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Mihnea COLTOIU

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Mihnea COLTOIU*)

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*) The National Institute for Scientific and Technical Creation Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, ROMANIA.

A positivity theorem for modifications of projective algebraic spaces

o. Introduction

Let Y be a projective algebraic manifold and $\pi: X \to Y$ a modification of Y. As it was pointed out by Hironaka([4] ,p.444) in general X fails to be Kählerian. In fact, since X is a Moišhezon manifold, the existence of a Kähler metric on X would imply that X is projective algebraic [6].

The aim of this note is to prove that any manifold which is a modification of a projective algebraic manifold carries a "quasi-Kählerian" metric. This means that there exist an open covering $\mathcal{U}=(U_i)$ of X and strongly plurisubharmonic functions $\varphi_i: U_i \to [-\infty,\infty)$, $\varphi_i \not\equiv -\infty$ on any component of U_i, such that $\varphi_i=\varphi_j+\lambda_{ij}$ on $U_i \cap U_j$ where λ_{ij} are pluriharmonic functions on $U_i \cap U_j$. One cannot take the functions φ_i to be real valued and continuous because, by a result of Varouchas [7], it would follow that X is a Kähler manifold. Hence the condition of upper semicontinuity of φ_i is the best possible one may expect.

All our results stated above hold for complex spaces.

1. The main result

All complex spaces are assumed to be reduced and with countable topology.

Let X be a complex space. An upper semicontinuous function $\varphi: X \to [-\infty,\infty)$ is said to be plurisubharmonic if for any holomorphic function $\tau: W \to X$ (W=the unit disc in $\mathbb{C}$) it follows that $\varphi\circ\tau$ is subharmonic on W (possibly $\equiv -\infty$). φ is called strongly plurisubharmonic if for any C^∞ real valued function θ with compact support in X there exists an $\varepsilon_o > o$ such that $\varphi+\varepsilon\theta$ is plurisubharmonic on X $|\varepsilon| \leq \varepsilon_o$.

By a theorem of Fornaess and Narasimhan [3] the above definitions agree with the usual ones i.e.any(strongly) plurisubharmonic function is locally the restriction of a (strongly) plurisubharmonic function on an open subset of C^N in which X is locally embedded.

A proper,holomorphic and surjective map of complex spaces $p:X\to Y$ is said to be a modification if there exists a rare analytic subset $Y'\subset Y$ such that $X'=p^{-1}(Y')$ is rare in X and the induced map $X\setminus X'\to Y\setminus Y'$ is an isomorphism.In the sequel a modification will be denoted by $(X,X')\to(Y,Y')$.

The following theorem due to Hironaka [5] shows that semi-locally (i.e. on relatively compact open subsets) the modifications are not far from being a blowing-up :

<u>Lemma of Chow</u> (Hironaka [5]) Let $(X,X')\xrightarrow{p}(Y,Y')$ be a modification and $U\subset\subset Y$ a relatively compact open subset of Y. Then there exist a coherent ideal $\mathcal{J}$ on U such that $A:=\operatorname{supp}(\mathcal{O}_U/\mathcal{J})$ $\subset Y'\cap U$ and a commutative diagram :

$$\begin{tikzcd} \tilde{U} \arrow[r, "\tau"] \arrow[dr, "\pi"'] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$

where $\pi:\tilde{U}\to U$ is the blowing-up of U with center $(A,\mathcal{O}_U/\mathcal{J}|_A)$ and τ is holomorphic,proper and surjective.

<u>Definition</u> Let X be a complex space.X is said to be quasi-Kählerian if there exist an open covering $\mathcal{U}=(U_i)$ of X and strongly plurisubharmonic functions $\varphi_i\to[-\infty,\infty)$, $\varphi_i\not\equiv-\infty$

on any irreducible component of U_i, and $\varphi_i=\varphi_j+\lambda_{ij}$ on $U_i\cap U_j$ where λ_{ij} is pluriharmonic on $U_i\cap U_j$ (i.e. locally the real part of a holomorphic function).

The following theorem provides examples of quasi-Kählerian spaces :

Theorem 1 Let X be a modification of a projective algebraic space. Then X is quasi-Kählerian.

For the proof of Theorem 1 we need some lemmas.

Lemma 1 ([1] Corollary 2.2.) Let X,Y be complex spaces and $p:X\to Y$ a proper, surjective, holomorphic map. Let $\psi:Y\to[-\infty,\infty)$ be an upper semicontinuous function such that $\psi\circ p$ is strongly plurisubharmonic on X. Then ψ is strongly plurisubharmonic on Y.

Lemma 2 Let $\pi:X\to Y$ be a blowing-up given by a coherent ideal $\mathcal{J}\subset\mathcal{O}_Y$ and assume that there exist $f_1,\dots,f_s\in\Gamma(Y,\mathcal{J})$ such that the germs $(f_1)_y,\dots,(f_s)_y$ generate $\mathcal{J}_y$ for any $y\in Y$. Then for any strongly plurisubharmonic function $\varphi\in C^\infty(Y,\mathbb{R})$ the function $\varphi\circ\pi+\log(|f_1\circ\pi|^2+\dots+|f_s\circ\pi|^2)$ is strongly plurisubharmonic on X.

Proof

If we set $f=(f_1,\dots,f_s):Y\to\mathbb{C}^s$ then there is a canonical sheaf epimorphism $f^*\mathfrak{m}\to\mathcal{J}$ where $\mathfrak{m}\subset\mathcal{O}_{\mathbb{C}^s}$ is the sheaf of ideals of the origin. Let $\xi_1:\mathbb{P}(\mathcal{J})\to Y$ and $\xi_2:\mathbb{P}(f^*\mathfrak{m})\to Y$ be the projective varietes over Y corresponding to $\mathcal{J}$, respectively to $f^*\mathfrak{m}$ (in general they are not reduced). The sheaf epimorphism $f^*\mathfrak{m}\to\mathcal{J}$ gives rise to an embedding $\mathbb{P}(\mathcal{J})\hookrightarrow\mathbb{P}(f^*\mathcal{J})$. By the very definition of the analytic blowing-up [2] there is an embedding $X\hookrightarrow\mathbb{P}(\mathcal{J})$ hence to prove the lemma it is enough to verify that $\tau\circ\xi_2$ is strongly plurisubharmonic on $\mathbb{P}(f^*\mathfrak{m})$ where $\tau=\varphi+\log(|f_1|^2+\dots+|f_s|^2)$. $\mathbb{P}(f^*\mathfrak{m})\hookrightarrow Y\times\mathbb{P}_{s-1}$ is the subspace given by the equations :

$$f_j(y)z_i-f_i(y)z_j=0 \quad 1\le i<j\le s$$

where $(z_1:\ldots:z_s)$ are the homogeneous coordinates on $\mathbb{P}_{s-1}$. Set $U_i=\{(y,z)\in Y\times\mathbb{P}_{s-1} \mid z_i\neq 0\}$, $\alpha_i: U_i\to Y\times\mathbb{C}^{s-1}$ $\alpha_i(y,z)=(y,z_1/z_i,\ldots,z_{i-1}/z_i,z_{i+1}/z_i,\ldots,z_s/z_i)$ and define $\psi_i: Y\times\mathbb{C}^{s-1}\to[-\infty,\infty)$ by $\psi_i=\varphi+\log(1+\sum_{j=1}^{s-1}|t_j|^2)+\log(|f_i|^2)$ where $(t_1,\ldots,t_{s-1})$ are the affine coordinates on $\mathbb{C}^{s-1}$.
Then ψ_i is strongly plurisubharmonic on $Y\times\mathbb{C}^{s-1}$ and $\tau\circ\xi_2=\psi_i\circ\alpha_i$ on $U_i\cap\mathbb{P}(f^*\mathcal{M})$. It follows that $\tau\circ\xi_2$ is strongly plurisubharmonic on $\mathbb{P}(f^*\mathcal{M})$ which proves the lemma.

<u>Lemma 3</u> Let X be a projective algebraic space and $\mathcal{J}$ a coherent ideal on X. Then there exist $s\in\mathbb{N}$, $\mathcal{U}=(U_i)$ an open covering of X and $(f_1^i,\ldots,f_s^i)\in\Gamma(U_i,\mathcal{J})^s$ such that :

1) the germs $(f_1^i)_x,\ldots,(f_s^i)_x$ generate $\mathcal{J}_x$ for any $x\in U_i$
2) $f_k^i=g_{ij}f_k^j$ with $g_{ij}\in\mathcal{O}^*(U_i\cap U_j)$

Proof

Let F be an ample line bundle on X and let $n_o\in\mathbb{N}$ be such that $\mathcal{J}\otimes\underline{F}^{n_o}$ is generated by global sections. Hence one may choose $t_1,\ldots,t_s\in\Gamma(X,\mathcal{J}\otimes\underline{F}^{n_o})$ whose germs $(t_1)_x,\ldots,(t_s)_x$ generate $\mathcal{J}_x\otimes\underline{F}_x^{n_o}\simeq\mathcal{J}_x$ at any point $x\in X$. If $\mathcal{U}=(U_i)$ is a sufficiently small open covering of X then $t_k=f_k^i\in\Gamma(U_i,\mathcal{J})$ on U_i, hence $(t_1,\ldots,t_s)=(f_1^i,\ldots,f_s^i)$ on U_i and g_{ij} are the transition functions for F^{n_o}.

<u>Proof of Theorem 1</u>

By Chow's Lemma there is a commutative diagram :

$$\begin{tikzcd} Y^* \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where π is the blowing-up given by a coherent ideal $\mathcal{J}$ and f is holomorphic and surjective.

Choose $\mathcal{U}=(U_i)$ and $(f_1^i,\ldots,f_s^i)$ with the properties 1)

and 2) in Lemma 3 (corresponding to $\mathcal{J}$ given by Chow's Lemma). Y is a Kähler space so there exist strongly plurisubharmonic functions $\psi_i \in C^\infty(U_i,\mathbb{R})$ such that $\mu_{ij}=\psi_i-\psi_j$ is pluriharmonic on $U_i \cap U_j$ (refining $\mathcal{U}$ if necessary).
On $\widetilde{U}_i=p^{-1}(U_i)$ we define $\lambda_i=\psi_i\circ\pi+\log(|f_1^i\circ\pi|^2+\ldots+|f_s^i\circ\pi|^2)$.
Since $(f_1^i)_x,\ldots,(f_s^i)_x$ generate $\mathcal{J}_x$ for any $x\in U_i$ it follows from Lemma 1 and Lemma 2 that λ_i is strongly plurisubharmonic on $\widetilde{U}_i$. Also it is clear that $\lambda_i \not\equiv -\infty$ on any irreducible component of $\widetilde{U}_i$. On $\widetilde{U}_i \cap \widetilde{U}_j$ we get $\lambda_i=\lambda_j+\mu_{ij}\circ\pi+\log|g_{ij}\circ\pi|^2$ and $\mu_{ij}\circ\pi+\log|g_{ij}\circ\pi|^2$ is pluriharmonic. It follows that $\{\lambda_i\}$ is a quasi-Kähler metric on X and the theorem is proved.

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Mihnea Colţoiu
Department of Mathematics INCREST
Bd. Păcii 220, R-79622 Bucharest
Romania