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MODELS FOR INFINITE SEQUENCES OF NONCOMMUTING OPERATORS

Gelu POPESCU

In [1], J.W.Bunce developed a model theory for n-tuples of not necessarily commuting operators, extending the work of A.E. Frazho [4] for pairs of operators.

He proved, for a finite number of operators on Hilbert space, versions of the Rota model theorem, the de Branges-Rovnyak model theorem, and the coisometric extension theorem.

The aim of this paper is to extend these results for infinite sequences of noncommuting operators, to generalize some results due to B.Sz.Nagy [8] and G.C.Rota [5, Problem 121] and to give some necessary and sufficient conditions for simultaneous similarity. We shall prove all these results without using the theorems above mentioned (for a finite number of operators).

1. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded operators in $\mathcal{H}$. We recall that a coisometry $V \in B(\mathcal{H})$ is called pure if $V^n \to 0$ (strongly) as $n \to \infty$.

In [1, Proposition 4] J.W.Bunce proved that for any finite family $\{A_i ; 1 \leq i \leq n\}$ ($n \in \mathbb{N}$) of operators such that $r(A_i) < 1$ for each i ($r(T)$ denoting the spectral radius of an operator $T \in B(\mathcal{H})$) and $\sum_{i=1}^{n} A_i^* A_i \leq I_{\mathcal{H}}$ ($I_{\mathcal{H}}$ is the identity on $\mathcal{H}$) there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and pure coisometries $\{S_i ; 1 \leq i \leq n\}$ acting on $\mathcal{K}$ such that $S_i(\mathcal{H}) \subset \mathcal{H}$, $S_i|_{\mathcal{H}} = A_i$ for each i and $S_i S_j^* = 0$ for $i \neq j$.

We begin with a theorem which generalizes the result above mentioned and Proposition 1 of [1], replacing the condition that $r(A_i) < 1$ by the condition $A_i^n \to 0$ (strongly) as $n \to \infty$.

Let us consider Λ and J to be sets of indices such that card $\Lambda \leq \aleph_0$.

<u>Proposition 1.1.</u> Let $\mathcal{A}_\alpha = \{A_{\alpha,i} ; i \in \Lambda\} \subset B(\mathcal{H})$ for each $\alpha \in J$.

Then the following two conditions are equivalent

a) $\sum_{i\in\Lambda} A^*_{\alpha,i} A_{\alpha,i} \leq I_{\mathcal{H}}$ for each $\alpha \in J$.

b) There exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and the families of coisometries $\mathcal{V}_\alpha = \{V_{\alpha,i} ; i \in \Lambda\} \subset B(\mathcal{K})$ $(\alpha \in J)$ such that $\sum_{i\in\Lambda} V^*_{\alpha,i} V_{\alpha,i} \leq I_{\mathcal{K}}$, $V_{\alpha,i}(\mathcal{H}) \subset \mathcal{H}$ and $V_{\alpha,i}|_{\mathcal{H}} = A_{\alpha,i}$ for each $\alpha \in J$, $i \in \Lambda$.

One can even require that $V_{\alpha,i}$ be a pure coisometry for every $\alpha \in J$ and $i \in \Lambda$ for which $A^n_{\alpha,i} \to 0$ (strongly) as $n \to \infty$.

<u>Proof.</u> It is easy to see that b) implies a).

Conversely, assume that for each $\alpha \in J$ $\sum_{i\in\Lambda} A^*_{\alpha,i} A_{\alpha,i} \leq I_{\mathcal{H}}$. Consider the Hilbert space $H = \bigoplus_{\substack{\alpha\in J\\ i\in\Lambda}} \mathcal{H}_{\alpha,i}$, where $\mathcal{H}_{\alpha,i}$ is a copy of the Hilbert space $\mathcal{H}$, and the operator $T \in B(\mathcal{H})$ defined by $T(\bigoplus_{\substack{\alpha\in J\\ i\in\Lambda}} h_{\alpha,i}) = \bigoplus_{\substack{\alpha\in J\\ i\in\Lambda}} A_{\alpha,i} h_{\alpha,i}$.

Note that T is a contraction. Indeed,

$$\| T(\bigoplus_{\substack{\alpha\in J\\ i\in\Lambda}} h_{\alpha,i}) \|^2 = \sum_{\alpha\in J} \Big(\sum_{i\in\Lambda} A^*_{\alpha,i} A_{\alpha,i} h_{\alpha,i}, h_{\alpha,i} \Big) \leq \sum \| h_{\alpha,i} \|^2 \leq \| \bigoplus_{\alpha\in J} h_{\alpha,i} \|^2,$$

for every $\bigoplus_{\substack{\alpha \in J \\ i \in \Lambda}} h_{\alpha,i} \in H$.

Let us determine a Hilbert space $K \supset H$ and a coisometry $V \in B(K)$ such that $V(H) \subset H$ and $V|_H = T$.

Let $K = H \oplus \mathcal{M}$, where $\mathcal{M}$ is a Hilbert space which we shall determine. With respect to this decomposition of K the matrix of V is

$$V = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix}$$

where $X : \mathcal{M} \to H$ and $Y : \mathcal{M} \to \mathcal{M}$ satisfy the relations:

(1.1) $\quad TT^* + XX^* = I_H,\ XY^* = 0,\ YY^* = I_{\mathcal{M}}$.

Since Y is a coisometry, the Wold decomposition of the Hilbert space $\mathcal{M}$ with respect to Y^* is $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ and $Y = S^* \oplus U$, where S is the unilateral shift acting on $\mathcal{M}_0 = \bigoplus_{n=0}^{\infty} Y^{*n}(\mathcal{L})$, $\mathcal{L} = \mathcal{M} \ominus Y^*(\mathcal{M})$ is the wandering subspace of S, and U is a unitary operator acting on $\mathcal{M}_1 = \mathcal{M} \ominus \mathcal{M}_0 = \bigcap_{n=0}^{\infty} Y^{*n}(\mathcal{M})$.

The relation $XY^* = 0$ implies $X|_{\mathcal{M}_1} = 0$. Therefore, with respect to the decomposition $K = H \oplus \mathcal{M}_0 \oplus \mathcal{M}_1$ the matrix of V is

$$V = \begin{pmatrix} T & X & 0 \\ 0 & S^* & 0 \\ 0 & 0 & U \end{pmatrix}$$

where X stands for $X|_{\mathcal{M}_0}$. Therefore $X : \mathcal{M}_0 \to H$ and the relations (1.1) become

(1.2) $\quad TT^* + XX^* = I_H,\ XS = 0$

Obviously, we can consider

$$\mathcal{M}_0 = \ell^2(\mathcal{L}) = \{ (x_1, x_2, \ldots); x_i \in \mathcal{L}, \sum_{i=1}^{\infty} \| x_i \|^2 < \infty \}$$

and $S : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$, $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$.

Since $XS = 0$, it follows that $X(0, y_1, y_2, \ldots) = 0$ for every $y_i \in \mathcal{L}$ such that $\sum_{i=1}^{\infty} \| y_i \|^2 < \infty$. We embed $\mathcal{L}$ in $\ell^2(\mathcal{L})$ by identifying the element $\ell \in \mathcal{L}$ with the element $(\ell, 0, 0, \ldots) \in \ell^2(\mathcal{L})$. In the sequel X stands for $X|_{\mathcal{L}}$. Thus the relations (1.2) become

(1.3) $$TT^* + XX^* = I_H ,$$

where $X : \mathcal{L} \longrightarrow H$.

The relation (1.3) holds for $\mathcal{L} = H$ and $X = (I_H - TT^*)^{1/2}$. With respect to the decomposition $H = \bigoplus_{\substack{\alpha \in J \\ i \in \Lambda}} \mathcal{H}_{\alpha, i}$ we have

$X = \bigoplus_{\alpha \in J, i \in \Lambda} X_{\alpha, i}$, where $X_{\alpha, i} : H \longrightarrow \mathcal{H}$. Taking into account (1.3) the following relations hold:

(1.4) $$A_{\alpha, i} A^*_{\alpha, i} + X_{\alpha, i} X^*_{\alpha, i} = I_{\mathcal{H}} \quad , \text{ for each } \alpha \in J,\ i \in \Lambda$$

$$A_{\alpha, i} A^*_{\alpha, j} + X_{\alpha, i} X^*_{\alpha, j} = 0 \quad , \text{ for } \alpha \in J,\ i \neq j, i, j \in$$

Let $\{1, 2, \ldots\} = \bigcup_{i \in \Lambda} N_i$ such that $N_i \cap N_j = 0$ $(i \neq j)$ and card $N_i = \aleph_0$ for each $i \in \Lambda$. Setting

$$N_i = \{ n_1^{(i)}, n_2^{(i)}, \ldots \} , \text{ where } n_1^{(i)} < n_2^{(i)} < \ldots$$

for each $i \in \Lambda$

we define $Z_i \in B(\ell^2(H))$ by

$$Z_i(h_1, h_2, \ldots) = (h_{n_1^{(i)}}, h_{n_2^{(i)}}, \ldots), \ h_j \in H \ (\sum_{j \in \Lambda} \| h_j \|^2 < \infty$$

Note that $Z_i Z_i^* = I_{\ell^2(H)}$ $(i \in \Lambda)$ and $Z_i Z_j^* = 0$ $(i \neq j)$.

Consider the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \ell^2(H)$ and define $V_{\alpha, i} \in B(\mathcal{K})$ $(\alpha \in J,\ i \in \Lambda)$ by the matrix

$$V_{\alpha, i} = \begin{pmatrix} A_{\alpha, i} & W_{\alpha, i} \\ & \end{pmatrix}$$

where $W_{\alpha,i}$ ($\alpha \in J$, $i \in \Lambda$) stands for the operator

$X_{\alpha,i} \oplus 0 \oplus 0 \oplus \ldots$

By (1.4) a simple computation shows that for every $\alpha \in J$, $i \in \Lambda$ we have:

$$V_{\alpha,i} V^*_{\alpha,i} = I_{\mathcal{K}}, \quad V_{\alpha,i} V^*_{\alpha,j} = 0 \ (i \neq j), \quad V_{\alpha,i}(\mathcal{H}) \subset \mathcal{H}$$

and $V_{\alpha,i}|_{\mathcal{H}} = A_{\alpha,i}$.

Let us prove that if $A^n_{\alpha,i} \longrightarrow 0$ (strongly) as $n \longrightarrow \infty$ for some $\alpha \in J$, $i \in \Lambda$, then $V^n_{\alpha,i} \longrightarrow 0$ (strongly) as $n \longrightarrow \infty$ for the same $\alpha \in J$, $i \in \Lambda$.

Note that, with respect to decomposition $\mathcal{K} = \mathcal{H} \oplus \ell^2(H)$, we have

$$V^n_{\alpha,i} = \begin{pmatrix} A^n_{\alpha,i} & \sum_{j=0}^{n} A^j_{\alpha,i} W_{\alpha,i} (Z_i S^*)^{n-j} \\ 0 & (Z_i S^*)^n \end{pmatrix}$$

Let $y = (\underbrace{0,\ldots,0}_{m-1 \text{ times}}, y_m, 0, 0, \ldots) \in \ell^2(H)$, where $m \geq 1$ and let $n > m$. Since $W_{\alpha,i}(0, h_2, h_3, \ldots) = 0$ for every $(0, h_2, h_3, \ldots) \in \ell^2(H)$, it follows that

$$\sum_{j=0}^{n} A^j_{\alpha,i} W_{\alpha,i} (Z_i S^*)^{n-j} y = A^{n-p}_{\alpha,i} W_{\alpha,i} (Z_i S^*)^p y$$ if there exists $1 \leq p \leq m$ such that $(Z_i S^*)^p y = (y_m, 0, 0, \ldots)$.

Otherwise, we have $\sum_{j=0}^{n} A^j_{\alpha,i} W_{\alpha,i} (Z_i S^*)^{n-i} y = A^n_{\alpha,i} W_{\alpha,i}$.

In both the cases, since $A^n_{\alpha,i} \longrightarrow 0$ (strongly) as $n \longrightarrow \infty$, it follows that $\sum_{j=0}^{n} A^j_{\alpha,i} W_{\alpha,i} (Z_i S^*)^{n-j} y \longrightarrow 0$ as $n \longrightarrow \infty$.

Since $(Z_i S^*)^n \longrightarrow 0$ (strongly) too, it follows that $V^n_{\alpha,i}(h \oplus y) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $h \in \mathcal{H}$ and all y of the

form above mentioned.

But, the span of all the vectors y of considered types is the Hilbert space $\ell^2(H)$ and $\| V^n_{\alpha,i} \| \leq 1$ for each $n \in \{1,2,\ldots$ Thus, we have that $V^n_{\alpha,i} \to 0$ (strongly) as $n \to \infty$ for the same $\alpha \in J$, $i \in \Lambda$ for which $A^n_{\alpha,i} \to 0$ (strongly) as $n \to \infty$.

This completes the proof.

We remark that if card Λ = card $J = 1$ we find again the coisometric extension theorem and the de Branges-Rovnyak model theorem (see [9], [5]). The result of E.Durszt and B.Sz-Nagy [2] is contained also in Proposition 1.1.

2. We say that a family $\mathcal{A} = \{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ is simultaneously similar to a family $\mathcal{B} = \{B_i\}_{i\in\Lambda} \subset B(\mathcal{K})$ if there exists an invertible operator $R \in B(\mathcal{H},\mathcal{K})$ so that $A_i = R^{-1} B_i R$ for every $i \in \Lambda$.

In what follows we shall obtain a generalization of a result due to B.Sz.-Nagy [8], that is, an operator $A \in B(\mathcal{H})$ is similar to an isometry if and only if there exists $a \geq b > 0$ such that

$$b \| h \|^2 \leq \| A^n h \|^2 \leq \| h \|^2 \text{ for every } h \in \mathcal{H},\ n \in \mathbb{N}.$$

We shall also obtain a generalization of Rota's model theorem, for infinite sequences of operators, and we shall give some necessary and sufficient conditions for a family $\{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ to be simultaneously similar to a family $\{T_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ of contractions with $\sum_{i\in\Lambda} T_i^* T_i \leq I_{\mathcal{H}}$.

Let us denote by $F(k,\Lambda)$ the set of all functions from the set $\{1,2,\ldots,k\}$ to the set Λ. For $\mathcal{A} = \{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ and $f \in F(k,\Lambda)$, let $A_f = A_{f(1)} A_{f(2)} \cdots A_{f(k)}$.

The following two lemmas are simple extensions of Lemmas 4 and 5 from [1] We omit the proofs.

Lemma 2.1. Let $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ such that the series $\sum_{i \in \Lambda} A_i^* A_i$ is convergent in the strong operator topology (if card $\Lambda = \aleph_0$)

a) If $1 \leq m < n$ then

$$\sum_{f \in F(n,\Lambda)} A_f^* A_f = \sum_{q \in F(n-m,\Lambda)} A_q^* \Big(\sum_{g \in F(m,\Lambda)} A_g^* A_g \Big) A_q$$

b) For any $m, n \geq 1$

$$\Big\| \sum_{f \in F(nm,\Lambda)} A_f^* A_f \Big\| \leq \Big\| \sum_{g \in F(n,\Lambda)} A_g^* A_g \Big\|^m$$

Lemma 2.2. Let $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ with $\sum_{i \in \Lambda} A_i^* A_i$ strongly convergent.

Then $\lim_{n \to \infty} \Big\| \sum_{f \in F(m,\Lambda)} A_f^* A_f \Big\|^{1/m} =$

$$= \inf_m \Big\{ \Big\| \sum_{f \in F(m,\Lambda)} A_f^* A_f \Big\|^{1/m} \Big\}.$$

Define $r(\mathcal{A}) = \inf_m \Big\{ \Big\| \sum_{f \in F(m,\Lambda)} A_f^* A_f \Big\|^{1/2m} \Big\}$.

For Λ with card $\Lambda = 1$ we find again the well-known formula for the spectral radius of an operator.

The case when card $\Lambda < \aleph_0$ is considered in [1].

Proposition 2.3. Let $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$. The following statements are equivalent:

a) There exists a family of contractions $\mathcal{T} = \{T_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ with $\sum_{i \in \Lambda} T_i^* T_i \leq I_{\mathcal{H}}$ such that $\mathcal{A}$ is simultaneously similar to $\mathcal{T}$.

b) There exists a positive invertible operator $P \in B(\mathcal{H})$ such that $\sum_{i \in \Lambda} A_i^* P A_i \leq P$.

Proof. Assume a) and let $R \in B(\mathcal{H})$ be an invertible operator such that $A_i = R^{-1} T_i R$ for each $i \in \Lambda$. Since

$\sum_{i\in\Lambda} T_i^* T_i = (R^{-1})^* \left(\sum_{i\in\Lambda} A_i^* R^* R A_i \right) R^{-1} \leq I_{\mathcal{H}}$ it follows that

$\sum_{i\in\Lambda} A_i^* P A_i \leq P$, where $P = R^* R$.

Conversely, assume b) and consider $T_i = RA_i R^{-1}$, where $R = P^{1/2}$. Thus, $\sum_{i\in\Lambda} T_i^* T_i \leq I_{\mathcal{H}}$ and the proof is complete.

A necessary condition for simultaneous similarity is the following:

<u>Proposition 2.4.</u> If a family $\mathcal{A} = \{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ is simultaneously similar to a family $\mathcal{T} = \{T_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ of contractions with $\sum_{i\in\Lambda} T_i^* T_i \leq I_{\mathcal{H}}$, then there exists $M > 0$ such that

$$\sum_{f\in F(n,\Lambda)} \| A_f h \|^2 \leq M \| h \|^2$$

for every $h \in \mathcal{H}$ and $n \in \mathbb{N}$.

In particular it follows that $r(\mathcal{A}) \leq 1$.

<u>Proof.</u> According to Proposition 2.3 there is a positive invertible operator $R \in B(\mathcal{H})$ such that $T_i = RA_i R^{-1}$ for each $i \in \Lambda$ and $\sum_{i\in\Lambda} T_i^* T_i \leq I_{\mathcal{H}}$.

By Lemma 2.1 we have

$$\left\| \sum_{f\in F(k,\Lambda)} T_f^* T_f \right\| \leq \left\| \sum_{i\in\Lambda} T_i^* T_i \right\|^k \leq 1 \quad \text{for any } k \in \mathbb{N}.$$

Since $A_f = R^{-1} T_f R$ $(f \in F(k,\Lambda))$ it follows that

$$\left\| \sum_{f\in F(k,\Lambda)} A_f^* A_f \right\| \leq \| R \|^2 \left\| \sum_{f\in F(k,\Lambda)} T_f^* (R^{-1})^2 T_f \right\| \leq$$

$$\leq \| R \|^2 \| R^{-1} \|^2$$

for any $k \in \mathbb{N}$.

By Lemma 2.2 is simple to deduce that $r(\mathcal{A}) \leq 1$.

The following theorem is a generalization of the result due to B.Sz.-Nagy [8] (for single operators) and the proof is on the same line.

Proposition 2.5. Let $\mathcal{A} = \{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$. The following two conditions are equivalent:

a) There exists $\mathcal{V} = \{V_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} V_i^* V_i = I_{\mathcal{H}}$ such that $\mathcal{A}$ is simultaneously similar to $\mathcal{V}$.

b). There exists $a \geq b > 0$ such that

$$b \|h\|^2 \leq \sum_{f\in F(n,\Lambda)} \|A_f h\|^2 \leq a \|h\|^2$$

for any $h \in \mathcal{H}$ and $n \in \mathbb{N}$.

Proof. Assume a) and let $R \in B(\mathcal{H})$ a positive invertible operator such that $A_i = R^{-1} V_i R$ for each $i \in \Lambda$.

Since $\sum_{i\in\Lambda} V_i^* V_i = I_{\mathcal{H}}$, we have, using Lemma 2.1,

$$\sum_{f\in F(n,\Lambda)} V_f^* V_f = I_{\mathcal{H}} \quad \text{for any } n \in \mathbb{N}. \tag{2.1}$$

As in the proof of Proposition 2.4, we obtain that

$$\sum_{f\in F(n,\Lambda)} \|A_f h\|^2 \leq \|R\|^2 \|R^{-1}\|^2 \|h\|^2$$

for any $h \in \mathcal{H}$, $n \in \mathbb{N}$.

On the other hand we have, using (2.1).

$$\sum_{f\in F(n,\Lambda)} \|A_f h\|^2 \geq \frac{1}{\|R\|^2} \sum_{f\in F(n,\Lambda)} \|V_f R h\|^2 =$$

$$= \frac{1}{\|R\|^2} \|Rh\|^2 \geq \frac{\|h\|^2}{\|R\|^2 \|R^{-1}\|^2}$$

for any $h \in \mathcal{H}$, $n \in \mathbb{N}$.

Conversely, assume condition b) is true.

If card $\Lambda = \aleph_0$ we can take $\Lambda = \mathbb{N} = \{1,2,\dots\}$. It is easy to see that the series $\sum_{f \in F(n,\Lambda)} \| A_f h \|^2$ is convergent for $h \in \mathcal{H}$ and $n \in \mathbb{N}$, whence, for every $h_1, h_2 \in \mathcal{H}$ and $n \in \mathbb{N}$, $\sum_{f \in F(n,\Lambda)} (A_f h_1, A_f h_2)$ is convergent.

Since for any $h_1, h_2 \in \mathcal{H}$ we have that

$$\left| \sum_{f \in F(n,\Lambda)} (A_f h_1, A_f h_2) \right| \leq \frac{a}{2} (\| h_1 \|^2 + \| h_2 \|^2)$$

for every $n \in \mathbb{N}$,

we can define for any $h_1, h_2 \in \mathcal{H}$

$$\langle h_1, h_2 \rangle = \operatorname*{LIM}_{n \to \infty} \sum_{f \in F(n,\Lambda)} (A_f h_1, A_f h_2), \text{ where LIM}$$

means Banach limit.

Taking into account the properties of the Banach limit we see that $\langle \cdot , \cdot \rangle$ is a hermitian bilinear form and

$$b \| h \|^2 \leq \langle h,h \rangle = \operatorname*{LIM}_{n \to \infty} \sum_{f \in F(n,\Lambda)} \| A_f h \|^2 \leq a \| h \|^2 \tag{2.2}$$

for each $h \in \mathcal{H}$.

By a well-known theorem on the bounded, hermitian, bilinear form, there exists a selfadjoint operator $P \in B(\mathcal{H})$ such that

$$\langle h_1, h_2 \rangle = (Ph_1, h_2) \quad \text{for all } h_1, h_2 \in \mathcal{H}$$

From (2.2) it follows that $bI_{\mathcal{H}} \leq P \leq a I_{\mathcal{H}}$, therefore P is a positive invertible operator.

Now we show that $P = \sum_{i=1}^{\infty} A_i^* P A_i$.

Since the series $\sum_{i \in \Lambda} \| A_i h \|^2$ is convergent, for every $\varepsilon >$ there exists $k_0 \in \mathbb{N}$ such that $\sum_{i=k+1}^{\infty} \| A_i h \|^2 \leq \frac{\varepsilon}{a}$ for any $k \geq k_0$.

Thus, for every $k \geqslant k_0$ and $n \in \mathbb{N}$ we have:

$$0 \leq \sum_{i=1}^{\infty} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 - \sum_{i=1}^{k} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 =$$

$$= \sum_{i=k+1}^{\infty} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 \leq a \sum_{i=k+1}^{\infty} \| A_i h \|^2 \leq \varepsilon ,$$

whence it follows

$$0 \leq \operatorname*{LIM}_{n \to \infty} \left(\sum_{i=1}^{\infty} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 \right) -$$

$$- \operatorname*{LIM}_{n \to \infty} \left(\sum_{i=1}^{k} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 \right) \leq \varepsilon$$

for any $k \geqslant k_0$.

Since $(Ph, h) = \operatorname*{LIM}_{n \to \infty} \sum_{f \in F(n+1, \Lambda)} \| A_f h \|^2 =$

$= \operatorname*{LIM}_{n \to \infty} \left(\sum_{i=1}^{\infty} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 \right)$ for any $h \in \mathcal{H}$, we have that:

$$0 \leq (Ph, h) - \sum_{i=1}^{k} \left(\operatorname*{LIM}_{n \to \infty} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 \right)$$ for any $k \geqslant k_0$

In other words

$$(Ph, h) = \lim_{k \to \infty} \sum_{i=1}^{k} \left(\operatorname*{LIM}_{n \to \infty} \sum_{f \in F(n, \Lambda)} \| A_f A_i h \|^2 \right) =$$

$$\sum_{i=1}^{\infty} \langle A_i h , A_i h \rangle = \left(\left(\sum_{i=1}^{\infty} A_i^* P A_i \right) h, h \right)$$ for every $h \in \mathcal{H}$.

Therefore, $\sum_{i=1}^{\infty} A_i^* P A_i = P$ and setting $R = P^{1/2}$, $V_i = R A_i R^{-1}$ for each $i \in \Lambda$, we deduce that $\sum_{i \in \Lambda} V_i^* V_i = I_{\mathcal{H}}$.

The case when Λ is a finite set is even simpler to deduce. The proof is complete.

We now give a necessary and sufficient condition for simultaneous similarity

Proposition 2.6. Let $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$. Then the following conditions are equivalent.

a) There exists $\mathcal{T} = \{T_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ with $\sum_{i \in \Lambda} T_i^* T_i \leq I_{\mathcal{H}}$ such that $\mathcal{A}$ is simultaneously similar to $\mathcal{T}$.

b) There exists $D \in B(\mathcal{H})$ and $a \geq b > 0$ such that

$$b \|h\|^2 \leq \sum_{f \in F(n, \Lambda)} \|A_f h\|^2 + \sum_{f \in F(n-1, \Lambda)} \|DA_f h\|^2 + \dots$$

$$+ \sum_{f \in F(1, \Lambda)} \|DA_f h\|^2 + \|Dh\|^2 \leq a \|h\|^2$$

for every $h \in \mathcal{H}$, $n \in \mathbb{N}$.

Proof. Assume condition a) is true. Then, according to Proposition 2.3 there exists a positive invertible operator $P \in B(\mathcal{H})$ such that $\sum_{i \in \Lambda} A_i^* P A_i \leq P$ (we can assume that $\|P\| \leq 1$

Let $D = (P - \sum_{i \in \Lambda} A_i^* P A_i)^{1/2}$ and for each $h \in \mathcal{H}$ and $n \in \mathbb{N}$

let $S_{n,h} = \sum_{f \in F(n, \Lambda)} \|A_f h\|^2 + \sum_{f \in F(n-1, \Lambda)} \|DA_f h\|^2 + \dots +$

$$+ \sum_{f \in F(1, \Lambda)} \|DA_f h\|^2 + \|Dh\|^2$$

An easy computation shows that :

$$S_{n,h} = \sum_{f \in F(n, \Lambda)} (A_f^* A_f h, h) - \sum_{f \in F(n, \Lambda)} (A_f^* P A_f h, h) +$$

$$+ (Ph, h).$$

By Proposition 2.4 there exists $M > 0$ such that

$$S_{n,h} \leq \sum_{f \in F(n, \Lambda)} (A_f^* A_f h, h) + (Ph, h) \leq (M + 1) \|h\|^2$$

for any $h \in \mathcal{H}$ and $n \in \mathbb{N}$.

On the other hand, since $P \leqslant I_{\mathcal{H}}$, we have

$$\sum_{f \in F(n,\Lambda)} (A_f^* A_f h, h) \geqslant \sum_{f \in F(n,\Lambda)} (A_f^* P A_f h, h), \text{ therefore}$$

$$S_{n,h} \geqslant (Ph, h) \frac{1}{\|R^{-1}\|} \|h\|^2 \quad (\text{where } R = P^{1/2}) \text{ for}$$

any $h \in \mathcal{H}$ and $n \in \mathbb{N}$.

Now assume b) and we shall prove that b) implies a). Let us consider the Hilbert space

$$K = \mathcal{H} \oplus \mathcal{D} \oplus \mathcal{D} \oplus \ldots$$

where $\mathcal{D} = \overline{D\mathcal{H}}$, and embed $\mathcal{H}$ in K by identifying the element $h \in \mathcal{H}$ with the element $(h,0,0,\ldots) \in K$.

Let $\lambda_i \in \mathbb{C}$ $(i \in \Lambda)$ with $\sum_{i \in \Lambda} |\lambda_i|^2 = 1$ and the operators $B_i \in B(K)$ $(i \in \Lambda)$ define by

$$B_i(h_0, h_1, \ldots) = (A_i h_0, \lambda_i D h_0, \lambda_i h_1, \ldots)$$

For each $f \in F(n,\Lambda)$ $(n \in \mathbb{N})$ we have:

$$B_f(h_0, h_1, \ldots) = (A_{f(1)} \cdots A_{f(n)} h_0, \lambda_{f(1)} D A_{f(2)} \cdots A_{f(n)} h_0,$$

$$\lambda_{f(1)} \lambda_{f(2)} D A_{f(3)} \cdots A_{f(n)} h_0, \ldots, \lambda_{f(1)} \cdots \lambda_{f(n-1)} D A_{f(n)} h_0$$

$$, \lambda_{f(1)} \cdots \lambda_{f(n)} D h_0, \lambda_f h_1, \ldots)$$

where λ_f stands for $\lambda_{f(1)} \cdots \lambda_{f(n)}$.

Since $\sum_{f \in F(n,\Lambda)} |\lambda_f|^2 = 1$ for any $n \in \mathbb{N}$, it is easy to show (by induction) that for each $k \in \{1, 2, \ldots, n-1\}$

$$\sum_{f \in F(n,\Lambda)} \| \lambda_{f(1)} \cdots \lambda_{f(k)} D A_{f(k+1)} \cdots A_{f(n)} h \|^2 =$$

$$= \sum_{g \in F(n-k,\Lambda)} \| D A_g h \|^2$$

for any $h \in \mathcal{H}$ and $n \in \mathbb{N}$.

Therefore, $\sum_{f \in F(n,\Lambda)} \| B_f(h_0, h_1, \ldots) \|^2 = \sum_{f \in F(n,\Lambda)} \|A_f h_0\|^2$

$$+ \sum_{f \in F(n-1, \Lambda)} \| DA_f h_0 \|^2 + \ldots + \sum_{f \in F(1, \Lambda)} \| DA_f h_0 \|^2 +$$

$$+ \| h_1 \|^2 + \ldots$$

for any $(h_0, h_1, \ldots) \in K$ and $n \in \mathbb{N}$.

By b) there exists $M \geq N > 0$ such that

$$N \| k \|^2 \leq \sum_{f \in F(n, \Lambda)} \| B_f k \|^2 \leq M \| k \|^2 \text{ for any } k \in K,$$

$n \in \mathbb{N}$.

According to Proposition 2.5, there exists $\mathcal{V} = \{V_i\}_{i \in \Lambda} \subset B(K)$ with $\sum_{i \in \Lambda} V_i^* V_i = I_K$ such that the family $\mathcal{B} = \{B_i\}_{i \in \Lambda}$ is simultaneously similar to $\mathcal{V}$.

Let us notice that $B_i^* | \mathcal{H} = A_i^*$, $B_i^*(\mathcal{H}) \subset \mathcal{H}$. Let $\Omega \in B(K)$ an invertible operator such that $B_i^* = \Omega V_i^* \Omega^{-1}$ $(i \in \Lambda)$ and consider the invertible operator $\Omega_0 : \mathcal{H} \longrightarrow \mathcal{H}_0, \Omega_0 = \Omega^{-1} | \mathcal{H}$, where $\mathcal{H}_0$ stands for $\Omega^{-1} \mathcal{H}$. Since $B_i^*(\mathcal{H}) \subset \mathcal{H}$ we have that $V_i^*(\mathcal{H}_0) \subset \mathcal{H}_0$ $(i \in \Lambda)$ and

$$A_i^* = B_i^* | \mathcal{H} = \Omega_0^{-1} (V_i^* | \mathcal{H}_0) \Omega_0$$

Writing the polar decomposition of Ω_0, that is $\Omega_0 = U |\Omega_0|$ where $|\Omega_0| = (\Omega_0^* \Omega_0)^{1/2}$ and $U = \Omega_0 |\Omega_0|^{-1}$ we obtain $A_i^* = R T_i^* R^{-1}$, where $R = |\Omega_0|^{-1}$ and $T_i^* = U^* (V_i^* | \mathcal{H}_0) U$.

Now an easy computation shows that $\sum_{i \in \Lambda} T_i^* T_i \leq I_{\mathcal{H}}$.

The proof is complete.

<u>Remark 2.7.</u> For Λ with card $\Lambda = 1$ the above proposition was proved in [3].

<u>Corollary 2.8.</u> Let $D_{\mathcal{A}} = | I_{\mathcal{H}} - \sum_{i \in \Lambda} A_i^* A_i |^{1/2}$. If there exists $a > 0$ such that

$$(2.3) \qquad \sum_{f \in F(n, \Lambda)} \| A_f h \|^2 + \sum_{f \in F(n-1, \Lambda)} \| D_{\mathcal{A}} A_f h \|^2 + \ldots +$$

$$+ \| D_{\mathcal{A}} h \|^2 \leq a \| h \|^2 \quad \text{for any } h \in \mathcal{H} \text{ and } n \in \mathbb{N},$$

then the condition b) of Proposition 2.6 is fulfilled.

Proof. Since for every $h \in \mathcal{H}$, $k \in \mathbb{N}$

$$\sum_{f \in F(k, \Lambda)} (A_f^* D_{\mathcal{A}} A_f h, h) \geq \sum_{f \in F(k, \Lambda)} (A_f^* A_f h, h) -$$

$$- \sum_{f \in F(k+1, \Lambda)} (A_f^* A_f h, h)$$

we have (2.3) with $a = 1$.

The following corollary is a generalization of Rota's model theorem [7].

Corollary 2.9. Let $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ such that there exists $a > 0$ and

(2.4) $$\sum_{n=1}^{\infty} \Big(\sum_{f \in F(n, \Lambda)} \| A_f h \|^2 \Big) \leq a \| h \|^2$$

for any $h \in \mathcal{H}$.

Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$, a family $\mathcal{S} = \{S_i\}_{i \in \Lambda} \subset B(\mathcal{K})$ of pure coisometries with orthogonal initial spaces, $S_i(\mathcal{H}) \subset \mathcal{H}$ $(i \in \Lambda)$ and an invertible operator $R \in B(\mathcal{H})$ such that

$$A_i = R^{-1} (S_i | \mathcal{H}) R \quad \text{for each } i \in \Lambda.$$

Proof. According to (2.4) and Proposition 2.6, there exists $\mathcal{T} = \{T_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ with $\sum_{i \in \Lambda} T_i^* T_i \leq I_{\mathcal{H}}$ and an invertible operator $R \in B(\mathcal{H})$ such that

(2.5) $$A_i = R^{-1} T_i R \qquad (i \in \Lambda)$$

On the other hand, for each $i \in \Lambda$

$$\sum_{n=1}^{\infty} \| A_i^n h \|^2 \leq \sum_{n=1}^{\infty} \Big(\sum_{f \in F(n, \Lambda)} \| A_f h \|^2 \Big) \leq a \| h \|^2$$

for any $h \in \mathcal{H}$.

Hence $A_i^n \to 0$ (strongly) as $n \to \infty$ and by (2.5) $T_i^n \to 0$ (strongly) for any $i \in \Lambda$. Applying Proposition 1.1, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$, a family $\mathcal{S} = \{S_i\}_{i \in \Lambda} \subset B(\mathcal{K})$ of pure coisometries with $\sum_{i \in \Lambda} S_i^* S_i \leq I_{\mathcal{K}}$, $S_i(\mathcal{H}) \subset \mathcal{H}$ and $S_i|_{\mathcal{H}} = T_i$ for each $i \in \Lambda$. Thus, $A_i = R^{-1}(S_i|_{\mathcal{H}})R$ $(i \in \Lambda)$ and the proof is complete.

Remark 2.10. For Λ with card $\Lambda = 1$ we find again the Rota model theorem, nomely, if there exists $a > 0$ such that $\sum_{n=1}^{\infty} \| A^n h \|^2 \leq a \| h \|^2$ for any $h \in \mathcal{H}$ (equivalent with $r(A) < 1$), then A is similar to a part of a backward shift.

We now give some conditions equivalent to condition (2.4) in Corollary 2.9. The proof of the following theorem is almost identical to the proof of [1, Proposition 6]. We omit the proof.

Proposition 2.11. Let $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$. Th following conditions are equivalent.

a) There is a positive operator $P \in B(\mathcal{H})$ such that

$$\left(\sum A_i^* P A_i \right) + I_{\mathcal{H}} = P$$

b) The series $\sum_{i \in \Lambda} A_i^* A_i$ and $\sum_{n=1}^{\infty} \left(\sum_{f \in F(n, \Lambda)} A_f^* \right.$ are strongly convergent.

c) The series $\sum_{i \in \Lambda} A_i^* A_i$ is strongly convergent and $r(\mathcal{A}) < 1$.

d) There is $a > 0$ such that $\sum_{n=1}^{\infty} \left(\sum_{f \in F(n, \Lambda)} \| A_f h \|^2 \right) \leq a \| h \|^2$ for any $h \in \mathcal{H}$.

Remark 2.12. If $\sum_{i \in \Lambda} A_i^* A_i \leq r I_{\mathcal{H}}$, where $r < 1$, then the family $\mathcal{A} = \{A_i\}_{i \in \Lambda}$ satisfies the condition d) of Proposition 2.11.

3. In this section we generalize the result from [5], namely, an operator $A \in B(\mathcal{H})$ is similar to a contraction

if and only if there exists $k \in \mathbb{N}$ such that A^k is similar to a contraction.

We shall also obtain a formula for $r(\mathcal{A})$, where $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ with $\sum_{i \in \Lambda} A_i^* A_i$ strongly convergent, which generalizes the well-known formula for the spectral radius of an operator $A \in B(\mathcal{H})$, that is,

$$r(A) = \inf_S \| S^{-1} AS \|$$

where the infimum is taken for all invertible operators $S \in B(\mathcal{H})$ (see [5, Problem 122]).

<u>Proposition 3.1.</u> Let $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$. The following statements are equivalent.

a) There is a family $\mathcal{C} = \{C_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ with $\sum_{i \in \Lambda} C_i^* C_i \leq I_{\mathcal{H}}$ such that $\mathcal{A}$ is simultaneously similar to $\mathcal{C}$.

b) There is $k \in \mathbb{N}$ and a family $\mathcal{T}_k = \{T_{(f)}\}_{f \in F(k,\Lambda)} \subset B(\mathcal{H})$ with $\sum_{f \in F(k,\Lambda)} T_{(f)}^* T_{(f)} \leq I_{\mathcal{H}}$ such that the family $\mathcal{A}_k = \{A_f\}_{f \in F(k,\Lambda)}$ is simultaneously similar to $\mathcal{T}_k$.

<u>Proof.</u> Assume condition a) is true. Let $R \in B(\mathcal{H})$ an invertible operator such that $A_i = R^{-1} C_i R$ $(i \in \Lambda)$. Hence $A_f = R^{-1} C_f R$ for $f \in F(k, \Lambda)$, $k \in \mathbb{N}$. Setting $T_{(f)} = C_f$ for $f \in F(k, \Lambda)$, $k \in \mathbb{N}$. We have that for each $k \in \mathbb{N}$ the family $\mathcal{A}_k$ is simultaneously similar to $\mathcal{T}_k$.

Conversely, assume b) is true. By Proposition 2.3 there is a positive invertible operator $P \in B(\mathcal{H})$ such that

$$\sum_{f \in F(k,\Lambda)} A_f^* P A_f \leq P. \tag{3.1}$$

Let us consider the positive invertible operator $Q \in B(\mathcal{H})$ given by the relation

$$Q = P + \sum_{n=1}^{k-1} \Big(\sum_{f \in F(n,\Lambda)} A_f^* P A_f \Big)$$

Taking into account (3.1) we have :

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$$\sum_{i\in\Lambda} A_i^* Q A_i = \sum_{n=1}^{k} \Big(\sum_{f\in F(n,\Lambda)} A_f^* P A_f \Big) \leq P + \sum_{n=1}^{k-1} \Big(\sum_{f\in F(n,\Lambda)} A_f^* P A_f \Big)$$

$$= Q.$$

It then follows from Proposition 2.3 that a) is true, so the proof is complete .

<u>Corollary 3.2.</u> Let $\mathcal{A} = \{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ be such that there exists $k \in \mathbb{N}$, $0 < r \leq 1$ and

(3.2) $$\sum_{f\in F(k,\Lambda)} \| A_f h \|^2 \leq r \| h \|^2 \text{ for any } h \in \mathcal{H}$$

Then there exists $\mathcal{T} = \{T_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} T_i^* T_i \leq I_{\mathcal{H}}$ and such that $\mathcal{A}$ is simultaneously similar to $\mathcal{T}$.

If $0 < r < 1$ one can even require that $\| T_i \| < 1$ for any $i \in \Lambda$.

<u>Proof.</u> Note that the condition (3.2) is equivalent to the condition

(3.3) $$\sum_{f\in F(k,\Lambda)} A_f^* A_f \leq r I_{\mathcal{H}}$$

By Proposition 3.1. there exists a family $\mathcal{T} = \{T_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} T_i^* T_i \leq I_{\mathcal{H}}$ and such that $\mathcal{A}$ is simultaneously similar to $\mathcal{T}$.

If $0 < r < 1$, then there is $\varepsilon > 1$ such that $\Big\| \varepsilon^{2k} \sum_{f\in F(k,\Lambda)} A_f^* A_f \Big\| \leq 1$. Considering $B_i = \varepsilon A_i$ $(i \in \Lambda)$ we have $\sum_{f\in F(k,\Lambda)} B_f^* B_f \leq I_{\mathcal{H}}$ and by Proposition 3.1. there exists a family $\mathcal{C} = \{C_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} C_i^* C_i \leq I_{\mathcal{H}}$ such that the family $\mathcal{B} = \{B_i\}_{i\in\Lambda}$ is simultaneously similar to $\mathcal{C}$. Hence, $\mathcal{A}$ is simultaneously similar to the family $\mathcal{T} = \{T_i\}_{i\in\Lambda}$, where $T_i = 1/\varepsilon \; C_i$ $(i \in \Lambda)$.

<u>Remark 3.3.</u> If $\mathcal{A} = \{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ and $r(\mathcal{A}) < 1$, then the condition (3.2) of Corollary 3.2. is fulfilled.

<u>Corollary 3.4.</u> If $\mathcal{A} = \{A_i\}_{i\in\Lambda} \subset B(\mathcal{H})$ and $r(\mathcal{A}) = 0$ then for every $\varepsilon > 0$, there is a family $\mathcal{T} = \{T_i\}_{i\in\Lambda} \subset B($

with $\sum_{i \in \Lambda} T_i^* T_i \leq I_{\mathcal{H}}$, $\| T_i \| \leq \varepsilon$ $(i \in \Lambda)$ such that $\mathcal{A}$ is simultaneously similar to $\mathcal{T}$.

Proof. For any $\varepsilon > 0$ $r(\frac{1}{\varepsilon}\mathcal{A}) = \frac{1}{\varepsilon} r(\mathcal{A}) = 0$, where $\frac{1}{\varepsilon}\mathcal{A} = \{\frac{1}{\varepsilon} A_i\}_{i \in \Lambda}$. Hence, $r(\frac{1}{\varepsilon}\mathcal{A}) < 1$ and by Remark 3.3 and Corollary 3.2. there is a family $\mathcal{C} = \{C_i\}_{i \in \Lambda}$ with $\sum_{i \in \Lambda} C_i^* C_i \leq I_{\mathcal{H}}$ such that $\frac{1}{\varepsilon}\mathcal{A}$ is simultaneously similar to $\mathcal{C}$. Setting $\mathcal{T} = \{T_i\}_{i \in \Lambda}$ where $T_i = \varepsilon C_i$ $(i \in \Lambda)$ the proof is complete.

We now use these results for proving the following

Proposition 3.5. If $\mathcal{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ and $\sum_{i \in \Lambda} A_i^* A_i$ is strongly convergent, then

$$r(\mathcal{A}) = \inf_S \left\{ \| \sum_{i \in \Lambda} (S^{-1} A_i S)^* (S^{-1} A_i S) \|^{1/2} \right\},$$

where the infimum is taken for all invertible operators $S \in B(\mathcal{H})$.

Proof. First, we show that for each invertible operator $S \in B(\mathcal{H})$ $r(\mathcal{A}) = r(S^{-1}\mathcal{A} S)$, where $S^{-1}\mathcal{A} S$ stands for the family $\{S^{-1} A_i S\}_{i \in \Lambda}$.

By Lemma 2.2 $r(S^{-1}\mathcal{A} S) \leq \inf_k \left\{ \| S \|^{1/k} \| S^{-1} \|^{1/k} \| \sum_{f \in F(k, \Lambda)} A_f^* A_f \|^{1/2k} \right\} \leq r(\mathcal{A})$. Hence, $r(\mathcal{A}) = r(S(S^{-1}\mathcal{A} S)S^{-1}) \leq r(S^{-1}\mathcal{A} S)$. Therefore,

$$r(\mathcal{A}) = r(S^{-1}\mathcal{A} S) \tag{3.4}$$

Using Lemma 2.1 and (3.4) we obtain

$$r(\mathcal{A}) \leq \inf_S \left\{ \| \sum_{i \in \Lambda} (S^{-1} A_i S)^* (S^{-1} A_i S) \|^{1/2} \right\} \tag{3.5}$$

According to Corollary 3.4, if $r(\mathcal{A}) = 0$ and $0 < \varepsilon < 1$, then there is a family $\mathcal{T} = \{T_i\}_{i \in \Lambda} \subset B(\mathcal{H})$ with $\sum_{i \in \Lambda} T_i^* T_i \leq I_{\mathcal{H}}$, $\| T_i \| \leq \varepsilon$ $(i \in \Lambda)$ such that $A_i = R^{-1}(T_i)R$ $(i \in \Lambda)$ for an invertible operator $R \in B(\mathcal{H})$. Therefore,

$$\| \sum_{i \in \Lambda} (RA_i R^{-1})^* (RA_i R^{-1}) \|^{1/2} \leq \varepsilon \quad \text{whence}$$

$$\inf_S \left\{ \left\| \sum_{i \in \Lambda} (S^{-1}A_iS)^*(S^{-1}A_iS) \right\|^{1/2} \right\} = 0$$

If $r(\mathcal{A}) \neq 0$, let us consider the family $\mathcal{B} = \{B_i\}_{i \in \Lambda}$ where $B_i = \frac{\varepsilon}{r(\mathcal{A})} A_i$, $0 < \varepsilon < 1$, $i \in \Lambda$. Since $r(\mathcal{B}) < 1$, by Remark 3.3. and Corollary 3.2. there exists a family $\mathcal{C} = \{C_i\}_{i \in \Lambda}$ with $\sum_{i \in \Lambda} C_i^* C_i \leq I_{\mathcal{H}}$ and a positive invertible operator $P \in B(\mathcal{H})$ such that $B_i = P^{-1} C_i P$ $(i \in \Lambda)$. An easy computation shows that

$$\sum_{i \in \Lambda} (PA_iP^{-1})^* (PA_iP^{-1}) = \left(\frac{r(\mathcal{A})}{\varepsilon}\right)^2 \sum_{i \in \Lambda} C_i^* C_i,$$

whence $r(\mathcal{A}) \geq \varepsilon \left\| \sum_{i \in \Lambda} (PA_iP^{-1})^*(PA_iP^{-1}) \right\|^{1/2} \geq \varepsilon \inf_S \left\{ \left\| \sum_{i \in \Lambda} (SA_iS^{-1})^* (SA_iS^{-1}) \right\|^{1/2} \right\}$ for every $0 < \varepsilon < 1$. Setting $\varepsilon \longrightarrow 1$ it follows that

$$r(\mathcal{A}) \geq \inf_S \left\{ \left\| \sum_{i \in \Lambda} (SA_iS^{-1})^* (SA_iS^{-1}) \right\|^{1/2} \right\} \tag{3.6}$$

The proof is complete.

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