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FEEDBACK THEORIES (A CALCULUS FOR ISOMORPHISM CLASSES OF FLOWCHART SCHEMES)

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Feedback theories (a calculus for isomorphism
classes of flowchart schemes)

by Gh. Ştefănescu

Abstract. A simple representation of multi-entry multi-exit
flowchart schemes is given. This shows that the basic operations
on flowchart schemes are: separated sum, composition with
'empty flowcharts,' and feedback. The main technical point is
giving a calculus for isomorphism classes of flowchart schemes.
This calculus is similar to that of polynomials and may be
considered as a framework of our calculi for deterministic and
nondeterministic flowchart schemes presented in [5].
Key words. feedback, semantics, flowchart schemes

1. Introduction. The reasons for the present note is twofolds:
(a) we are trying to prove that feedback is more natural than iteration;
(b) we give a calculus for isomorphism classes of flowchart schemes.
This paper may be seen as a natural extension of the last paper of
Elgot [4]. The characteristic feature of [4] is the attempt to
weaken the 'algebraic theory' structure (in the sense of Lawvere),
widely used in semantics of flowchart algorithms.

(a) The feedback is 'scalar' and all usual flowchart schemes
can be built up from atomic flowchart schemes and trivial ones
(which may be thought of as redirecting flow of control) by means of
sum, composition and feedback. This is no longer true for scalar
iteration (cf. [4], only flowcharts fulfiling 'for every closed path

C there is a vertex v_C of C such that every begin path to a

vertex of C meets v_C' can be obtained).

(b) There is some interest in axiomatization of isomorphism

classes of flowchart schemes [2, 3, 4]. Our calculus extend these;

its characteristic features are:

- the operations on flowcharts are defined by simple formulae

rather by some 'verbal descriptions' as in [2,4];

- the restriction to sum and composition of our algebraic structure

is more general than 'algebraic theories' used in [2,3] and essentially

corresponds to flow theories in [4], but instead of surjective

functions we need only bijective ones;

- our calculus work in a more general (and usful) case, e.g.

instead of trivial flowchart schemes we can use arbitrary known

flowchart algorithms in which a change of memory state can accompany
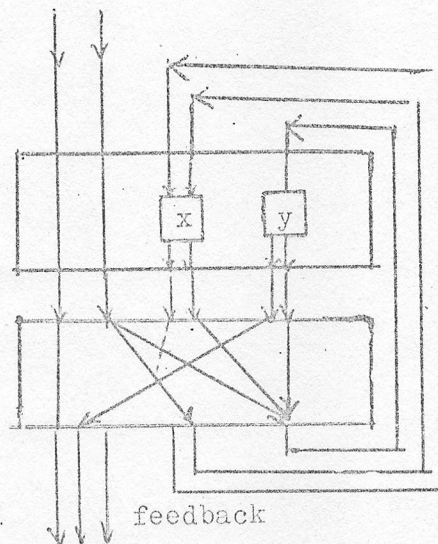
redirecting flow of control.

2. On the choice of operations. Every multi-entry multi-exit

flowchart scheme can be ordered as it is shown in FIG. 2.1.

$1_2 \oplus x \oplus y$

c
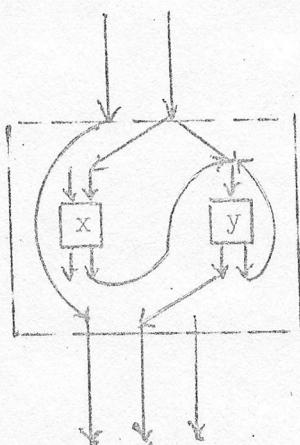
feedback

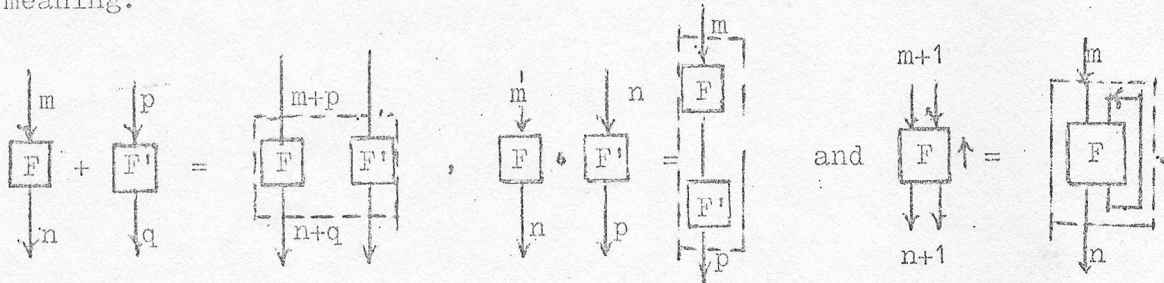2.1.a   2.1.b

FIG. 2.1. The standard form of a flowchart schemes

This shows that the basic operations on flowchart schemes are:

separated sum (or parallel composition) +, composition (or serial

composition) ∘ and feedback ↑; these have the following intuitive

meaning:

In the sequal $\uparrow^k$ denotes k-times application of $\uparrow$. By FIG. 2.1,

composition can be restricted to composition with 'empty flowcharts,'

that is flowcharts schemes without internal vertices. Generally, we

define the composition by
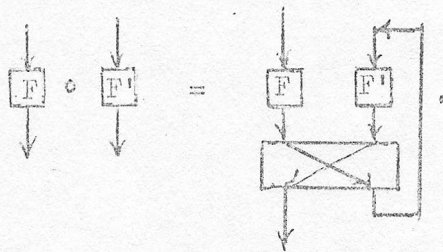
3. The categories Fn and Bi.

The category Fn has the set

of natural numbers $\mathbb{N}$ as its class of objects. The set of morphisms

of Fn with source n and target p is the set of all functions

$f : [n] = \{1,\ldots,n\} \longrightarrow [p]$. If $x \in [n]$ we write xf for the value

of f applied to x and if $g : [p] \longrightarrow [q]$ is a function, we write

$f \circ g : [n] \longrightarrow [q]$ for the composite.

Given a pair of functions $f_i : [n_i] \longrightarrow [p_i]$, $i \in [2]$ we define

the function $f_1 + f_2 : [n_1 + n_2] \longrightarrow [p_1 + p_2]$ as

$x(f_1 + f_2) = $ 'if $x \in [n_1]$ then xf_1 else $(x - n_1)f_2 + p_1$' for $x \in [n_1 + n_2]$.

The class of bijective functions in Fn is closed under $\circ$ and contains the identities, hence it gives a subcategory Bi of Fn. Moreover, Bi is closed under $+$. We write $m \leftrightarrow n$ for the block permutation function $m \leftrightarrow n : [m+n] \longrightarrow [n+m]$ given by

$$x(m \leftrightarrow n) = \text{'if } x \in [m] \text{ then } n+x \text{ else } x-m' \text{ for } x \in [m+n].$$

Given a bijective function $f : [n+1] \longrightarrow [p+1]$ we define the function $f\uparrow : [n] \longrightarrow [p]$ by

$$xf\uparrow = \text{'if } xf \neq p+1 \text{ then } xf \text{ else } (n+1)f' \text{ for } x \in [n].$$

4. Feedback theories. Our basic algebraic structure is defined as follows.

4.1. A biflow $(T,+,\circ,\uparrow)$ is an extension of $(Bi,+,\circ,\uparrow;I_m, m \leftrightarrow n)$ such that:

(4.1.1) $(T,+,I_0)$ is a monoid;

(4.1.2) Block permutation axiom: for $f_i \in T(m_i,n_i)$, $i \in [2]$

$$(f_1+f_2)\circ(n_1 \leftrightarrow n_2) = (m_1 \leftrightarrow m_2)\circ(f_2+f_1);$$

(4.1.3) $(T,\circ,I_m)$ is a category, having the same objects as Bi;

(4.1.4) Composition and sum are related by: for $f_i \in T(m_i,n_i)$, $g_i \in T(n_i,p_i)$, $i \in [2]$

$$(f_1+f_2)\circ(g_1+g_2) = (f_1\circ g_1) + (f_2\circ g_2);$$

(4.1.5) Feedback is context free:

(4.1.5.1) $f + g\uparrow^p = (f+g)\uparrow^p;$

(4.1.5.2) for $f \in T(m+p, n+p)$, $g \in T(m',n')$

$$f \uparrow^p + g = ((I_m + m' \leftrightarrow p) \circ (f + g) \circ (I_n + p \leftrightarrow n')) \uparrow^p;$$

(4.1.5.3) $f \uparrow^p \circ g = (f \circ (g+I_p)) \uparrow^p;$

(4.1.5.4) $f \circ g \uparrow^p = ((f+I_p) \circ g) \uparrow^p;$

(4.1.6) Shifting block on feedback: for $f \in T(m+p,n+q)$, $g \in T(q,p)$

$$(f \circ (I_n+g)) \uparrow^p = ((I_m+g) \circ f) \uparrow^q.$$

4.2. Remark. (i) According to [4], this should be called a (scalar) feedback flow theory over Bi.

(ii) The axioms are not independent. In fact, (4.1.5.2) follows from (4.1.5.1) using (4.1.2), (4.1.5.3), (4.1.5.4).

(iii) For $f \in T(m,n)$, $g \in T(n,p)$ we have

(4.2.1) $f \circ g = (\quad f \circ (n \leftrightarrow n) \uparrow^n \circ g =) \quad ((f+g) \circ (n \leftrightarrow p)) \uparrow^n.$

4.3. EXAMPLES. Bi with the operations defined in 3 is a biflow. All iteration theories cf. [1,2], strong iteration theories cf. [5] and theories with iterate cf. [3], naturally are biflows (as feedback we take: $(1_m+0_p)(f(1_n+0_m+1_p))^\dagger$, for $f \in T(m+p,n+p)$ — † is the right iteration).

4.4. In practice, it is usful to have a simpler characterization of this algebraic structure. Such a simplification can be obtained. using only composition with morphisms in Bi and taking (4.2.1) as a definition for general composition. More precisely, if T endowed with sum, left and right composition with morphisms in Bi, and feedback, extends Bi and fulfils:

- (4.1.1), (4.1.2), (4.1.5.1);

- $(T, \diamond, I_m)$ is a bimodul over $\underline{Bi}$, i.e.

$$f \diamond I_n = I_m \diamond f = f, \quad \text{for} \quad f \in T(m,n), \quad \text{and}$$

$$f \diamond (g \diamond h) = (f \diamond g) \diamond h, \quad \text{whenever two morphisms are in } \underline{Bi};$$

- (4.1.4), (4.1.5.3), (4.1.5.4), (4.1.6) whenever the gs morphisms

are in $\underline{Bi}$;

then T is a biflow (composition being extended using (4.2.1)).

4.5. The category $\underline{BFl}$ has as objects biflows and as morphisms

functors which preserve morphisms in $\underline{Bi}$, sum and feedback.

5. Abstract theories of flowchart schemes. Such a theory is

given by:

- a double indexed set X of variables for atomic flowchart schemes

(that is, every $x \in X$ has a number of entries $^{\circ}x$ and a number of

exits $_{\bullet}x$ — another way to specify this is $x \in X(^{\circ}x, _{\bullet}x)$);

- a 'support theory' T consisting of a family of sets $T(m,n)$, $m,n \in \mathbb{N}$

(an element $c \in T(m,n)$ is considered as a known computation process

with m entries and n exits).

Note. The type of T corresponds to the type of flowchart

schemes we consider. While in the case of deterministic flowcharts

the basic support theory is $\underline{Pfn}$ given by $\underline{Pfn}(m,n) = $ 'the set

of all partial functions from $[m]$ to $[n]$,' in the nondeterministic

case this is $\underline{Rel}$ given by $\underline{Rel}(m,n) = $ 'the set of all relations

included in $[m] \times [n]$.' Actually, Elgot & Shepherdson [4] use $\underline{Sur}$,

the subtheory of all surjective functions in $\underline{Pfn}$. In this paper we

use $\underline{Bi}$. All these theories model only redirecting flow of control.

Note that more complicated theories can also be used, e.g. $\underline{Pfn}_D$

given by $\underline{Pfn}_D(m,n) = $ 'the set of all partial functions from $D \times [m]$ to $D \times [n]$' in which a change of memory state $d \in D$ may accompany redirecting flow of control. $\square$

A flowchart schemes is abstracted to an $\underline{X\text{-flownomial over } T}$ defined as an expression

$$((I_m + \Sigma x_i) \circ c) \uparrow^{\Sigma^\circ x_i}$$

where $x_i \in X$, the sum is finite and $c \in T(m + \Sigma_\circ x_i, n + \Sigma^\circ x_i)$; denote by $\underline{Fl}_{X,T}(m,n)$ their set.

The $\underline{\text{interpretation}}$ of an X-flownomial over T in a structure Q in which I_m, sum, composition and feedback have sense, is specified by a rank-preserving function $\varphi_X : X \longrightarrow Q$ (i.e. $\varphi_X(x) \in Q(^\circ x, _\circ x)$) and a 'morphism' $\varphi_T : T \longrightarrow Q$ (i.e. it preserves I_m and operations); it is

$$\varphi^{\#}(((I_m + \Sigma x_i) \circ c) \uparrow^{\Sigma^\circ x_i}) = ((I_m + \Sigma \varphi_X(x_i)) \circ \varphi_T(c)) \uparrow^{\Sigma^\circ x_i}.$$

T and X can naturally be embedded in $\underline{Fl}_{X,T}$ as follows:

$$c = ((I_m) \circ c) \uparrow^0 \quad \text{and} \quad x = ((I_{\circ x} + x) \circ (^\circ x \leftrightarrow _\circ x)) \uparrow^{^\circ x}$$

(the latter follows by (4.2.1)).

In the sequal we shall frequently write x, $^\circ x$, $_\circ x$ instead of Σx_i, $\Sigma^\circ x_i$, $\Sigma_\circ x_i$, respectively.

6. Operations on flownomials. Sum, composition and feedback in T extend themselves to X-flownomials over T:

(i) for $F = ((I_m + \underline{\underline{x}}) \circ c) \uparrow {}^{\cdot}\underline{\underline{x}} : m \longrightarrow n$ and $F' = ((I_{m'} + \underline{\underline{x}}') \circ c') \uparrow {}^{\cdot}\underline{\underline{x}}' : m' \longrightarrow n'$ we define $F + F' : m+m' \longrightarrow n+n'$ as

$$F + F' = ((I_{m+m'} + \underline{\underline{x}} + \underline{\underline{x}}') \circ (I_m + m' \leftrightarrow {}_{\cdot}\underline{\underline{x}} + I_{\circ\underline{\underline{x}}'}) \circ (c+c') \circ (I_n + {}^{\cdot}\underline{\underline{x}} \leftrightarrow n' + I_{\circ\underline{\underline{x}}'})) \uparrow {}^{\cdot}\underline{\underline{x}} + {}^{\cdot}\underline{\underline{x}}'.$$

(ii) for $F = ((I_m + \underline{\underline{x}}) \circ c) \uparrow {}^{\cdot}\underline{\underline{x}} : m \longrightarrow n,\ f \in T(p,m),$ and $g \in T(n,q)$ we define

$$F \circ g = ((I_m + \underline{\underline{x}}) \circ c \circ (g + I_{\cdot\underline{\underline{x}}})) \uparrow {}^{\cdot}\underline{\underline{x}}\ , \quad \text{and}$$

$$f \circ F = ((I_p + \underline{\underline{x}}) \circ (f + I_{\cdot\underline{\underline{x}}}) \circ c) \uparrow {}^{\cdot}\underline{\underline{x}}\ .$$

(iii) for $F = ((I_{m+1} + \underline{\underline{x}}) \circ c) \uparrow {}^{\cdot}\underline{\underline{x}} : m+1 \longrightarrow n+1$ we define $F \uparrow : m \longrightarrow n$ as

$$F \uparrow = ((I_m + \underline{\underline{x}}) \circ ((I_m + \underline{\underline{x}} \leftrightarrow 1) \circ c \circ (I_n + 1 \leftrightarrow {}^{\cdot}\underline{\underline{x}})) \uparrow) \uparrow {}^{\cdot}\underline{\underline{x}}.$$

Suppose T is a biflow. From these basic operations we derive the general feedback and composition, namely for $F = ((I_{m+k} + \underline{\underline{x}}) \circ c) \uparrow {}^{\cdot}\underline{\underline{x}} : m+k \longrightarrow n+k$ the flownomial $F \uparrow^k : m \longrightarrow n$ is

$$F \uparrow^k = ((I_n + \underline{\underline{x}}) \circ ((I_m + {}_{\cdot}\underline{\underline{x}} \leftrightarrow k) \circ c \circ (I_n + k \leftrightarrow {}_{\cdot}\underline{\underline{x}})) \uparrow^k) \uparrow {}^{\cdot}\underline{\underline{x}}$$

and for $F = ((I_m + \underline{\underline{x}}) \circ c) \uparrow {}^{\cdot}\underline{\underline{x}} : m \longrightarrow n,\ F' = ((I_n + \underline{\underline{x}}') \circ c') \uparrow {}^{\cdot}\underline{\underline{x}}' : n \longrightarrow p$ the flownomial $F \circ F' : m \longrightarrow p$ is

$$F \circ F' = ((I_m + \underline{\underline{x}} + \underline{\underline{x}}') \circ (c + I_{\cdot\underline{\underline{x}}'}) \circ (I_n + {}^{\cdot}\underline{\underline{x}} \leftrightarrow {}_{\cdot}\underline{\underline{x}}') \circ (c' + I_{\cdot\underline{\underline{x}}}) \circ (I_p + {}^{\cdot}\underline{\underline{x}}' \leftrightarrow {}^{\cdot}\underline{\underline{x}})) \uparrow {}^{\cdot}\underline{\underline{x}} + {}^{\cdot}\underline{\underline{x}}'.$$

Remark that all these formulae are rules of computation in a biflow, namely their instances obtained by replacing $\underline{\underline{x}}$s with elements in T are identities in T. This gives a half of the main theorem, i.e.

UNIQUE EXTENSION LEMMA. For every morphism $\varphi_T : T \longrightarrow Q$ in BF1 and every rank-preserving function $\varphi_X : X \longrightarrow Q$ the extension $\varphi^\# : \underline{\underline{Fl}}_{X,T} \longrightarrow Q$ preserves the operations. Moreover, this is the unique extension of (φ_X, φ_T) with respect to this property. $\square$

$$T \longrightarrow \underline{\underline{Fl}}_{X,T} \longleftarrow X$$

with φ_T, $\varphi^\#$, φ_X mapping to Q.

7. Isomorphic flownomials. Given $(x_1, \ldots, x_k)$, $(x'_1, \ldots, x'_{k'})$ a function $y : [k] \longrightarrow [k']$ such that $x_i = x'_{iy}$, $\forall i \in [k]$ has a unique 'block extension' to entries $\cdot y : [{}^\circ x_1 + \ldots + {}^\circ x_k] \longrightarrow [{}^\circ x'_1 + \ldots + {}^\circ x'_{k'}]$ and a unique block extension to exits $\cdot y : [{}_\circ x_1 + \ldots + {}_\circ x_k] \longrightarrow [{}_\circ x'_1 + \ldots + {}_\circ x'_{k'}]$ (see [4] for more details).

We say two flownomials $F = ((I_m + x_1 + \ldots + x_k) \circ c) \uparrow^{\sum {}^\circ x_i} : m \longrightarrow n$ and $F' = ((I_m + x'_1 + \ldots + x'_k) \circ c') \uparrow^{\sum {}^\circ x'_i} : m \longrightarrow n$ are isomorphic if there is a bijection $y : [k] \longrightarrow [k]$ such that

(i) $x_i = x'_{iy}$, $\forall i \in [k]$;

(ii) $c \circ (I_n + {}^\circ y) = (I_m + {}_\circ y) \circ c'$.

The isomorphism relation $\approx$ is a congruence relation, hence the operations are well defined in the quotient structure $\underline{\underline{Fl}}_{X,T}/\approx$. On the other hand, two isomorphic flownomials have the same interpretation in a biflow, hence the interpretation $\varphi^\# : \underline{\underline{Fl}}_{X,T} \longrightarrow Q$ induces one $\varphi^\wedge : \underline{\underline{Fl}}_{X,T}/\approx \longrightarrow Q$ on isomorphism classes of flownomials.

8. The algebraic structure of $Fl_{X,T}/\approx$. Suppose T is a biflow. A simple computation shows that $(\underline{\underline{Fl}}_{X,T}, +, I_0)$ is a monoid, $(\underline{\underline{Fl}}_{X,T}, \circ, I_m)$ is a bimodul over $\underline{Bi}$ and $(4.1.5.1)$ holds in $\underline{\underline{Fl}}_{X,T}$.

The identities (4.1.4), (4.1.5.3), (4.1.5.4), and (4.1.6) hold, whenever the g_5 morphisms are in T. In addition, the two sides in (4.1.2) give isomorphic flownomials. By 4.4 these give the other half of the main result, i.e.

STRUCTURE PRESERVING LEMMA. If T is a biflow, then $\underline{\underline{Fl}}_{X,T}/\approx$ is a biflow.

9. The main result.

This shows why we have asserted that this calculus is similar to that of polynomials. It follows from the above lemmas and the last sentence in 7.

THEOREM. If T is a biflow, then $\underline{\underline{Fl}}_{X,T}/\approx$ is the coproduct of T and the biflow freely generated by X in $\underline{\underline{BFl}}$. □
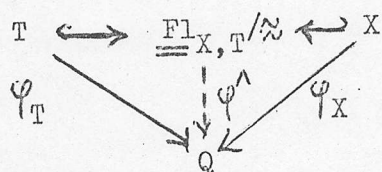
$$T \hookrightarrow \underline{\underline{Fl}}_{X,T}/\approx \hookleftarrow X$$

This means that $\underline{\underline{Fl}}_{X,T}$ is the biflow freely generated by adding X to T.

10. Extensions.

The class of biflows T which extends $\underline{\underline{Fn}}$ and fulfils

(10.1) $\quad 0_m f = f$, for $f \in T(m,n)$

(10.2) $\quad (m \vee m) \circ f = (f+f)(n \vee n)$, for $f \in T(m,n)$

where 0_m is the unique function in $\underline{\underline{Fn}}(0,m)$ and $m \vee m$ is the function in $\underline{\underline{Fn}}(m+m,m)$ given by $x(m \vee m) =$ 'if $x \in [m]$ then x else $x-m$,' for $x \in [m+m]$ equals the class of algebraic theories with iterate in [3].

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