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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS

No. 26/1986

BUCURESTI

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April 1986

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ON THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF CERTAIN SINGULAR PLANE CURVES

by András NÉMETHI

Let C be a complex algebraic curve in the projective space $\mathbb{P}^2$. The purpose of this paper is to calculate the fundamental group G of the complement of C in the case when $C = X \cap H_1 \cap \dots \cap H_{n-2}$ where

$X = \{ [z_o : \dots : z_n] \in \mathbb{P}^n : f_1(z_o, \dots, z_m) + f_2(z_{m+1}, \dots, z_n) = 0 \}$

and H_i are generic hyperplanes $(i=1,\dots,n-2)$.

This family of curves contains the Zariski's example [11] and the curves studied by Oka [6].

In the end we consider some examples with G abelian, but C with "bad" singularities.

1§. Some general results

1.1. Suppose $f(z_o, \dots, z_n)$ is a complex homogeneous polynomial of degree d. Let X be the algebraic hypersurfaces defined by f in the complex projective space $\mathbb{P}^n$ and denote the affine hypersurface $f^{-1}(1) \subset \mathbb{C}^{n+1}$ by F.

(1.1.1) The following statements are equivalent :

(a) F is not connected

(b) $f = g^r$ $(r > 1)$

Proof. The part (b) $\Rightarrow$ (a) is immediate. The part (a) $\Rightarrow$ (b) is the consequence of the following algebraical result [Serge Lang : Algebra , p.221]:

Let k be a field and d an integer ≥ 2. Let $a \in k$, $a \neq 0$. Assume that for all prime numbers p such that $p|d$ we have $a \notin k^p$, and if $4|d$ then $a \notin -4k^4$. Then $X^d - a$ is irreducible in $k[X]$.

Let k be the quotient field of the polynomial ring

$R = \mathbb{C}[z_o,\dots,z_n]$. Because F is not connected, the polynomial $z_{n+1}^d - f(z_o,\dots,z_n)$ is not irreducible in $k[z_{n+1}]$, hence there exist polynomials $Q, P \in R$, such that $Q^r \cdot f = P^r$ $(r>1)$. Since the ring R is unique factorization domain, we obtain (b).

(1.1.2) As a consequence of (1.1.1) we obtain, that F has r connected components if and only if $f = g^r$, and g is not a power of polynomial : $g \neq h^s$ with $s>1$.

Because $\pi_1(\mathbb{P}^n - Z(g)) = \pi_1(\mathbb{P}^n - Z(g^r))$ we can assume that f is not a power of polynomial. Therefore F is connected.

1.2. By Milnor [4] the affine hypersurface F is diffeomorphic to the Milnor fiber associated to f. If we take the monodromy $M: F \longrightarrow F$ associated to the Milnor fibration : $M(z) = \exp(2\pi i/d)\cdot z$, then $\mathbb{Z}_d$ acts on F via M. The map $p : F \longrightarrow \mathbb{P}^n - X$ defined by $p(z) = [z]$ is a cyclic d-fold cover and M generates the group of covering transformations. We obtain a short exact sequence :

(1.2.1) $$1 \longrightarrow \pi_1(F, z_o) \xrightarrow{p_*} \pi_1(\mathbb{P}^n - X, [z_o]) \xrightarrow{pr} \mathbb{Z}_d \longrightarrow 1$$

(where z_o is a fixed point on the fiber F)
(This short exact sequence was obtained and studied by Oka [7] and Randell [9] in the case when f is a square-free complex homogeneous polynomial).

Hence $G = \pi_1(\mathbb{P}^n - X, [z_o])$ is an extension of $H = \pi_1(F, z_o)$ by $\mathbb{Z}_d$.

1.3. By the theory of extensions of groups, the extension G is uniquely determined by the group H and the factor set associated with the extension G. (see for example: M. Suzuki [1o, § 7])

Let $g \in G$ such that $pr(g) = \hat{1}$ (the generator of the group $\mathbb{Z}_d$). Then the conjugation by g induces an automorphism T of H. Because $pr(g^d) = \hat{0}$, we obtain that $h_o = g^d \in H$.

(1.3.1) G is uniquely determined by the automorphism $T : H \longrightarrow H$ and the distinguished element $h_o \in H$.

The proof is easy.

Assume that H is defined by the generators $(g_i)_{i\in I}$ and the relations $(r_j)_{j\in J}$: $H = \langle (g_i)_{i\in I} \mid (r_j)_{j\in J} \rangle$

We define a new group by

$$(H,T,h_o,d) = \langle (g_i)_{i\in I}, \omega \mid (r_j)_{j\in J}, (T(g_i)\omega^{-1}g_i^{-1}\omega)_{i\in I}, h_o^{-1}\omega^d \rangle$$

(1.3.2) $\quad G = (H,T,h_o,d)$

Proof. (H,T,h_o,d) is an extension of H by $\mathbb{Z}_d, pr(\omega) = \hat{1}$ and ω determines exactly the automorphism T and the element h_o.

In particular (H,T,h_o,d) does not depend on the choice of g.

In the following we describe T and h_o by the monodromy M.

1.4. Let S be a topological space and let $N : S \longrightarrow S$ be a homeomorphism with period p. Let $s_o \in S$, $s_1 = N(s_o)$ and let $\alpha : [0,1] \longrightarrow S$ be a path with $\alpha(i) = s_i$ for $i=0,1$. We define the automorphism $T_{N,\alpha} : \pi_1(S,s_o) \longrightarrow \pi_1(S,s_o)$ by $T_{N,\alpha}([\beta]) = [\alpha^{-1}\circ N(\beta)\circ\alpha]$ where

$$(\alpha^{-1}\circ N(\beta)\circ\alpha)(t) = \begin{cases} \alpha(3t) & t\in[0,1/3] \\ N(\beta(3t-1)) & t\in[1/3,2/3] \\ \alpha(3-3t) & t\in[2/3,1] \end{cases}$$

For any positive integer d with $p|d$ we define a distinguished element $h_{N,\alpha} \in \pi_1(S,s_o)$ by

$h_{N,\alpha} = [N^{d-1}(\alpha)\circ \ldots \circ N(\alpha)\circ\alpha]$ where

$(N^{d-1}(\alpha)\circ \ldots \circ\alpha)(t) = N^k(\alpha(dt-k))$ if $t\in[k/d,(k+1)/d]$.

(1.4.1) The group $(\pi_1(S,s_o), T_{N,\alpha}, h_{N,\alpha}, d)$ does not depend on the choice of the path α.

Proof. Let α' an another path with $\alpha'(i) = s_i$ for $i=0,1$. Then $g_o = [\alpha'^{-1}\circ\alpha]$ is an element of the group $\pi_1(S,s_o)$. We define the isomorphism

$\Phi : (\pi_1(S,s_o), T_{N,\alpha}, h_{N,\alpha}, d) \longrightarrow (\pi_1(S;s_o), T_{N,\alpha'}, h_{N,\alpha'}, d)$
by $\Phi(g_i) = g_i$ $(i \in I)$, and $\Phi(\omega) = \omega' \cdot g_o$. The computations are standards and there are not given here.

Hence the group $(\pi_1(S,s_o), T_{N,\alpha}, h_{N,\alpha}, d)$ depend omly on the space S, the homeomorphism N and the enteger d.

1.5. The relation between the algebraic and the topological description is the following

<u>(1.5.1) Theorem</u>

$$\pi_1(\mathbb{P}^n - X, [z_o]) = (\pi_1(F,z_o), T_{M,\alpha}, h_{M,\alpha}, d)$$

Proof. We take $g = [p(\alpha)]$ and use that $p \circ M = p$.

Therefore $\pi_1(\mathbb{P}^n - X, [z_o])$ is completly determined by the fiber F and the action $M : F \longrightarrow F$.

(1.5.2) By some calculations we obtain the following consequences

(a) $D(H) \triangleleft D(G) \triangleleft H \triangleleft G$.

($D(H)$ and $D(G)$ are the commutator subgroups of H and G, and $\triangleleft$ denotes a normal subgroup).

(b) $G/D(G) = \mathbb{Z}^k/(d_1, \dots, d_k)$

(where X has k irreducible components with degree d_i)

$$D(G)/D(H) = \text{im}(M_* - 1)$$

$$H/D(G) = \text{coker}(M_* - 1)$$

(where $M_* : H_1(F,\mathbb{Z}) \longrightarrow H_1(F,\mathbb{Z})$ is induced by M.)

(c)

$$G \text{ abelian} \Leftrightarrow \begin{Bmatrix} H \text{ abelian} \\ T = 1 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} H \text{ abelian} \\ M_* = 1 \end{Bmatrix}$$

(d) G finite $\Leftrightarrow$ H finite

(e) If X is irreducible $(k=1)$, then

$$D(G) = H$$

$$Z(G) \cap D(G) = Z(H) \cap \{h \in H : T(h) = h\}.$$

($Z(G)$ and $Z(H)$ are the centers of G and H)

1.6. Examples:

(a) Suppose that $\dim \mathrm{Sing}(X) \leq \dim X - 2$ (in the case of curves : $\mathrm{Sing} X = \emptyset$). Then by Kato and Matsumoto [3] F is 1-connected. Therefore $G = \mathbb{Z}_d$.

(b) Suppose that $n=2$ and the curve X has only nodes as singularities. Then by Oka [7] the rational monodromy $M_* \otimes 1_{\mathbb{Q}}$ is the identity , by Randell [9] $M_*=1$ and by Fulton-Deligne [1] G is abelian.

2§. The group G[s,t,d]

2.1. Let Ω_a denote the set consisting of all a-th roots of unity , and we define the transformation $M_a : \Omega_a \longrightarrow \Omega_a$ by $M_a(z) = \exp(2\pi i/a) \cdot z$. We consider the join space $\Omega_s * \Omega_t$, the join transformation $M_s * M_t : \Omega_s * \Omega_t \longrightarrow \Omega_s * \Omega_t$ and a positive integer d such that $s|d$ and $t|d$. Then we define the group $G[s,t,d]$ by :

$$G[s,t,d] := (\pi_1(\Omega_s * \Omega_t, z_o), T_{M_s * M_t, \alpha}, h_{M_s * M_t, \alpha}, d)$$

It is easy to see, that $\pi_1(\Omega_s * \Omega_t, z_o)$ is the free group of rank $(s-1)(t-1)$. The computation of the relations of the group $G[s,t,d]$ is simple but tedions and are not given here.

2.2. Examples

(a) $G[1,t,d] \approx \mathbb{Z}_d$

(b) $G[2,2,d] \approx (\mathbb{Z}, \mathrm{id}, d/2, d) \approx \mathbb{Z} \times \mathbb{Z}_{d/2}$

(c) $G[2,3,6] \approx \mathbb{Z}_2 * \mathbb{Z}_3 := \langle g_1, g_2 \mid g_1^2 = g_2^3 = 1 \rangle$

3§. Our special case

3.1. Suppose that $f(z_o,\ldots,z_n) = f_1(z_o,\ldots,z_m) + f_2(z_{m+1},\ldots,z_n)$ $(0<m<n)$. Let $F_1 = f_1^{-1}(1) \subset \mathbb{C}^{m+1}$ and $F_2 = f_2^{-1}(1) \subset \mathbb{C}^{n-m}$. Then by Oka [8] and Némethi [5] $F = F_1 * F_2$ (the join of F_1 and F_2). By an analysis of the fundamental group of join space we obtain that $\pi_1(F, z_o)$ is a free group of rank $(p_1-1)(p_2-1)$ where

p_i is the number of connected components of F_i $(i=1,2)$.

We can identify the set of connected components of F_i with Ω_{p_i}.

By (1.1) the action induced by the monodromy M is exactly the action M_{p_i}.

Because $\pi_1(F,z_o) = \pi_1(\Omega_{p_1} * \Omega_{p_2}, z_o)$ and the automorphism $T_{M,\alpha}$ and the element $h_{M,\alpha}$ correspond exactly to $T_{M_{p_1} * M_{p_2},\alpha}$ and $h_{M_{p_1} * M_{p_2},\alpha}$ we obtain that

$$\pi_1(\mathbb{P}^n - X, [z_o]) = G[p_1, p_2, d].$$

In particular $\pi_1(\mathbb{P}^n - X, [z_o])$ depend only on the numbers p_1, p_2 and d.

<u>(3.1.1) MAIN THEOREM</u>

Let $C \subset \mathbb{P}^2$ a curve of degree d. Suppose that $C = X \cap H_1 \cap \ldots \cap H_{n-2}$ where

$$X = \left\{ [z_o : \ldots : z_n] \in \mathbb{P}^n : f_1^{p_1}(z_o, \ldots, z_m) + f_2^{p_2}(z_{m+1}, \ldots z_n) = 0 \right\}, \quad (0<m<n)$$

H_i are generic hyperplanes, and f_i are not power of polynomials: $f_i \neq g_i^{q_i}$, $q_i > 1$ $(i=0,1)$. Then

$$\pi_1(\mathbb{P}^2 - C) = G[p_1, p_2, d]$$

For the proof we use our results and the following theorem of Hamm and Lê [2] :

Let X be a hypersurface in $\mathbb{P}^n$ and H a generic hyperplane. Then $(\mathbb{P}^n - X, H - X)$ is $(n-1)$ - connected.

3.2. Examples

(a) An example of Zariski : $C = \left\{ (y^3+z^3)^2 + (x^2+z^2)^3 = 0 \right\}$

If we take $f = (y^3+z^3)^2 + (x^2+w^2)^3$ and $H = \{z=w\}$ we obtain $\pi_1(\mathbb{P}^2 - C) = G[2,3,6]$.

(H intersect X-SingX and SingX transversally)

(b) The Oka's examples: $C=X\cap H$ where

$$X: \prod_{j=1}^{c}(y-\beta_j z)^{\nu_j} - \prod_{i=1}^{e}(x-\alpha_i w)^{\lambda_i} = 0$$

Let $\nu=(\nu_1,\ldots,\nu_c)$, $\lambda=(\lambda_1,\ldots,\lambda_e)$ and $d=\sum_{j=1}^{c}\nu_j=\sum_{i=1}^{e}\lambda_i$

Then $\pi_1(\mathbb{P}^2-C)=G[\nu,\lambda,d]$.

(c) If in (3.2.b) $\nu=1$ then $\pi_1(\mathbb{P}^2-C)=\mathbb{Z}_d$, but the singularities of the curve C can be very "bad" : for all pair (ν_j,λ_i) with $\nu_j \geqslant 2$, $\lambda_i \geqslant 2$ C has a singular point S_{ij} given in local analytic coordinates by the equation $y^{\nu_j}-x^{\lambda_i}=0$. Hence $\mathrm{Sing}C=\bigcup S_{ij}$

In particular, if we take $X: yz^{d-1}-xw^{d-1}=0$, then C has only one singular point with local equation $z^{d-1}-w^{d-1}=0$.

(d) Not all curves can be obtained by our method : Let C be the 3-cuspidal quartic. Then $\pi_1(\mathbb{P}^2-C)$ is finit of order 12 [11], $\pi_1(F)=\mathbb{Z}_3$ and hence $\pi_1(F)$ is not free as in our case.

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