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by

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COCYCLES ON TREES

by Mihai V.Pimsner

Let X be an oriented tree, endowed with an orientation preserving action of the locally compact group G. In their paper [2], Julg and Valette constructed in a geometric way a Kasparov bimodule $\gamma \in \mathcal{E}_G(\mathbb{C},\mathbb{C})$ [4] and showed that the class of γ in $KK_G(\mathbb{C},\mathbb{C})$ coincides with 1_G, the unit of the ring $KK_G(\mathbb{C},\mathbb{C})$. They achieved this by exhibiting a homotopy γ_t, $t \in [0,\infty]$, (i.e. an element in $\mathcal{E}_G(\mathbb{C},\mathbb{C}([0,\infty]))$.) such that $\gamma_\infty = \gamma$ and $\gamma_0 = 1_G$. If $G = \mathbb{F}_n$ is the free group on n-generators acting on its natural tree, the Kasparov bimodule γ reduces to a construction done in [7] for the computation of the K-groups of the reduced crossed products by $\mathbb{F}_n$. Moreover, the fact that $\gamma = 1_{\mathbb{F}_n}$ was needed in order to carry over the above computation. (To be more specific one needs that $j_r(\gamma)$ is the identity in the ring $KK(\mathbb{F}_n \ltimes_r A, \mathbb{F}_n \ltimes_r A)$, where $j_r : KK_G(\mathbb{C},\mathbb{C}) \longrightarrow KK(G \ltimes_r A, G \ltimes_r A)$ is the map defined by Kasparov in [4] and [5]. (see [6])). Lacking G-equivariant KK-theory, most constructions in [7] seem complicated and unnatural.

The aim of this paper is to show that the homotopy exhibited in [7] for the free group, can be naturally constructed out of the tree X. The idea that was behind this homotopy is to perturb the action of G on the vertices of X by an explicit cocycle, so we construct the cocycle first. This leads to a continuous family γ_w of Kasparov bimodules, with $w = (w_y)$ running over the direct product $\prod U(2)$ of $N = \operatorname{card} \Sigma^1$ copies of the unitary group $U(2)$, where Σ^1 is the orbit space of the edges of X. If $w = (w_y)$ is the constant N-tuple $w_y = 1$ we get $\gamma_w = \gamma$ and if w_y is the transposition matrix $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for every $y \in \Sigma^1$, we get $\gamma_w = 1_G$. The explicit knowledge of the cocycle is useful for computations. We illustrate this by computing the character of the 1-summable Fredholm module γ_w [1]. This simplifies some of the computations of Julg and Valette in [3] and shows that their results remain true even if one drops the condition that the tree is uniform locally finite.

Let X^0 denote the set of vertices of the tree X. By an action of the locally compact group G on X, we shall mean a continuous action $G \times X^0 \longrightarrow X^0$, denoted $(g,x) \longmapsto gx$, that preserves the natural distance d on X^0. This determines a continuous action $(g,y) \longmapsto gy$ of G on X^1, the set of edges of the tree X. We shall denote by $\Sigma^1 \subset X^1$ a fixed transversal for this action of G on X^1, that is a complete system of representatives of the orbit space $G \backslash X^1$, and we shall denote by $\hat{y} \in \Sigma^1$ the class of the edge $y \in X^1$. The set Y of oriented edges will be identified with the pairs (x',x'') of vertices such that $d(x',x'') = 1$. If y is an oriented edge $y = (x',x'')$ we shall denote by $\bar{y}$ the oriented edge (x'',x') and by $|y|$ the (unoriented) edge determined by y. We shall also fix an orientation of the tree, i.e. a cross-section of the map $Y \ni y \longmapsto |y| \in X^1$, and consider X^1 as a subset of Y. We shall moreover assume that the action of G preserves this orientation, that is the action of G on Y leaves X^1 invariant. If $x',x'' \in X^0$ are two vertices we shall denote by $[x',x'']$ the geodesic joining x' with x''. We shall say that the oriented edge y belongs to the geodesic $[x',x'']$ if y is of the form $y = (x^i,x^{i+1})$, $x'=x^1,x^2,\ldots,x^n=x''$ being the path that defines this geodesic.

<u>Definition</u> : Suppose that $\pi : G \longrightarrow L(H)$ is some unitary representation of the group G on the Hilbert space H. We shall say that a function

$$c : X^0 \times X^0 \longrightarrow L(H)$$

is a cocycle on X for π , if c satisfies the following conditions

1) $c(x,x) = 1$, $c(x^2,x^1) = c(x^1,x^2)^*$
2) $c(x^1,x^2)\, c(x^2,x^3) = c(x^1,x^3)$
3) $c(gx^1,gx^2) = \pi(g)\, c(x^1,x^2)\, \pi(g^{-1})$

for every $x,x^1,x^2,x^3 \in X^0$ and every $g \in G$.

Note that if we fix a vertex x^0 , then the function $g \longmapsto c(x^0,gx^0)$ is a cocycle for π , that is $g \longmapsto c(x^0,gx^0)\pi(g)$ is a unitary representation of G on H .

The cocycle c on X is completely determined by its values on Σ^1 . We shall call the restriction of c to Σ^1 the generating function of the cocycle. It satisfies

$$c(y)\, \pi(g) = \pi(g)\, c(y)$$

for every g in the stabilizer of the edge y , and every $y \in \Sigma^1$. Conversely, every unitary valued function $c : \Sigma^1 \longrightarrow U(H)$ with the above property determines a cocycle on X for π , still denoted by c , for properties 1) and 3) uniquely determine c on the set of oriented edges and since X is a tree, properties 1) and 2) show that the value

$$c(x^1,x^n) := c(x^1,x^2)\, c(x^2,x^3) \ldots c(x^{n-1},x^n)$$

does not depend on the particular path $x^1,x^2,\ldots,x^n$ with fixed end points x^1 and x^n .

We shall construct now a particular class of cocycles on X for the representation $\pi^o : G \longrightarrow U(l^2(X^o))$ induced by the action of G on X^o.

If $u \in U(\mathbb{C}^2)$ is a unitary matrix and (x',x'') is an oriented edge, we shall denote by

$$u(x',x'') \in U(l^2(X^o))$$

the unitary defined by

$$u(x',x'') = v\, u\, v^* + 1 - vv^*$$

where $v = v(x',x'') : \mathbb{C}^2 \longrightarrow l^2(X^o)$ is the isometry defined on the basis of $\mathbb{C}^2$ by $v(e_1) = \delta_{x'}$, $v(e_2) = \delta_{x''}$.

If $w = (w_y)$, $y \in \Sigma^1$, is a card Σ^1-tuple of unitary matrices belonging to $U(\mathbb{C}^2)$, we shall denote by

$$c_w : X^o \times X^o \longrightarrow U(l^2(X^o))$$

the cocycle on X for the representation π^o, given by the generating function $c_w : \Sigma^1 \longrightarrow U(l^2(X^o)$ defined for $y = (x',x'') \in \Sigma^1$ by

$$c_w(y) = w_y(x',x'') .$$

It is easy to see that if we extend the function $w : \Sigma^1 \longrightarrow U(\mathbb{C}^2)$ to all oriented edges by

$$w_y = w_{\hat{y}} \qquad \text{for } y \in X^1$$
$$w_y = \tau\, w_{\bar{y}}^* \,\tau \qquad \text{for } y \in Y \setminus X^1$$

(τ being the transposition matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$), then the value of c_w is given on every oriented edge $y = (x',x'')$ by

$$c_w(x',x'') = w_y(x',x'') .$$

<u>Remark 1.</u> If we fix a unitary matrix $w \in U(\mathbb{C}^2)$ such that $\tau w \tau = w^*$, then the formula

$$c_w(x',x'') = w(x',x'')$$

defines the cocycle corresponding to the constant function $w_y = w$, for every $y \in \Sigma^1$, without any reference to Σ^1.

Let us record for further use the following two straightforward properties of the cocycle c_w.

i) If x',x'' are arbitrary vertices of the tree X and $p(x',x'')$ denotes the orthogonal projection on the space spanned by the vectors $\{\delta_x \mid x \in [x',x'']\}$, then $p(x',x'')$ commutes with $c_w(x',x'')$ and

$$p(x',x'')\, c_w(x',x'') + 1 - p(x',x'') = c_w(x',x'') .$$

ii) *If $x' \neq x''$ belong to X°, then:*

$$\langle c_w(x',x'')\,\delta_{x''} , \delta_{x'} \rangle = \prod_{y\in[x',x'']} (w_y)_{1,2}$$

where $(w_y)_{1,2}$ is the $1,2$ entry of the matrix w_y.

Let now $x^\circ \in X^\circ$ be a fixed origin of the tree X.

<u>Definition</u> We shall denote by π_w° the unitary representation of G on $l^2(X^\circ)$ defined by

$$\pi_w^\circ(g) = c_w(x^\circ, gx^\circ)\,\pi^\circ(g) .$$

In the sequel, $V : l^2(X^1) \longrightarrow l^2(X^o)$ will denote the isometry defined by Julg and Valette in [2], induced by the map that sends each vertex $x \in X^o \setminus \{x^o\}$ to the (unique) edge y that has one extremity x and lies on the geodesic $[x, x^o]$.

<u>Lemma 1.</u> a) If $w : \Sigma^1 \longrightarrow U(\mathbb{C}^2)$ is the constant function $w_y = 1$, for every $y \in \Sigma^1$, then $\pi_w^o = \pi^o$. If w is the constant function $w_y = \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\pi_w^o(g) = V \pi^1(g) V^* + p(x^o)$, where $\pi^1 : G \longrightarrow U(l^2(X^1))$ is the unitary representation induced by the action of G on X^1, and $p(x^o)$ is the orthogonal projection on the space $\mathbb{C}\,\delta_{x^o}$.

b) $\pi_w^o(g) - \pi_{w'}^o(g) = p(x^o, gx^o)(\pi_w^o(g) - \pi_{w'}^o(g))$

for every $w, w' : \Sigma^1 \longrightarrow U(\mathbb{C}^2)$ and every $g \in G$.

c) The function

$$U(\mathbb{C}^2)^{\Sigma^1} \ni w \longmapsto \pi_w^o(g) \in U(l^2(X^o))$$

is norm continuous for every $g \in G$.

<u>Proof</u> : a) If $w_y = \tau$ for every $y \in \Sigma^1$, then it is easy to see that

$$c_w(x', x'')\,\delta_x = \delta_x \qquad \text{if} \quad x \notin [x', x'']$$

while if $x = x^i$ belongs to the geodesic $x' = x^1, x^2, \ldots, x^n = x''$ then

$$c_w(x',x'')\,\delta_{x^i} = \begin{cases} \delta_{x^{i+1}} & \text{for } i<n \\ \delta_{x^1} & \text{for } i=n. \end{cases}$$

An easy computation shows now that $c_w(x^o,gx^o)\,\pi^o(g) = V\,\pi^1(g)\,V^* + p(x^o)$.

Point b) follows from property i) of the cocycle c_w, while c) is obvious.

q.e.d.

We shall denote by H_w, $w \in U(\mathbb{C}^2)^{\Sigma^1}$, the graded Hilbert space $l^2(X^o) \oplus l^2(X^1)$ endowed with the continuous G action $\pi_w(g) = \pi^o_w(g) \oplus \pi^1(g)$. The Kasparov bimodule determined by the isometry V will be denoted by $\gamma_w \in \mathcal{E}_G(\mathbb{C},\mathbb{C})$. The preceding lemma obviously implies the following proposition

<u>Proposition 2.</u> The above constructed family γ_w, $w \in U(\mathbb{C}^2)^{\Sigma^1}$, belongs to $\mathcal{E}_G(\mathbb{C}, C(U(\mathbb{C}^2)^{\Sigma^1}))$. Its restriction to the point $w_y = 1$, for every $y \in \Sigma^1$, is the Kasparov bimodule γ constructed by Julg and Valette, while its restriction to the point $w_y = \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for every $y \in \Sigma^1$ is 1_G.

<u>Remark 2.</u> Julg and Valette defined in [2] besides V the isometry U by $U = VS$, where $S : l^2(X^1) \longrightarrow l^2(X^1)$ is given by $S\,\delta_{(x',x'')} = \varepsilon(x',x'')\,\delta_{(x',x'')}$, ε being $\varepsilon(x',x'') = 1$ (resp. -1) if $d(x',x^o) > d(x'',x^o)$ (resp. $d(x',x^o) < d(x'',x^o)$). If we denote by γ'_w the Kasparov bimodules determined on H_w by U we still get an element $\gamma' \in \mathcal{E}_G(\mathbb{C}, C(U(\mathbb{C}^2)^{\Sigma^1}))$. In this case 1_G corresponds to

the constant function $w_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for every $y \in \Sigma^1$, while the restriction to the constant function $w_y = 1$ yields an element that obviously equals γ in $KK_G(\mathbb{C},\mathbb{C})$. Moreover, it is easy to see that their homotopy corresponds to the path $[0,\infty] \ni t \longmapsto w^t \in U(\mathbb{C}^2)^{\Sigma^1}$, where w^t is the constant function

$$w_y^t = \begin{pmatrix} (1-e^{-2t})^{1/2} & e^{-t} \\ -e^{-t} & (1-e^{-2t})^{1/2} \end{pmatrix}.$$

for every $y \in \Sigma^1$. Note that the above matrices satisfy $\tau w_y^t \tau = w_y^{t\,*}$ (see remark 1.)

It is easy to see from lemma 1. point b) that the Kasparov bimodules γ_w are 1-summable Fredholm modules in the sense of Connes [1.] , so we pass to the computation of their characters.

<u>Lemma 3.</u> a) The trace of the finite rank operator

$$p(x,gx)\,\pi_w^0(g)$$

does not depend on the vertex $x \in X^0$.

b) The trace of the above operator is equal to

$$\prod_{y \in [x,gx]^\circ} (w_y)_{1,2}\ .$$

for every vertex $x \in X^o$ satisfying $x \neq gx$ and $[g^{-1}x,x] \cap \cap [x,gx] = \{x\}$. If x is fixed by g ; then the above trace is 1 .

<u>Proof</u> Let $x \in X^o$ be a vertex such that $[g^{-1}x,x] \cap \cap [x,gx] = \{x\}$. Using property i) of the cocycle c_w we get :

$$tr(p(x,gx)\pi_w^o(g)) = tr(p(x,gx)\ c_w(x,gx)\ p(x,gx)\pi^o(g)) = tr(p(x,gx)\ c_w(x,gx)\pi^o(g)\ p(g^{-1}x,x)) = \langle c_w(x,gx)\pi^o(g)\delta_x,\delta_x\rangle = \langle c_w(x,gx)\ \delta_{gx}\ ,\ \delta_x\rangle .$$

Thus point b) of the lemma follows from property ii) of the cocycle c_w .

To prove a) note first that if $x \in X^o$ satisfies $gx \neq x$ and $[g^{-1}x,x] \cap [x,gx] = \{x\}$, then every $x' \in X^o$ satisfying $[g^{-1}x',x'] \cap [x',gx'] = \{x'\}$ has to lie on the geodesic $[g^i x, g^{i+1}x]$ for some $i \in \mathbb{Z}$. Point b) then implies that

$$tr(p(x',gx')\pi_w^o(g)) = tr(p(x,gx)\pi_w^o(g)).$$

This shows that the trace of the operator $p(x,gx)\pi_w^o(g)$ is constant on the set of vertices satisfying $[g^{-1}x,x] \cap \cap [x,gx] = \{x\}$.

Let now $x \in X^o$ be any vertex and let $x=x^1,\dots,x^n=gx$ be the path that defines the geodesic $[x,gx]$. If $[g^{-1}x,x] \cap \cap [x,gx] \neq x$, it follows that $g(x^2) = x^{n-1}$. The properties of the cocycle c_w imply that

$$tr(p(x,gx)\ c_w(x,gx)\,\pi^o(g)) = tr(p(x,gx)\ c_w(x,x^2)\ c_w(x^2,gx)\,\pi^o(g)) =$$

$$tr(c_w(x,x^2)\ p(x,gx)\ c_w(x^2,gx)\,\pi^o(g)) =$$

$$tr(p(x,gx)\ c_w(x^2,gx)\,\pi^o(g)\ c_w(x,x^2)) =$$

$$tr(p(x,gx)\ c_w(x^2,gx)\ c_w(gx,gx^2)\,\pi^o(g)) =$$

$$tr(p(x,gx)\ c_w(x^2,gx^2)\,\pi^o(g)).$$

It is now easy to see that the last trace is equal

$$tr(p(x^2,gx^2)\ c_w(x^2,gx^2)\,\pi^o(g))\ .$$

Since G acts without inversion we may proceed inductively until we find $x^i \in [x,gx]$ such that $[g^{-1}x^i,x^i] \cap [x^i,gx^i] = \{x^i\}$.

q.e.d.

<u>Proposition 4.</u> The character of the 1-summable Fredholm module γ_w is given by

$$\tau_w(g) = \begin{cases} 1 & \text{if } g \text{ has a fixed point in } X^o \\ \prod_{y \in [x,gx]} (w_y)_{1,2} & \text{otherwise, where } x \in X^o \text{ is any vertex satisfying } gx \neq x \text{ and } [g^{-1}x,x] \cap [x,gx] = \{x\}\ . \end{cases}$$

<u>Proof</u> We have to compute the trace of $\pi^o_w(g) - V\,\pi^1(g)\,V^*$. If we denote by w_o the constant function $(w_o)_y = \tau$ for every $y \in \Sigma^1$, then

$$\mathrm{tr}(\pi^o_w(g) - V\pi^1(g)V^*) = \mathrm{tr}(\pi^o_w(g) - \pi^o_{w_o}(g) + p(x^o)) =$$

$$\mathrm{tr}(\pi^o_w(g) - \pi^o_{w_o}(g)) + 1 .$$

By lemma 1. $\mathrm{tr}(\pi^o_w(g) - \pi^o_{w_o}(g)) = \mathrm{tr}(p(x^o,gx^o)(\pi^o_w(g) - \pi^o_{w_o}(g))$ and by the preceding lemma $\mathrm{tr}(p(x^o,gx^o)\pi^o_{w_o}(g)) = 1$.
This shows that

$$\tau_w(g) = \mathrm{tr}(p(x^o,gx^o)\pi^o_w(g))$$

which together with the preceding lemma concludes the proof of the proposition.

q.e.d.

Remark 3. The same formula (with the same proof) holds for the characters τ'_w of the 1-summable Fredholm modules γ'_w of remark 2.

Corollary 5. If $w : \Sigma^1 \longrightarrow U(\mathbb{C}^2)$ satisfies $(w_y)_{1,2} = \lambda \in \mathbb{R}$, for every $y \in \Sigma^1$, then the character of γ_w is given by

$$\tau_w(g) = \lambda^{p(g)}$$

where $p(g) = \inf_{x\in X^o} d(x,gx)$.

Proof The proof is a straightforward consequence of the preceding proposition once we notice that $(\tau w^* \tau)_{1,2} = \bar{w}_{1,2}$, for every $w \in L(\mathbb{C}^2)$, and that $p(g)$ is equal to the length of the geodesic $[x,gx]$, for every vertex satisfying $[g^{-1}x,x] \cap [x,gx] = \{x\}$.

q.e.d.

<u>Remark 4.</u> In view of remark 3. the preceding corollary holds for the 1-summable Fredholm modules γ'_W too. Remark 2. then shows that the formula obtained in chapter 5. of [3] holds for arbitrary trees.

Moreover, corollary 5. shows that the results of Julg and Valette in [3] concerning the Selberg principle are true without any assumption on the tree X .

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