e(\gamma))) & \xrightarrow[\sim]{(r_2)_*} & \tilde{H}_{q-1}(u^{-1}(\Gamma)) \\ \downarrow \text{id.}\ \wr & & & & \downarrow ((\Gamma I)_o^{-1}\circ(\Gamma I)_1)_* \\ \tilde{H}_q(F) & \xrightarrow{\partial_*} & \tilde{H}_{q-1}(u^{-1}(\gamma)) & \xrightarrow{(r_1)_*} & \tilde{H}_{q-1}(u^{-1}(\Gamma))\end{array}$$

Summed this diagram with the diagram from Theorem 2.7. we obtain that $m_e=((\Gamma I)_o^{-1}\circ(\Gamma I)_1)_* = m_g * m_h$ ∎

Remarks 2.9.

1.) The construction of the generic fiber F as the join space $G*H$ is independent on the local structure of the singularities of the polynomials g and h.

For example, if we take $g(x_1,x_2)=x_1(x_1x_2+1)$ then $\Lambda_g=\{0\}$ (the point 0 is not a critical value), the generic fiber is diffeomorph with $\mathbb{C}^*$, and the monodromy m_g is trivial. Therefore, for any $h(y_1,\dots,y_m)$ with $\Lambda_h=\{0\}$, the generic fiber of the sum function f=g+h has the homotopy type of S^1*H, $\Lambda_f=\{0\}$ and the monodromy of f is $1*m_h$

2.) If Λ_g and Λ_h have more elements, then we can construct also a set S, as union of small disks and curves , with the following proprieties

- F is homotopic equivalent with $u^{-1}(S)$
- $\gamma \cap S = m(\gamma) \cap S$
- m and $m(\gamma)$ intersect (transversally) only the curves.

If γ intersects S in the points $(I_i)_i$, then we have the following decomposition of the $a \in \mathrm{im}((r_1)_* \circ \partial_*)$, $a = \sum_i a_i$:

$$\begin{array}{ccccccc}
\tilde{H}_*(F) & \xleftarrow{\partial_*} & \tilde{H}_*(u^{-1}(\gamma)) & \xrightarrow{(r_1)_*} & & & \tilde{H}_*(u^{-1}(\Gamma)) \\
\downarrow \wr & & & & & & \uparrow \sum_i \\
\tilde{H}_*(S) & \xrightarrow{\partial_*} & \bigoplus_i \tilde{H}_*(u^{-1}(I_i)) & \xrightarrow{\oplus \mathrm{ind}_{*i}} & \bigoplus_i \tilde{H}_*(u^{-1}(\gamma)) & \xrightarrow{\oplus (r_1)_*} & \bigoplus_i \tilde{H}_*(u^{-1}(\Gamma))
\end{array}$$

The monodromy action of f on a component a_i of this decomposition is the monodromy action induced by a path in $L_{e+\rho}$ (which depend on the i) , hence can be described by the monodromy actions of the g and h.

For example, if we have the following picture (Figure 3. then $m_e(a) = m_e(a_o + \sum_{k=1}^{2} \sum_{l=1}^{r} a_l^k) = a_o + \sum_{k=1}^{2} \sum_{l=1}^{r} (\gamma_l^k)_* (a_l^k)$, where γ_l^k is the path $(\Gamma I_{j_l}^k)_o^{-1} \circ (\Gamma I_{j_l}^k)_1$.

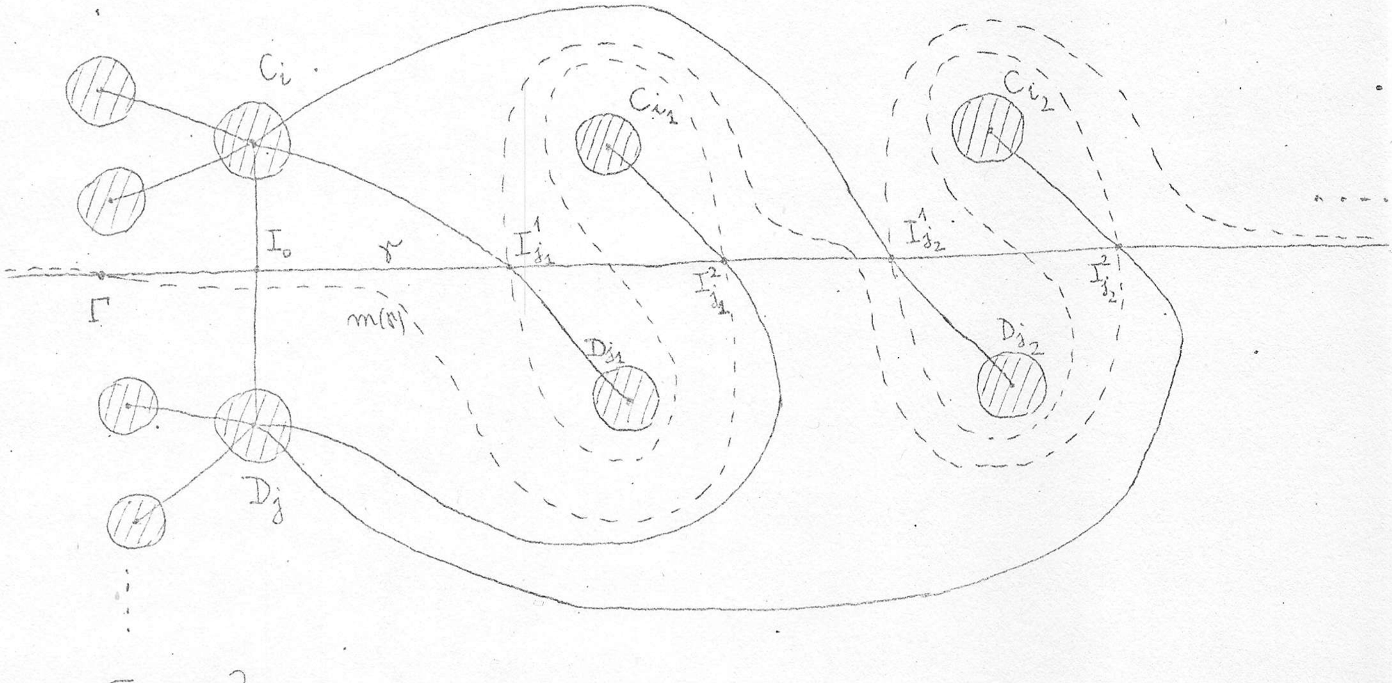


Figure 3.

3. The product case : $P(c,d)=c\cdot d$

The proofs of the affirmations in this section can be obtained by adapting the proofs of 2§. Therefore, they are omited here.

Let $g : \mathbb{C}^n \longrightarrow \mathbb{C}$ and $h : \mathbb{C}^m \longrightarrow \mathbb{C}$ be as in 2§. and we define $f : \mathbb{C}^n \times \mathbb{C}^m \longrightarrow \mathbb{C}$ by $f(x,y)=g(x)\cdot h(y)$.

If we take $\Lambda_f=\{0\}\cup\Lambda_g\cdot\Lambda_h=\{0\}\cup\{c_i\cdot d_j \mid c_i\in\Lambda_g,\ d_j\in\Lambda_h\}$ then the restricted map $f : \mathbb{C}^n\times\mathbb{C}^m - f^{-1}(\Lambda_f) \longrightarrow \mathbb{C}-\Lambda_f$ is a C^∞ locally trivial fibration.

If $e\notin\Lambda_f$, then $Q_e=\{(c,d)\in\mathbb{C}\times\mathbb{C} \mid c\cdot d=e\}$ is diffeomorph with $D=\{z\in\mathbb{C} : 1<|z|<2\}$. Let $\{D_j\}_{j=\overline{1,s}}=Q_e\cap(\mathbb{C}\times\Lambda_h)$ and $\{C_i\}_{i=\overline{1,t}}=Q_e\cap(\Lambda_g\times\mathbb{C})$. Then there exists a C^∞ diffeomorphism $v : D \longrightarrow Q_e$ such that $\lim\limits_{|z|\to 1} v(z)=(0,\infty)$, $\lim\limits_{|z|\to 2} v(z)=(\infty,0)$,

$D_j\in v(D\cap\{z : |z-1|\le\frac{1}{3}\}) = \mathcal{D}$ $\qquad (j=1,\dots,s)$

$C_i\in v(D\cap\{z : |z-2|\le\frac{1}{3}\}) = \mathcal{C}$ $\qquad (i=1,\dots,t)$

It is easy to see, that $\Lambda_g-\{0\}\subset pr_1\mathcal{C}$ and $0\notin pr_1\mathcal{C}$.

If $Q_e\cap(\mathbb{C}\times\Lambda_h)=\emptyset$ (respectively $Q_e\cap(\Lambda_g\times\mathbb{C})=\emptyset$), then we don't construct the set $\mathcal{D}$ (respectively $\mathcal{C}$) . .

If we denote $\gamma_D=\partial\mathcal{D}=\{z\in D : |z-1|=\frac{1}{3}\}$, $\gamma_C=\partial\mathcal{C}$, and $\mathcal{E}=Q_e-\mathrm{int}(\mathcal{D}\cup\mathcal{C})$, then :

$$f^{-1}(e) = u^{-1}(Q_e) = u^{-1}(\mathcal{C}) \bigcup_{u^{-1}(\gamma_C)} u^{-1}(\mathcal{E}) \bigcup_{u^{-1}(\gamma_D)} u^{-1}(\mathcal{D}) .$$

(A). Some proprieties of the set $u^{-1}(\mathcal{C})$:

i.) $u^{-1}(\mathcal{C}) \approx g^{-1}(pr_1\mathcal{C})\times H$

ii.) If $0\notin\Lambda_g$,then $g^{-1}(pr_1\mathcal{C})\approx g^{-1}(\mathbb{C})$, hence $u^{-1}(\mathcal{C})\approx\mathbb{C}^n\times H$

iii.) If $\{0\}=\Lambda_g$, then $g^{-1}(pr_1\mathcal{C})\approx g^{-1}(\text{point})\times pr_1\mathcal{C}$, hence $u^{-1}(\mathcal{C})\approx G\times H\times pr_1\mathcal{C}$.

iv.) If $0\in\Lambda_g\neq\{0\}$, then (by Mayer-Vietoris argument)

$\widetilde{H}_*(g^{-1}(pr_1\mathcal{C}))\oplus\widetilde{H}_*(g^{-1}(\mathbb{C}-pr_1\mathcal{C}))=\widetilde{H}_*(g^{-1}(pr_1\gamma_C))=\widetilde{H}_*(G)$ and $\widetilde{H}_*(g^{-1}(\mathbb{C}-pr_1\mathcal{C}))$ depend only on the singularity of g at 0.

(A'). The set $u^{-1}(\mathcal{D})$ has similar proprieties.

(B). By the triviality of the fibrations $u^{-1}(\gamma_C) \longrightarrow \gamma_C$ and $u^{-1}(\gamma_D) \longrightarrow \gamma_D$ we have $u^{-1}(\gamma_C) \approx \mathbb{R}\times G\times H$ and $u^{-1}(\gamma_D) \approx \mathbb{R}\times G\times H$.

(C). Some proprieties of the set $u^{-1}(\mathcal{E})$:

i.) $u:u^{-1}(\mathcal{E}) \longrightarrow \mathcal{E}$ is a C^{∞} locally trivial fibration with fiber $G\times H$ and base space $\mathcal{E} \approx S^1\times\mathbb{R}$.

ii.) The path $\alpha: [0,1] \longrightarrow \mathcal{E}$, $\alpha(t)=v(\frac{3}{2}\cdot\exp(2\pi i t))$ is a generator of $\pi_1(\mathcal{E}, v(\frac{3}{2})) = \mathbb{Z}$.

iii.) Because $(pr_1 \circ v)_* : H_1(D) \longrightarrow H_1(\mathbb{C}^*)$ is a isomorphism of the group $\mathbb{Z}$, we have $(pr_1\circ v)_*(1) = \varepsilon$, where $\varepsilon = \pm 1$. Then $(pr_2 \circ v)_*(1) = -\varepsilon$, and the monodromy $\alpha_*: H_*(G\times H) \longrightarrow H_*(G\times H)$ is equal with $m_g(0)^{\varepsilon} \times m_h(0)^{-\varepsilon}$, where $m_g(0)$ (respectively $m_h(0)$) is the monodromy of g (of h) around at 0.

iv.) If $0 \notin \Lambda_g$, then $u^{-1}(\mathcal{E}) \approx G\times h^{-1}(pr_2 \mathcal{E})$.

If $0 \notin \Lambda_g$, $0 \notin \Lambda_h$, then $u^{-1}(\mathcal{E}) \approx \mathcal{E}\times F\times G$.

Now, the topological proprieties of the generic fiber $f^{-1}(e) = F$ can be determined summing (A),(B),(C). For example :

I. If $\Lambda_g \neq \emptyset$, then $F \approx G\times h^{-1}(\mathbb{C}^*)$

II. If $\Lambda_g = \{0\}$, then $F \approx u^{-1}(\mathcal{E}) \bigcup_{G\times H} G\times h^{-1}(pr_2 \mathcal{D})$

a.) $0 \notin \Lambda_h \Longrightarrow F \approx g^{-1}(pr_1 \mathcal{E})\times H \bigcup_{G\times H} G\times C^m$

b.) $\{0\} = \Lambda_h \Longrightarrow F \approx u^{-1}(\mathcal{E})$ ($\mathcal{E} = Q_e$), hence there is a fibration $F \xrightarrow{u} S^1\times\mathbb{R} = Q_e$ with fiber $G\times H$.

III. If $0 \notin \Lambda_g$ and $0 \notin \Lambda_h$, then

$$F \approx \mathbb{C}^n\times H \bigcup_{G\times H} G\times H\times\mathcal{E} \bigcup_{G\times H} G\times\mathbb{C}^m \approx G*H \bigcup_{G\times H} G\times H\times \mathcal{E} .$$

Therefore $H_q(F) = H_q(G*H) \oplus H_{q-1}(G\times H)$.

The description of the monodromy of f in the general case is delicate, and we do not clarify it in this paper. For example, the change $\gamma_D \mapsto m_e(\gamma_D)$, $\gamma_C \mapsto m_e(\gamma_C)$ $(e \neq 0)$ is illustrated by the following "polypdance" figure :

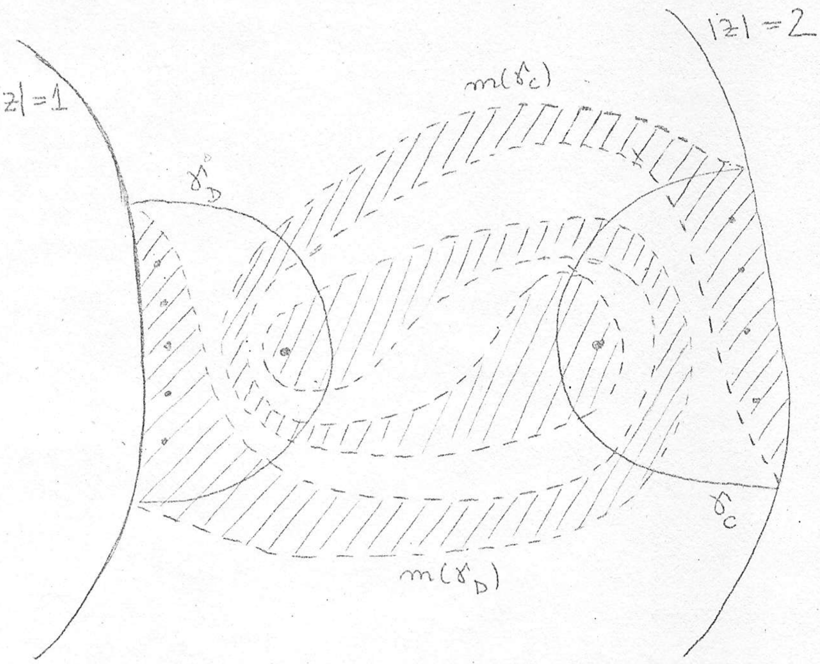


Figure 4.

In the case $\Lambda_g = \{0\}$, $\Lambda_h = \{0\}$ the monodromy of f can be described easy. In this case $F \longrightarrow Q_1 \approx S^1 \times \mathbb{R}$ is a fibration with fiber $G \times H$. The path $\gamma_o : [0,1] \longrightarrow Q_1$

$\gamma_o(t) = (\exp(2\pi i t), \exp(-2\pi i t))$ induces the monodromy of u

$(\gamma_o)_* : H_*(G \times H) \longrightarrow H_*(G \times H)$, $(\gamma_o)_* = m_g(0) \times m_h(0)^{-1}$.

If we use the map $\gamma_s : [0,1] \longrightarrow Q_{\exp(2\pi i s)}$,

$\gamma_s(t) = (\exp(2\pi i t) \cdot \exp(2\pi i s), \exp(-2\pi i t))$, we obtain that the monodromy action of f is induced by $m_g(0) \times 1$.

(Because $(m_g(0) \times 1) \circ (m_g(0) \times m_h(0)^{-1}) = (m_g(0) \times m_h(0)^{-1}) \circ (m_g(0) \times 1)$ the map $m_g(0) \times 1$ defines a isomorphism on $H_*(F)$.)

It is easy to see , that $m_g(0) \times 1$ and $1 \times m_h(0)$ induced the same action.

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