INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

CHARACTERIZATION OF REGULAR RIESZ SPACES USING ORU-COMPACT OPERATORS

by

Dan Tudor VUZA

PREPRINT SERIES IN MATHEMATICS

No.28/1986

BUCURESTI

CHARACTERIZATION OF REGULAR RIESZ SPACES USING ORU-COMPACT OPERATORS

by

Dan Tudor VUZA*)

May 1986

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii nr.220,79622 Bucharest,ROMANIA

Mea 23732

CHARACTERIZATION OF REGULAR RIESZ SPACES USING ORU-COMPACT OPERATORS

Dan Tudor VUZA

0. Introduction

In our paper [7] we have characterized Banach lattices with order continuous norm as being those order complete Banach lattices F with the property that for any Banach lattice E, the Riesz space $L_{oru}(E,F)$ of all oru-compact operators from E to F is a band in the Riesz space $L_r(E,F)$ of all order bounded operators from E to F; the class of oru-compact operators was introduced there as a natural enlargement of the class of finite rank operators. The purpose of the present paper is to extend this result to the case when F is no more a Banach lattice: namely, the main theorem asserts that an order complete Riesz space F has the property that $L_{oru}(E,F)$ is a band in $L_r(E,F)$ for any Riesz space E if and only if F is order separable and regular. It is known that regular Riesz spaces are especially important for the theory of integration of vector valued functions with respect to vector measures (see [2]). As every Banach lattice with order continuous norm is σ-regular and there exist regular non σ-regular Riesz spaces (see §1), our result is indeed a generalization of the result in [7].

When applied to the Riesz space of all equivalence classes of measurable functions on a σ-finite measure space, our theorem yields an extension of Schep's well-known result on kernel operators [4], as the set of kernel operators is in general a proper subset of the set of oru-compact operators (see §3).

As the establishment of the main theorem requires the proof of the fact that every super order complete regular Riesz space has the Egoroff property, a discussion of some aspects involving the Egoroff property forms the object of §2. Namely, we give a study of a class of Riesz spaces, the so called Riesz spaces satisfying the abstract Egoroff theorem. The main results in the section are the following: every σ-order complete Riesz space satisfying the abstract Egoroff theorem is weakly σ-distributive; also, assuming the continuum hypothesis to hold, every order complete Riesz space satisfying the abstract Egoroff theorem has the Egoroff property. As each σ-order complete regular Riesz space satisfies the abstract Egoroff theorem, we obtain as corollaries the facts that every σ-order complete regular Riesz space is weakly σ-distributive, every super order complete regular Riesz space has the Egoroff property and, assuming the continuum hypothesis to hold, every order complete regular Riesz space is order separable. We draw the attention on the fact that we make no use of the continuum hypothesis except for the already mentioned results which are proved only for their own interest and are not employed in the proof of the main theorem in §3; consequently, we shall not think of the phrase "super order complete regular Riesz space" as being redundant.

1. Preliminaries

Throughout the section, E and F will be Riesz spaces.

Whenever F is order complete we shall denote by $L_r(E,F)$ the Riesz space of all order bounded linear operators from E to F. The notation $E^\sim$ will be used for $L_r(E,\mathbb{R})$; $E^\times$ will be the order ideal in E consisting of order continuous functionals. $\mathcal{J}(E,F)$ (respectively $\mathcal{J}^\times(E,F)$) will denote the band in $L_r(E,F)$ generated by the operators of the form $x \longmapsto f(x)y$ with $f \in E^\sim$ (respectively $f \in E^\times$) and $y \in F$.

1_E will stand for the identity map on E.

The projection on the band generated by an element x in a σ-order complet Riesz space will be denoted by $[x]$. $C(x)$ will be the set of all components of some positive x in E (that is, the set $\{y \mid y \in E,\ y \wedge (x - y) = 0\}$.

For a double sequence (x_{nm}), the symbol $(x_{nm})_m$ will be used to emphasize that we want n fixed and m as a running index.

The symbol $\longrightarrow$ will be used to denote order convergence in E: $x_n \longrightarrow x$ if there is $(y_n) \subset E$ such that $|x_n - x| \leq y_n$ and $y_n \downarrow 0$.

A Riesz subspace F of E is called: order closed, if whenever (x_δ) is a net in F, $x \in E$ and $x_\delta \uparrow x$ it follows that $x \in F$; relatively σ-order closed, if whenever $(x_n) \subset F_+$, $x \in E$ and $x_n \downarrow x$ it follows that $x \in F$.

For every $x \in E_+$, E_x will be the order ideal generated by x; the seminorm $\| \ \|_x$ on E_x is given by

$$\|y\|_x = \inf\{\alpha \mid \alpha \in \mathbb{R}_+,\ |y| \leq \alpha x\} .$$

In case E is Archimedean, $\| \ \|_x$ is a norm.

The symbol $\xrightarrow{u}$ will be used to denote the convergence with respect to the regulator $u \in E_+$: $x_n \xrightarrow{u} x$ if $\|x_n - x\|_u \longrightarrow 0$. If we do not want to specify the regulator we write $x_n \Longrightarrow x$; by definition, this means that there is $u \in E_+$ such that $x_n \xrightarrow{u} x$. We also say that (x_n) is relatively uniformly convergent to x.

A subset M of an Archimedean Riesz space E is called relatively uniformly totally bounded if it is contained in a principal order ideal E_x and it is totally bounded for $\| \ \|_x$.

Suppose F Archimedean. A linear operator $U: E \longrightarrow F$ is called oru-compact (see [7]) if it maps order bounded subsets of E onto relatively uniformly totally bounded subsets of F. The vector space $L_{oru}(E,F)$ of all oru-compact operators from E to F is a Riesz space for the usual order relation whenever F is relatively uniformly complete; $L^\times_{oru}(E,F)$ will denote the order ideal in $L_{oru}(E,F)$ formed by those U for which $|U|$ is order continuous.

E is called <u>order separable</u> if whenever $M \subset E$, $x \in E$ and $M \uparrow x$ there is an at most countable subset $N \subset M$ such that $N \uparrow x$. By theorem 29.3 in [3], an Archimedean Riesz space is order separable iff every order bounded subset of nonzero pairwise disjoint elements is at most countable. It is customary to call super order complete an order complete order separable Riesz space.

E is called weakly σ-distributive (see [8]) if it is σ-order complete and if for every order bounded double sequence $(x_{nk}) \subset E$ such that $(x_{nk})_k$ is decreasing for each $n \geq 0$ we have

$$\sup_{n\geq 0} \inf_{k\geq 0} x_{nk} = \inf_{\varphi : \mathbb{N} \to \mathbb{N}} \sup_{n \geq 0} x_{n,\varphi(n)}$$

Wright's criterion (lemma L in) shows that in order to prove that a σ-order complete Riesz space is σ-distributive, it suffices to show that

$$\inf_{\varphi : \mathbb{N} \to \mathbb{N}} \sup_{n\geq 0} x_{n,\varphi(n)} = 0$$

whenever $x \in E_+$, $(x_{nk}) \subset C(x)$ and $(x_{nk})_k \downarrow 0$ for each n.

E has the Egoroff property (see [3]) if for every order bounded double sequence $(x_{nk}) \subset E$ such that $(x_{nk})_k \downarrow 0$ for each $n \geq 0$, there is $(x_n) \subset E$ and $\varphi : \mathbb{N} \to \mathbb{N}$ such that $x_{n,\varphi(n)} \leq x_n$ for $n \geq 0$ and $x_n \downarrow 0$.

We mention the following connection between the above properties:

PROPOSITION 1.1 [5]. Every super order complete weakly σ-distributive Riesz space has the Egoroff property.

Finally, E is called regular (see [2]) if $x_n \Rightarrow 0$ whenever $(x_n) \subset E$ and $x_n \to 0$. Regular spaces are called in [3] Riesz spaces with stable order convergence. It is well known that for an order complete Banach lattice, regularity is equivalent to the order continuity of the norm; in fact, order complete regular Banach lattices are already σ-regular, that is, for every double sequence (x_{nk}) such that $(x_{nk})_k \to 0$ for each n, there is $u \geq 0$ such that $(x_{nk})_k \xrightarrow{u} 0$ for each n (see [2]). The space of all real sequences (x_n) such that $x_n = 0$ for all but a finite number of choices of n, is an well known example of regular non σ-regular Riesz space; however, it has no weak order unit. An example of regular non σ-regular Rieaz space with a weak order unit and without atoms is provided by the order ideal in the space of all equivalence classes of Lebesgue measurable functions on $[0,1]$ formed by the functions x with the property that there is $a \in (0,1)$ (depending on x) such that $\int_a^1 |x(t)| \, dt < \infty$.

For every set X, $l_\infty(X)$ will be the Riesz space of all bounded real functions on X; for every compact space X, C(X) will be the Riesz space of all continuous real functions on X.

2. Riesz spaces satisfying the abstract Egoroff theorem

We say that a σ-order complete Riesz space E satisfies the abstract Egoroff theorem if whenever $(x_n) \subset E_+$, $e \in E_+$ and $x_n \downarrow 0$ there is $(e_m) \subset E_+$ such that $e_m \uparrow e$ and $([e_m]x_n)_n \xrightarrow{e} 0$ for each $m \geq 0$.

By theorems 74.3 and 74.5 in [3], every σ-order complete Riesz space with the Egoroff property satisfies the abstract Egoroff theorem.

PROPOSITION 2.1. Every relatively σ-order closed Riesz subspace of a σ-order complete Riesz space satisfying the abstract Egoroff theorem also satisfies it.

PROOF. Let E be a σ-order complete Riesz space satisfying the abstract Egoroff theorem and let F be a relatively σ-order closed Riesz subspace of E. Let $(x_n) \subset F_+$ and $e \in F_+$ be such that $x_n \downarrow 0$. By the hypothesis there is $(e_m) \subset E_+$ such that $([e_m]x_n)_n \xrightarrow{e} 0$ for each m. We shall prove that to every m there corresponds $f_m \in F$ such that $e_m \leq f_m \leq e$ and $([f_m]x_n)_n \xrightarrow{e} 0$; this will conclude the proof, since if

$$e'_m = \bigvee_{i=0}^{m} f_i ,$$

then $e'_m \in F_+$, $e'_m \uparrow e$ and $([e'_m]x_n)_n \xrightarrow{e} 0$ for each m (remark that $[e'_m]x_n$ equals the order projection of x_n on e'_m computed in F).

So let $m \geq 0$ be given; as $([e_m]x_n)_n \xrightarrow{e} 0$ there is a sequence (n_k) such that $[e_m]x_{n_k} \leq k^{-1}e$ for any $k \geq 2$. Put $f_{mk} = [((k-1)^{-1}e - x_{n_k})_+]e$; as F is relatively σ-order closed we have $f_{mk} \in F$. Let $\alpha_k = (k-1)^{-1} - k^{-1}$. From the relation

$$((k-1)^{-1}e - x_{n_k})_+ \geq (k-1)^{-1}e - x_{n_k} = \alpha_k e + k^{-1}e - x_{n_k} \geq$$
$$\geq \alpha_k e - (k^{-1}e - x_{n_k})_-$$

we obtain

$$\alpha_k e \leq ((k-1)^{-1}e - x_{n_k})_+ + (k^{-1}e - x_{n_k})_- .$$

Then, observing that we always have $[u+v] \leq [u] + [v]$ and $[\alpha u] = [u]$ for $\alpha > 0$, we derive from the above relation

$$[e] \leq [((k-1)^{-1}e - x_{n_k})_+] + [(k^{-1}e - x_{n_k})_-] . \tag{1}$$

Applying $[e_m]$ to both sides of the inequality $[e_m]x_{n_k} \leq k^{-1}e$ we obtain $[e_m]x_{n_k} \leq k^{-1}[e_m]e$; hence

$$[e_m](k^{-1}e - x_{n_k})_- = (k^{-1}[e_m]e - [e_m]x_{n_k})_- = 0.$$

This implies that $e_m \wedge (k^{-1}e - x_{n_k}))_- = 0$ and therefore, $[(k^{-1}e - x_{n_k})_-]e_m = 0$. Consequently, we obtain from (1)

$$e_m = [e]e_m \leq [((k-1)^{-1}e - x_{n_k})_+]e_m \leq [((k-1)^{-1}e - x_{n_k})_+]e = f_{mk} .$$

Now put

$$f_m = \bigwedge_{k=2}^{\infty} f_{mk} ;$$

as F is relatively σ-order closed, $f_m \in F$ and $e_m \leq f_m \leq e$. To prove the relation $([f_m]x_n)_n \xrightarrow{e} 0$ we note that

$$[f_m]x_{n_k} \leq [f_{mk}]x_{n_k} = [f_{mk}]((k-1)^{-1}e + (x_{n_k} - (k-1)^{-1}e)_+ -$$
$$- (x_{m_k} - (k-1)^{-1}e)_-) \leq (k-1)^{-1}e$$

as $[f_{mk}](x_{n_k} - (k-1)^{-1}e)_+ = 0$ by the definition of f_{mk}.

PROPOSITION 2.2. A σ-order complete Riesz space satisfying the abstract Egoroff theorem is weakly σ-distributive.

PROOF. If a σ-order complete Riesz space E satisfies the abstract Egoroff theorem, then all its principal order ideal also do; as E is weakly σ-distributive iff all its principal order ideals are, it suffices to consider the case when E has a strong order unit. But in this case, E is order isomorphic to a space $C(X)$ for some σ-stonean space E; hence, by Wright's criterion, it suffices to show that every σ-meagre subset of X is nowhere dense. So let

$$M = \bigcup_{n=1}^{\infty} M_n$$

where the M_n's are closed nowhere dense G_δ subsets of X. If M is not nowhere dense there is a closed-open subset $N \subset X$ such that $M \cap N$ is dense in N; replacing X by N we may assume that $N = X$. We may also assume that the M_n's are pairwise disjoint: indeed, let

$$X_n = X \setminus \bigcup_{i=1}^{n} M_i .$$

As X_n is an open F_σ subset there is a sequence $(X_{nm})_{m\geq 1}$ of pairwise disjoint closed-open subsets such that

$$X_n = \bigcup_{m=1}^{\infty} X_{nm} .$$

Put $M_{nm} = M_{n+1} \cap X_{nm}$; then the M_{nm}'s are pairwise disjoint and $\bigcup_{n,m\geq 1} M_{nm}$ is dense in X.

Let $f: X \longrightarrow \mathbb{R}$ be defined by

$$f(t) = n^{-1}, \quad t \in M_n$$
$$f(t) = 0 \quad , \quad t \in X \setminus M.$$

f is an upper semicontinuous Baire function, hence there is a sequence $(f_n) \subset C(X)$ such that $f_m(t) \downarrow f(t)$ for every $t \in X$. As the complement of a σ-meagre subset in a compact space is dense, it follows that $f_n \downarrow 0$ in $C(X)$; as $C(X)$ satisfies the abstract Egoroff theorem, there is a sequence $(e_m) \subset C(X)$ such that $e_m \uparrow e$ and $([e_m]f_n)_n \xrightarrow{e} 0$ for each m (e being the function identically one on X). Each e_m must vanish on M: indeed, if $e_m(t_0) \neq 0$ for some $t_0 \in M_{n_0}$, there are $\alpha > 0$ and a closed-open neighborhood V of t_0 such that $\alpha \chi_V \leq e_m$ (χ_V being the characteristic function of V). It follows then from the relation

$$\chi_V f_n = [\chi_V]f_n = [\alpha \chi_V]f_n \leq [e_m]f_n$$

that

$$n_0^{-1} = f(t_0) \leq f_n(t_0) = (\chi_V f_n)(t_0) \leq ([e_m]f_n)(t_0) .$$

But this contradicts the relation $([e_m]f_n)_n \xrightarrow{e} 0$.

As M is dense in X we obtain that $e_m = 0$, which contradicts the relation $e_m \uparrow e$. The contradiction so obtained completes the proof.

PROPOSITION 2.3. Assume that the continuum hypothesis holds. Then every order complete Riesz space satisfying the abstract Egoroff theorem is order separable.

PROOF. Suppose that the order complete Riesz space E satisfying the abstract Egoroff theorem is not order separable. Then there is an uncountable order bounded subset $M \subset E_+ \setminus \{0\}$ consisting of pairwise disjoint elements. Define $H: l_\infty(M) \longrightarrow E$ by

$$H(f) = \sup_{\substack{F \subset M \\ F \text{ finite}}} \sum_{t \in F} f(t)t \quad , \quad f \in l_\infty(M)_+ .$$

H is an order isomorphism of $l_\infty(M)$ onto a relatively σ-order closed Riesz subspace of E; hence, by proposition 2.1, $l_\infty(M)$ satisfies the abstract Egoroff theorem. As M is uncountable and we assume the continuum hypothesis to hold, a result of Banach and Kuratowski [1] asserts that there is a double sequence (M_{nk}) of subsets of M such that $M_{0k} = M$, $M_{nk} \uparrow M$ for each n and $\bigcap_{n=0}^{\infty} M_{n,\varphi(n)}$ is at most countable for each $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$. Replacing, if necessary, M_{nk} by $\bigcap_{i=0}^{n} M_{ik}$ we may assume that $M_{n+1,k} \subset M_{nk}$. Define $f_k \in l_\infty(M)$ by

$$f_k(t) = (n+1)^{-1}, \quad t \in M_{nk} \setminus M_{n+1,k}$$

$$f_k(t) = 0 \quad , \quad t \in \bigcap_{n=0}^{\infty} M_{nk} .$$

The monotony properties of (M_{nk}) imply $f_{k+1} \leq f_k$. As $f_k(t) \leq (n+1)^{-1}$ for $t \in M_{nk}$ and $M_{nk} \uparrow M$ for each n, we have $f_k \downarrow 0$. Consequently, there is by hypothesis $(e_m) \subset l_\infty(M)_+$ such that $e_m \uparrow e$ and $([e_m]f_k)_k \xrightarrow{e} 0$ for each m (e being the function identically one on M). If we let $N_m = \{t \mid t \in M, e_m(t) > 0\}$ then $N_m \uparrow M$ and $\sup f_k(N_m) \longrightarrow 0$ as $k \longrightarrow \infty$ for each m. We shall see that each N_m must be at most countable; this will contradict the relation $N_m \uparrow M$ and the proof will be complete. Indeed, given $m \geq 0$, we can find $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\sup f_{\varphi(n)}(N_m) < (n+1)^{-1}$ for each $n \geq 0$. By the definition of the f_k's, this implies $N_m \subset M_{n,\varphi(n)}$ for each n; consequently, $N_m \subset \bigcap_{n=0}^{\infty} M_{n,\varphi(n)}$ and therefore, it is at most countable.

COROLLARY 2.1. Assume that the continuum hypothesis holds. Then every order complete Riesz space satisfying the abstract Egoroff theorem has the Egoroff property.

PROOF. Follows from propositions 1.1, 2.2 and 2.3.

PROPOSITION 2.4. A σ-order complete Riesz space is regular iff it satisfies the abstract Egoroff theorem and every order bounded disjoint sequence in E is relatively uniformly convergent to 0.

PROOF. Let E be a σ-order complete regular Riesz space and let $(x_n) \subset E_+$, $e \in E_+$ be such that $x_n \downarrow 0$. By the hypothesis there is $u \in E_+$ such that $x_n \xrightarrow{u} 0$. Put $e_m = [(me - u)_+]e$; (e_m) is an increasing sequence in $C(e)$. If

$$e' = \bigvee_{m=0}^{\infty} e_m$$

then $(e - e') \wedge e_m = 0$ for $m \geq 0$, which implies

$$0 = (e - e') \wedge \bigvee_{m=1}^{\infty} (e - m^{-1}u)_+ = (e - e') \wedge e = e - e' .$$

Thus, $e_m \uparrow e$. The relation

$$[e_m]u = [e_m](me + (u - me)_+ - (u - me)_-) \leq me$$

implies that $([e_m]x_n)_n \xrightarrow{e} 0$ for each m; consequently, E satisfies the abstract Egoroff theorem.

On the other side, let $(x_n) \subset E_+$ be an order bounded disjoint sequence. Put

$$y_n = \bigvee_{m=n}^{\infty} x_n .$$

As $y_n \downarrow 0$ it follows by the hypothesis that $y_n \Longrightarrow 0$; consequently, $x_n \Longrightarrow 0$.

Conversely, let E satisfy the conditions in the statement of the proposition and let $(x_n)_{n\geq 0} \subset E$ be such that $x_n \downarrow 0$. As E satisfies the abstract Egoroff theorem, there is $(e_m) \subset E_+$ such that $e_m \uparrow x_0$ and

$$([e_m]x_n)_n \xrightarrow{x_0} 0$$

for each $m \geq 0$. Replacing, if necessary, e_m by $[e_m]x_0$ we may assume that $e_m \in C(x_0)$. The sequence $(e_m - e_{m-1})_{m\geq 1}$ is an order bounded disjoint sequence; therefore, there is by hypothesis $u \in E_+$ such that $e_m - e_{m-1} \xrightarrow{u} 0$. We may assume that $x_0 \leq u$. The proof will be concluded if we show that $x_n \xrightarrow{u} 0$. Indeed, given $\varepsilon > 0$, there is p such that $e_m - e_{m-1} \leq 2^{-1}\varepsilon u$ whenever $m \geq p$. Then

$$e_q - e_p = \bigvee_{i=p+1}^{q} (e_i - e_{i-1}) \leq 2^{-1}\varepsilon u$$

for any $q \geq p+1$; as $e_q \uparrow e$ we obtain from the above relation $e - e_p \leq 2^{-1}\varepsilon u$. As $[e_p]x_n \xrightarrow{x_0} 0$ there is n such that $[e_p]x_n \leq 2^{-1}\varepsilon x_0$. We have

$$x_n = [e_p]x_n + [x_0 - e_p]x_n \leq [e_p]x_n + x_0 - e_p \leq \varepsilon u,$$

which proves our assertion.

COROLLARY 2.2. Every σ-order complete regular Riesz space is weakly σ-distributive. If we assume the continuum hipothesis to hold, then every order complete regular Riesz space has the Egoroff property.

The following proposition shows in particular that super order complete regular Riesz spaces behave like σ-regular spaces with respect to order bounded double sequences.

PROPOSITION 2.5. For any super order complete Riesz space the following are equivalent:

i) E is regular.

ii) For any order bounded double sequence $(x_{nk}) \subset E$ such that $(x_{nk})_k \downarrow 0$ for each n there is $u \in E_+$ such that $(x_{nk})_k \xrightarrow{u} 0$ for each n.

iii) For any $x \in E_+$ and any double sequence $(x_{nk}) \subset C(x)$ such that each sequence $(x_{nk})_k$ is disjoint there is $u \in E_+$ such that $(x_{nk})_k \xrightarrow{u} 0$ for each n.

PROOF.

i) $\Longrightarrow$ ii) Replacing, if necessary, x_{nk} by $\bigvee_{i=0}^{n} x_{ik}$ we may assume that $x_{nk} \leq x_{n+1,k}$. By proposition 1.1 and corollary 1.2, E has the Egoroff property; hence there are $(x_n) \subset E_+$ and $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $x_n \downarrow 0$ and $x_{n,\varphi(n)} \leq x_n$. As E is regular, there is $u \in E_+$ such that $x_n \xrightarrow{u} 0$. As $x_{m,\varphi(n)} \leq x_{n,\varphi(n)} \leq x_n$ whenever $m \leq n$ it follows that $(x_{nk})_k \xrightarrow{u} 0$ for each m.

ii) $\Longrightarrow$ iii) Let

$$y_{nk} = \bigvee_{i=k}^{\infty} x_{ni} .$$

(y_{nk}) is an order bounded double sequence such that $(y_{nk})_k \downarrow 0$ for each n; consequently, there is $u \in E_+$ such that $(y_{nk})_k \xrightarrow{u} 0$ for each n. As $x_{nk} \leq y_{nk}$, the same is true for (x_{nk}).

iii) $\Longrightarrow$ i) We prove first that E is weakly σ-distributive. So let $x \in E_+$ and let $(y_{nk}) \subset C(x)$ verify $(y_{nk})_k \downarrow 0$ for each n. Put

$$y = \inf_{\varphi : \mathbb{N} \longrightarrow \mathbb{N}} \ \sup_{n \geq 0} y_{n,\varphi(n)} .$$

If $x_{nk} = y_{nk} - y_{n,k+1}$ then $x_{nk} \in C(x)$ and each sequence $(x_{nk})_k$ is disjoint. By the hypothesis there is $u \in E_+$ with the property that $(x_{nk})_k \xrightarrow{u} 0$ for each n; consequently, given $\varepsilon > 0$, there is $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $x_{nk} \leq \varepsilon u$ whenever $k \geq \varphi(n)$. As

$$y_{n,\varphi(n)} = \bigvee_{k=\varphi(n)}^{\infty} x_{nk}$$

it follows that $y_{n,\varphi(n)} \leq \varepsilon u$ for each n. Hence $y \leq \varepsilon u$ be the definition of y; as ε is arbitrary, $y = 0$.

As E is also order separable, proposition 1.1 implies that E has the Egoroff property, hence it satisfies the abstract Egoroff theorem. On the other side, if $(x_n) \subset E_+$ is an order bounded disjoint sequence and if $x = \bigvee_{n=0}^{\infty} x_n$, then $x_n \in C(x)$ and consequently, $(x_n) \Longrightarrow 0$ by the hypothesis. An application of proposition 2.4 concludes the proof.

3. The operatorial characterization

The proof of the main result will rely on two lemmas and a proposition which presents interest in its own, as it gives a non topological version of a principality theorem for modules of operators (theorem 2.5 in [6]).

LEMMA 3.1. Let E be an Archimedean Riesz space and let $M \subset E_+ \setminus \{0\}$ be a subset of pairwise disjoint elements which is totally bounded for $\| \ \|_u$. Then M is at most countable; if M is infinite and if (x_n) is any arrangement of M into a sequence, then $x_n \xrightarrow{u} 0$.

PROOF. Our assertions will follow if we prove that for every $\varepsilon > 0$, the set $\{x \mid x \in M, \|x\|_u \geq \varepsilon\}$ is finite. Indeed, there are $x_1, \ldots, x_n \in M$ (depending on ε) such that $\inf_{1 \leq i \leq n} \|x - x_i\|_u < \varepsilon$ for any $x \in M$. If $x \in M \setminus \{x_1, \ldots, x_n\}$ then $x \wedge x_i = 0$ for $1 \leq i \leq n$; consequently, the inequality

$$x \leq x + x_i = |x - x_i|$$

shows that $\|x\|_u < \varepsilon$ and the proof is complete.

LEMMA 3.2. Let E be a super order complete regular Riesz space and let (M_n) be a sequence of relatively uniformly totally bounded subsets of E such that $\bigcup_{n=0}^{\infty} M_n$ is order bounded. Then there is $u \in E_+$ such that each M_n is totally bounded for $\| \ \|_u$.

PROOF. Let $x \in E_+$ be such that $\bigcup_{n=0}^{\infty} M_n \subset [-x, x]$. For each n there is $u_n \in E_+$ such that M_n is totally bounded for $\| \ \|_{u_n}$. Put $x_{nk} = [(u_n - kx)_+]x$; (x_{nk}) is an order bounded double sequence decreasing in k for each n. In fact, $(x_{nk})_k \downarrow 0$: indeed, if

$$x_n = \bigwedge_{k=0}^{\infty} x_{nk}$$

then

$$kx_n \leq kx_{nk} = [(u_n - kx)_+]((kx - u_n)_+ - (kx - u_n)_- + u_n) \leq u_n$$

for each k, wich implies that $x_n = 0$. Consequently, there is by proposition 2.5 $u \in E_+$ such that $(x_{nk})_k \xrightarrow{u} 0$ for each n; we may assume that $x \leq u$. We shall prove that each M_n is totally bounded for $\| \ \|_u$. Indeed, let $\varepsilon > 0$ and n be given. As $(x_{nk})_k \xrightarrow{u} 0$ there is k such that $x_{nk} \leq 4^{-1}\varepsilon u$. As M_n is totally bounded for $\| \ \|_{u_n}$ there are $y_1, \ldots, y_m \in M_n$ such that any $y \in M_n$ satisfies $|y - y_i| \leq (2k)^{-1}\varepsilon u_n$ for some $i \in \{1, \ldots, m\}$. Taking into account the fact that $x_{nk} \in C(x)$ we obtain that any $y \in M_n$ satisfies

$$|y - y_i| = [x]|y - y_i| = [x - x_{nk}]|y - y_i| + [x_{nk}]|y - y_i| \leq$$

$$\leq (2k)^{-1}\varepsilon[x - x_{nk}]u_n + 2[x_{nk}]x$$

for some $i \in \{1,\ldots,m\}$. The identity $[[u]v] = [u][v]$ enables us to derive the relations

$$[x - x_{nk}]u_n = ([x] - [x_{nk}])u_n = [x](1_E - [(u_n - kx)_+])u_n \leq$$
$$\leq (1_E - [(u_n - kx)_+])u_n =$$
$$= (1_E - [(u_n - kx)_+])((u_n - kx)_+ - (u_n - kx)_- + kx) \leq kx$$

and

$$[x_{nk}]x = x_{nk} .$$

Hence

$$|y - y_i| \leq 2^{-1}\varepsilon x + 2^{-1}\varepsilon u \leq \varepsilon u$$

and the proof is complete.

Recall that an f-algebra is a Riesz space A endowed with a structure of algebra such that $A_+A_+ \subset A_+$ and $ac \wedge b = ca \wedge b = 0$ whenever $a,b,c \in A_+$ and $a \wedge b = 0$.

Any space $C(X)$ is an f-algebra for the usual algebraic and order structure

For any order complete Riesz space E we denote by $Z_p(E)$ the algebra of operators on E generated by order projections; it is well-known that $Z_p(E)$ is an f-algebra for the usual algebraic and order structures.

If X is a compact space and E is an order complete Riesz space, the cone

$$\left\{ \sum_{i=1}^{n} f_i \otimes \pi_i \mid f_i \in C(X)_+,\ \pi_i \in Z_p(E)_+ \right\}$$

defines an order relation on the tensor product algebra $C(X) \otimes Z_p(E)$ turning it into an f-algebra. To verify for instance that $C(X) \otimes Z_p(E)$ is a Riesz space, note that every $a \in C(X) \otimes Z_p(E)$ can be written as

(1) $$a = \sum_{i=1}^{n} f_i \otimes P_i$$

where $f_i \in C(X)$ and the P_i's are mutually disjoint order projections; it is then readily seen that the modulus of a is given by

(2) $$|a| = \sum_{i=1}^{n} |f_i| \otimes P_i \quad .$$

There is a unique structure of $C(X) \otimes Z_p(E)$-module on $L_r(C(X),E)$ such that $((f \otimes \pi)U)(g) = \pi U(fg)$ whenever $f,g \in C(X)$, $\pi \in Z_p(E)$ and $U \in L_r(C(X),E)$; one can verify (by using (1) and (2)) that, for the module structure so defined, we have $|aU| = |a||U|$ whenever $a \in C(X) \otimes Z_p(E)$ and $U \in L_r(C(X),E)$. From this relation we may obtain for instance that

(3) $$(a \wedge b)U = aU \wedge bU$$

whenever $a,b \in C(X) \otimes Z_p(E)$ and $U \in L_r(C(X),E)_+$.

PROPOSITION 3.1. Let X be a compact space, let E be a super order complete Riesz space with the Egoroff property and let e denote the function identically one on X. Then for any $U,V\in L_r(C(X),E)$ such that $0\leq U\leq V$ there is a sequence $(a_n)\subset C(X)\otimes Z_p(E)$ such that $|U-a_nV|(e)\longrightarrow 0$.

PROOF. We assume first that U is a component of V. Put $V_1=U$, $V_2=V-U$. As $V_1\wedge V_2=0$ we have

$$\left\{\sum_{i=0}^{n} V_1(f_i)\wedge V_2(f_i)\ \middle|\ n\geq 0,\ f_i\in C(X)_+,\ \sum_{i=0}^{n} f_i=e\right\}\downarrow 0\ .$$

As E is order separabèe, there is a double sequence $(f_{ni})_{n\geq 0,\ 0\leq i\leq k_n}\subset C(X)_+$ such that

$$\sum_{i=0}^{k_n} f_{ni}=e$$

for each n and

$$\sum_{i=0}^{k_n} V_1(f_{ni})\wedge V_2(f_{ni})\downarrow 0$$

as $n\longrightarrow\infty$. Since

$$(V_1(f_{ni})-V_2(f_{ni}))_+\wedge(V_2(f_{ni})-V_1(f_{ni}))_+=0$$

there is an order projection P_{ni} such that

$$(1_E-P_{ni})((V_1(f_{ni})-V_2(f_{ni}))_+)=P_{ni}((V_2(f_{ni})-V_1(f_{ni}))_+)=0.$$

Let

$$a_n=\sum_{i=0}^{k_n} f_{ni}\otimes P_{ni}\ .$$

The relation

$$e\otimes 1_E-a_n=\sum_{i=0}^{k_n} f_{ni}\otimes(1_E-P_{ni})$$

shows that

$$0\leq a_nV_i\leq V_i,\quad i=1,2\ .$$

We also have

$$0\leq(V_1-a_nV_1)(e)=\sum_{i=0}^{k_n}(1_E-P_{ni})V_1(f_{ni})=$$
$$=\sum_{i=0}^{k_n}(1_E-P_{ni})((V_1(f_{ni})-V_2(f_{ni}))_+ + V_1(f_{ni})\wedge V_2(f_{ni}))\leq$$
$$\leq\sum_{i=0}^{k_n} V_1(f_{ni})\wedge V_2(f_{ni})\ .$$

Consequently, $(V_1-a_nV_1)(e)\longrightarrow 0$; a similar computation shows that $(a_nV_2)(e)\longrightarrow 0$. It follows then from the relation

$$|U - a_n V|(e) = |V_1 - a_n V_1 - a_n V_2|(e) \leq (V_1 - a_n V_1)(e) + (a_n V_2)(e)$$

that $|U - a_n V|(e) \longrightarrow 0$.

For the general case, by Freudenthal's theorem there is a sequence (V_n) of finite linear combinations of components of V such that $0 \leq V_n \leq V$ and $|U - V_n| \leq n^{-1}V$. By the previous argument, there is a double sequence $(a_{nk}) \subset C(X) \otimes Z_p(E)$ such that $(|V_n - a_{nk}V|(e))_k \longrightarrow 0$ for each n. Replacing, if necessary, a_{nk} by $(a_{nk})_+ \wedge (e \otimes 1_E)$ and taking into account (3), we may assume that $0 \leq a_{nk}V \leq V$. Therefore, the double sequence $(|V_n - a_{nk}V|(e))$ is order bounded; the Egoroff property gives then a $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $|V_n - a_{n,\varphi(n)}V|(e) \longrightarrow 0$. The relation

$$|U - a_{n,\varphi(n)}V|(e) \leq n^{-1}V(e) + |V_n - a_{n,\varphi(n)}V|(e)$$

shows that $|U - a_{n,\varphi(n)}V|(e) \longrightarrow 0$ and the proof is complete.

We may state now the main result of the paper.

THEOREM 3.1. For any Riesz space E and any super order complete regular Riesz space F, $L_{oru}(E,F)$ is a band in $L_r(E,F)$.

Conversely, let F be an order complete Riesz space such that $L_{oru}(l_\infty(M),F)$ is order closed in $L_r(l_\infty(M),F)$ for any set M. Then F is super order complete and regular.

PROOF. Let F be a super order complete regular Riesz space. In order to prove that $L_{oru}(E,F)$ is a band in $L_r(E,F)$, it suffices to do it only in the case when E has a strong order unit $u \in E_+$. Let $G = \{x \mid x \in E, \|x\|_u = 0\}$; the quotient Riesz space E/G is Archimedean and has a strong order unit $v > 0$. If H denotes the completion of E/G for the norm $\| \ \|_v$, it is straightforward to verify that $L_r(E,F)$ is order isomorphic to $L_r(H,F)$ through an isomorphism taking $L_{oru}(E,F)$ onto $L_{oru}(H,F)$. As H is order isomorphic to a space C(X), it suffices finally to consider only the case $E = C(X)$ for some compact space X.

We show first that $L_{oru}(C(X),F)$ is an order ideal in $L_r(C(X),F)$; as $L_{oru}(C(X),F)$ is a Riesz subspace, it suffices to prove that the relations $U \in L_r(C(X),F)$, $V \in L_{oru}(C(X),F)$, $0 \leq U \leq V$ imply $U \in L_{oru}(C(X),F)$. Denote by e the function identically one on X. There is $x \in F_+$ such that $V([-e,e])$ is totally bounded for $\| \ \|_x$. By proposition 1.1 and corollary 2.2, F has the Egoroff property; we may then apply proposition 3.1 and obtain a sequence $(a_n) \subset C(X) \otimes Z_p(F)$ such that $|U - a_n V|(e) \longrightarrow 0$. The regularity of F implies that there is $y \in F_+$ such that $|U - a_n V|(e) \overset{y}{\Longrightarrow} 0$; we may assume that $x \leq y$. As $a_n V([-e,e])$ is totally bounded for $\| \ \|_y$ and

$$\sup_{f \in [-e,e]} \|U(f) - a_n V(f)\|_y \longrightarrow 0$$

as $n \longrightarrow \infty$, it follows that $U([-e,e])$ is totally bounded for $\| \ \|_y$, the oru-compacity of U being thus established.

To prove that $L_{oru}(C(X),F)$ is a band, let $U \in L_r(C(X),F)$ be the supremum of a subset of $L_{oru}(C(X),F)$. As F is order separable, we may find a sequence

$(U_n) \subset L_{oru}(C(X),F)$ such that $0 \leq U_n \leq U$ and $U_n(e) \uparrow U(e)$. The regularity of F implies the existence of $x \in F_+$ such that $U_n(e) \xrightarrow{x} U(e)$. As $U_n([-e,e])$ is relatively uniformly totally bounded and

$$\bigcup_{n=0}^{\infty} U_n([-e,e]) \subset [-U(e),U(e)] ,$$

lemma 3.2 gives an $y \in F_+$ such that eabh set $U_n([-e,e])$ is totally bounded for $\| \ \|_y$ we may assume that $x \leq y$. As

$$\sup_{f \in [-e,e]} \|U(f) - U_n(f)\|_y \leq \|U(e) - U_n(e)\|_y ,$$

it follows that $U([-e,e])$ is totally bounded for $\| \ \|_y$, the oru-compacity of U being thus established.

Conversely, let F satisfy the conditions in the statement of the theorem. We prove first that F is order separable. To this purpose, let $M \subset F_+ \setminus \{0\}$ be an order bounded subset of pairwise disjoint elements. Define $U \in L_r(l_\infty(M),F)$ and, for every finite $N \subset M$, $U_N \in L_{oru}(l_\infty(M),F)$ by

$$U_N = \sum_{t \in N} f(t)t ,$$

$$U = \sup_{\substack{N \subset M \\ N \text{ finite}}} U_N .$$

As $L_{oru}(l_\infty(M),F)$ is by hypothesis order closed, it follows that $U \in L_{oru}(l_\infty(M),F)$. Consequently, if e denotes the function identically one on M, then $M \subset U([0,e])$ which implies that M is relatively uniformly totally bounded; by lemma 3.1, M must be at most countable and the order separability of F is proved.

Now we may apply proposition 2.5 in order to establish that F is regular: in fact, we shall see that F satisfies condition iii) in that proposition.

Let $x \in F_+$ and let $(x_{nk}) \subset C(x)$ be a double sequence such that each sequence $(x_{nk})_k$ is disjoint. Define $U_n \in L_{oru}(l_\infty(\mathbb{N} \times \mathbb{N}),F)$ and $U \in L_r(l_\infty(\mathbb{N} \times \mathbb{N}),F)$ by

$$U_n(f) = \sum_{0 \leq i,j \leq n} 2^{-i} f(i,j) x_{ij} ,$$

$$U = \sup_{n \geq 0} U_n .$$

The definition of U is correct because

$$U_n(e) = \sum_{0 \leq i,j \leq n} 2^{-i} x_{ij} \leq \sum_{i=0}^{n} 2^{-i} x \leq 2x$$

(e being the function identically one on $\mathbb{N} \times \mathbb{N}$). As $L_{oru}(l_\infty(\mathbb{N} \times \mathbb{N}),F)$ is order closed by hypothesis, U is oru-compact; consequently, there is $u \in F_+$ such that $U([0,e])$ is totally bounded for $\| \ \|_u$. Let e_{nk} be the function defined by

$$e_{nk}(n,k) = 1,$$

$$e_{nk}(m,p) = 0 \text{ for } (m,p) \neq (n,k) .$$

Then $U(e_{nk}) = 2^{-n}x_{nk}$. By lemma 3.1 it follows that

$$(2^{-n}x_{nk})_k = (U(e_{nk}))_k \xrightarrow{u} 0 \quad ;$$

consequently, we also have $(x_{nk})_k \xrightarrow{u} 0$ for each n and our assertion is proved.

THEOREM 3.2. Let E,F be Riesz spaces such that F is super order complete and regular and at least one of the following conditions holds:

i) E has a strong order unit.

ii) $F^{\sim}$ separates F.

Then $L_{oru}(E,F) = \mathcal{J}(E,F)$ and $L^{\times}_{oru}(E,F) = \mathcal{J}^{\times}(E,F)$.

PROOF. By theorem 3.1, $L_{oru}(E,F)$ is a band in $L_r(E,F)$; by corollary 3.2 in [7], $L^{\times}_{oru}(E,F)$ is a band in $L_{oru}(E,F)$, hence a band in $L_r(E,F)$. As every operator of the form $x \longmapsto f(x)y$ with $f \in E^{\sim}$ (respectively $f \in E^{\times}$) and $y \in F$ is in $L_{oru}(E,F)$ (respectively $L^{\times}_{oru}(E,F)$), it follows that $\mathcal{J}(E,F) \subset L_{oru}(E,F)$ and $\mathcal{J}^{\times}(E,F) \subset$ $\subset L^{\times}_{oru}(E,F)$.

On the other side, theorem 3.4 in [7] shows that $L_{oru}(E,F) \subset \mathcal{J}(E,F)$ and $L^{\times}_{oru}(E,F) \subset \mathcal{J}^{\times}(E,F)$.

We close the section by giving a characterization of kernel operators in terms of oru-compacity.

Let (S_i, Σ_i, μ_i) be σ-finite measure spaces and let $L_0(\mu_i)$ $(i=1,2)$ be the Riesz space of all equivalence classes of measurable functions on S_i. Consider an order ideal E in $L_0(\mu_1)$. Recall that an operator $U: E \longrightarrow L_0(\mu_2)$ is called a kernel operator if there is a measurable function k on $S_1 \times S_2$ with the following property:

For every $x \in E$ there is a μ_2-null set N_x such that the function $s_1 \longmapsto k(s_1, s_2)x(s_1)$ is μ_1-integrable and

$$U(x)(s_2) = \int_{S_1} k(s_1, s_2)x(s_1)d\mu_1(s_1)$$

whenever $s_2 \in S_2 \setminus N_x$.

THEOREM 3.3. Let (S_i, Σ_i, μ_i) $(i=1,2)$ and E be as above. Then the set of all kernel operators from E into $L_0(\mu_2)$ is precisely $L^{\times}_{oru}(E, L_0(\mu_2))$.

PROOF. As it was already pointed out in [4], an operator $U: E \longrightarrow L_0(\mu_2)$ is a kernel operator iff the restriction U_x of U to E_x is so for every $x \in E_+$; similarly, we remark that $U \in L^{\times}_{oru}(E, L_0(\mu_2))$ iff $U_x \in L^{\times}_{oru}(E_x, L_0(\mu_2))$ for every $x \in E_+$. Then the result follows observing that the set of kernel operators from E_x into $L_0(\mu_2)$ equals, by theorem 2.5 in [4], $\mathcal{J}^{\times}(E_x, L_0(\mu_2))$ and the latter equals, by theorem 3.2, $L^{\times}_{oru}(E_x, L_0(\mu_2))$ (as $L_0(\mu_2)$ is super order complete and regular).

Schep's theorem [4] asserts that the set of all kernel operators is a band in $L_r(E, L_0(\mu_2))$. Our characterization shows that in general, the set of all kernel operators is a proper subset of $L_{oru}(E, L_0(\mu_2))$. Hence our results may be viewed as an improvement of Schep's theorem as they show that not only the set of kernel operators (= order continuous oru-compact operators) but the whole space $L_{oru}(E,L_0(\mu_2))$ is a band in $L_r(E,L_0(\mu_2))$.

COROLLARY 3.1. If μ_1 has no atoms, then there is no nontrivial kernel operator defined on the whole space $L_0(\mu_1)$.

PROOF. By theorems 4.1 and 4.2 in [7], $L_{oru}(L_0(\mu_1),L_0(\mu_2)) = \{0\}$; then apply theorem 3.3.

REFERENCES

1. S. BANACH and C. KURATOWSKI, Sur une generalisation du probleme de mesure, Fund. Math. 14 (1929), 127-131.
2. R. CRISTESCU, Ordered Vector Spaces and Linear Operators, Abacus Press (1976).
3. W.A.J. LUXEMBURG and A.C. ZAANEN, Riesz Spaces I, North Holland Publ. Co., Amsterdam (1971).
4. A.R. SCHEP, Kernel operators, Indag. Math. 41, 1 (1979), 39-53.
5. D. VUZA, La propriete faible de Fatou et la propriete d'Egoroff, C.R.Acad.Sc.Paris, 301 (1985), 115-117.
6. D. VUZA, Ideals and bands in principal modules, Arch. Math. 45 (1985), 306-322.
7. D. VUZA, Oru-compact operators, Preprint Series Math. INCREST, București, 67 (1985).
8. J.D.M. WRIGHT, The measure extension problem for vector lattices, Ann. Inst. Fourier, Grenoble, 21, 4 (1971), 65-85.