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On the homology groups of Stein spaces and Runge pairs

1. Introduction

In [7] Narasimhan has proved that if X is a Stein space of dimension n then $H_r(X;\mathbb{Z})=0$ for $r>n$ and $H_n(X;\mathbb{Z})$ is torsion free. In the same paper he asked whether $H_n(X;\mathbb{Z})$ is free. An affirmative answer has been given by Hamm [3], [4] which has proved the stronger result that X has the homotopy type of a CW complex of dimension $\leq n$. Hamm's proof is based on Morse theory on singular spaces.

In this paper, using elementary methods, we give a short proof to Narasimhan's problem (Theorem 3) and in the same time we obtain :

<u>Theorem 1</u> Let X be a Stein space of dimension n and $Y\subset X$ an open set such that (X,Y) is a Runge pair. Then :

$$H_r(X,Y;\mathbb{Z})=0 \quad \text{for } r>n \quad \text{and}$$

$$H_n(X,Y;\mathbb{Z}) \text{ is torsion free.}$$

This theorem was proved by Andreotti and Narasimhan when X has isolated singularities [1].

Theorem 1 and Narasimhan's problem follow from :

<u>Theorem 2</u> Let X be a Stein space of dimension n, $\varphi: X\to\mathbb{R}$ a real analytic plurisubharmonic exhaustion function and let $X_\gamma=\{\varphi<\gamma\}$ for $\gamma\in\mathbb{R}$. If $\gamma_1<\gamma_2$ then $H_r(X_{\gamma_2},X_{\gamma_1};G)=0$ for $r>n$ and any abelian group G.

The proof of this theorem needs only the vanishing theorem for the relative homology groups for Runge pairs of Stein manifolds, result which is usually deduced using the classical Morse theory (i.e. on nonsingular spaces).

2. Preliminaries

All complex spaces are supposed to be reduced and countable at infinity. Given a Stein space X and $Y\subset X$ an open subset, (X,Y) is said to be a Runge pair if Y is Stein and the restriction map $\Gamma(X,\mathcal{O})\to\Gamma(Y,\mathcal{O})$ has dense image. If $K\subset X$ is a compact subset we set $\hat{K}_X=\{x\in X\ /\ |f(x)|\leq \sup_K |f| \text{ for any } f\in\Gamma(X,\mathcal{O})\}$. K is called holomorphically convex if $K=\hat{K}_X$. Any holomorphically convex compact set K has a fundamental system of Runge neighbourhoods. It follows that the restriction map $\Gamma(X,\mathcal{F})\to\Gamma(K,\mathcal{F})$ has dense image for any $\mathcal{F}\in Coh(X)$ (see [2], p.29). The Oka-Weil theorem asserts that (X,Y) is a Runge pair iff for any compact set $K\subset Y$ it follows that $\hat{K}_X\subset Y$. If $\varphi:X\to\mathbb{R}$ is plurisubharmonic according to a theorem of Narasimhan [6] the pair $(X,\{\varphi<c\})$ is Runge for any $c\in\mathbb{R}$.

We shall need also :

Proposition 2.1. Let X be a Stein space, $A\subset X$ an analytic subset and $K\subset X$ a holomorphically convex compact subset. Then $L=K\cup A$ has a fundamental system of Runge neighbourhoods.

Proof

Let $\{K_\nu\}$ be a sequence of holomorphically convex compact subsets of X such that $K_1=K$, $K_\nu\subset\mathring{K}_{\nu+1}$ for any ν and $X=\bigcup_\nu K_\nu$. We fix an open neighbourhood V of L and define $P_1=K_3\cap\complement V$,

$P_\nu=(K_{\nu+2}\setminus \mathring{K}_{\nu+1})\cap \complement V$ for $\nu\geq 2$. Let $\mathcal{J}$ be the ideal sheaf of A.

Remark that given any $x\notin L$ the set $K_x=K\cup\{x\}$ is still holomorphically convex and the function which is o in a neighbourhood of K and 2 in a neighbourhood of x defines a section in $\Gamma(K_x,\mathring{\mathcal{J}})$. Since $\Gamma(X,\mathring{\mathcal{J}})\to\Gamma(K_x,\mathring{\mathcal{J}})$ has dense image there exists $f\in\Gamma(X,\mathring{\mathcal{J}})$ such that $\sup_K|f|<1$ but $|f(x)|>1$. Using repeatedly this fact we can choose for any ν $f_{\nu,1},\ldots,f_{\nu,l_\nu}\in\Gamma(X,\mathring{\mathcal{J}})$ such that :

$$|f_{\nu,1}|^2+\ldots+|f_{\nu,l_\nu}|^2\leq 1/2^{\nu+1} \quad \text{on } K_\nu$$

and $|f_{\nu,1}|^2+\ldots+|f_{\nu,l_\nu}|^2>1$ on P_ν.

Then $\varphi=\sum_{\nu,j}|f_{\nu,j}|^2$ is a real analytic plurisubharmonic function on X and $L\subset\{\varphi<1\}\subset V$. But $\{\varphi<1\}$ is Runge in X so the proof of Proposition 2.1. is complete.

Remark When $K=\emptyset$ the above result is proved by Narasimhan in [7].

3. Proof of the main results

Lemma 3.1. Let X be a Stein space of dimension n and let $f\in\Gamma(X,\mathcal{O})$ be such that $\mathrm{Sing}(X)\subset A=\{f=0\}$. Let $\varphi:X\to\mathbb{R}$ be a real analytic plurisubharmonic exhaustion function and $X_\gamma=\{\varphi<\gamma\}$. Then $H_r(X,X_\gamma\cup A;G)=0$ for $r>n$ and any abelian group G.

Proof

$X_\gamma\cup A$ being semi-analytic there exists an open neighbourhood V' of $X_\gamma\cup A$ such that $X_\gamma\cup A$ is a strong deformation retract of V'. In particular the natural maps $\theta_r:H_r(X,X_\gamma\cup A;G)\to H_r(X,V';G)$ are injective for any r.

On the other hand given V a Runge neighbourhood of $X_\gamma\cup A$ we have (*) $H_r(X,V;G)=0$ for $r>n$. Indeed $(X\setminus A,V\setminus A)$ is a Runge pair in the Stein manifold $X\setminus A$. By [1] $H_r(X\setminus A, V\setminus A;G)=0$ if $r>n$. Now (*) follows by excision.

Since X_γ is a holomorphically convex compact subset of X according to Proposition 2.1. there is a Runge neighbourhood V of $X_\gamma \cup A$ such that $V\subset V'$. Hence θ_r are the null maps for $r>n$ and the lemma is proved.

Lemma 3.2. Let X be a Stein space, $f\in\Gamma(X,\mathcal{O})$ such that $\mathrm{Sing}(X)\subset A=\{f=0\}$ and $\varphi: X\to\mathbb{R}$ a real analytic plurisubharmonic exhaustion function. If $\gamma_1\le\gamma_2$ then $H_r(X_{\gamma_2}\cup A, X_{\gamma_1}\cup A;G)=0$ for $r>n$ and any abelian group G.

Proof

This follows from the exact sequence of the triple $(X, X_{\gamma_2}\cup A, X_{\gamma_1}\cup A)$ and Lemma 3.1.

Lemma 3.3. Let X be a complex space and A, B semi-analytic closed subsets of X. Then $H_r(A\cup B, B;G)\cong H_r(A, A\cap B;G)$ for any abelian group G and any $r\geq 0$.

Proof

From the triangulation theorem [5] for the pair of semi-analytic sets $(B, A\cap B)$ there is an open neighbourhood T of $A\cap B$ in B and a strong deformation retraction $R: T\times[0,1]\to T$ for the inclusion $A\cap B\hookrightarrow T$. By excision $H_r(A\cup B, B;G)\cong H_r(A\cup T, T;G)$ for any $r\geq 0$. But obviously R extends to a strong deformation retraction $\widetilde{R}:(A\cup T)\times[0,1]\to A\cup T$ for the inclusion $A\hookrightarrow A\cup T$. Hence $(A, A\cap B)$ is a deformation retract of $(A\cup T, T)$. Consequently $H_r(A\cup T, T;G)\cong H_r(A, A\cap B;G)$ for any $r\geq 0$. This concludes the proof of Lemma 3.3.

Proof of Theorem 2

We use induction on $n=\dim(X)$. For $n=o$ it is obvious. Assume that the theorem holds for all Stein spaces of dimension $\leq n-1$. Let $f\in\Gamma(X,\mathcal{O})$ be such that $\mathrm{Sing}(X)\subset\{f=o\}$ and f does not vanish identically on any irreducible component of X of positive dimension. Hence $A=\{f=o\}$ is a Stein space of dimension $\leq n-1$ and $\varphi|_A$ is a real analytic plurisubharmonic exhaustion function on A. Consequently by the induction hypothesis it follows that $H_r(X_\beta\cap A,X_\alpha\cap A;G)=o$ if $\alpha\leq\beta$ and $r>n-1$. Choosing $\beta\to\infty$ we get $H_r(A,X_\alpha\cap A;G)=o$ for any $\alpha\in\mathbb{R}$ and $r>n-1$. From Lemma 3.3. we obtain :

$$(*)\quad H_r(X_{\gamma_1}\cup A,X_{\gamma_1};G)=H_r(X_{\gamma_2}\cup A,X_{\gamma_2};G)=o \quad \text{if } r>n-1.$$

The exact sequence of the triple $(X_{\gamma_2}\cup A,X_{\gamma_1}\cup A,X_{\gamma_1})$ gives :

$$\ldots\to H_r(X_{\gamma_1}\cup A,X_{\gamma_1};G)\to H_r(X_{\gamma_2}\cup A,X_{\gamma_1};G)\to H_r(X_{\gamma_2}\cup A,X_{\gamma_1}\cup A;G)\to\ldots$$

Therefore $H_r(X_{\gamma_2}\cup A,X_{\gamma_1};G)=o$ for $r>n$, by Lemma 3.2. and $(*)$.

Now the exact sequence of the triple $(X_{\gamma_2}\cup A,X_{\gamma_2},X_{\gamma_1})$ gives :

$$\ldots\to H_{r+1}(X_{\gamma_2}\cup A,X_{\gamma_2};G)\to H_r(X_{\gamma_2},X_{\gamma_1};G)\to H_r(X_{\gamma_2}\cup A,X_{\gamma_1};G)\to\ldots$$

and similarly we obtain $H_r(X_{\gamma_2},X_{\gamma_1};G)=o$ for $r>n$ and any abelian group G, as desired.

<u>Theorem 3</u> Let X be a Stein space of dimension n and $A\subset X$ an analytic subset. Then :

$$H_r(X,A;Z)=o \quad \text{for } r>n \quad \text{and}$$

$$H_n(X,A;Z) \text{ is free.}$$

In particular $H^r(X,A;Z)=o$ for $r>n$.

Proof

a) We assume first that $A=\emptyset$. Let $\varphi: X\to\mathbb{R}$ be a real analytic plurisubharmonic exhaustion function and set $X_\gamma=\{\varphi\le\gamma\}$ for $\gamma\in\mathbb{R}$. For any $\nu\in\mathbb{N}$ the group $H_r(X_\nu;\mathbb{Z})$ is finitely generated (since X_ν is a finite polyhedra) hence $H_r(X_\nu,X_\mu;\mathbb{Z})$ are also finitely generated. By Theorem 2 and the universal coefficient theorem : $H_r(X_{\nu+1},X_\nu;\mathbb{Z})=H_r(X_\nu;\mathbb{Z})=0$ for $r>n$ and the groups $H_n(X_{\nu+1},X_\nu;\mathbb{Z}), H_n(X_\nu,\mathbb{Z})$ are torsion free and in fact free (being finitely generated). Because $H_r(X;\mathbb{Z})=\varinjlim_\nu H_r(X_\nu;\mathbb{Z})$ we obtain $H_r(X;\mathbb{Z})=0$ for $r>n$. To see that $H_n(X;\mathbb{Z})$ is free we consider the exact sequence :

$$\ldots\to H_{n+1}(X_{\nu+1},X_\nu;\mathbb{Z})\to H_n(X_\nu;\mathbb{Z})\to H_n(X_{\nu+1};\mathbb{Z})\to H_n(X_{\nu+1},X_\nu;\mathbb{Z})\to\ldots$$

Since $H_{n+1}(X_{\nu+1},X_\nu;\mathbb{Z})=0$ and $H_n(X_{\nu+1},X_\nu;\mathbb{Z})$ is free it follows easily that $H_n(X_\nu;\mathbb{Z})$ is a direct summand in $H_n(X_{\nu+1};\mathbb{Z})$. Hence $H_n(X;\mathbb{Z})=\varinjlim_\nu H_n(X_\nu;Z)$ is free.

b) We assume that $A\subset X$ is rare (i.e. A does not contain any irreducible component of X). Hence $\dim(A)\le n-1$.

From the exact sequence :

$$\ldots\to H_r(A;\mathbb{Z})\to H_r(X;\mathbb{Z})\to H_r(X,A;\mathbb{Z})\to H_{r-1}(A;\mathbb{Z})\to\ldots$$

and a) it follows that $H_r(X,A;\mathbb{Z})=0$ if $r>n$ and $H_n(X,A;Z)$ is free.

c) The general case. Let $\tilde{X}$ be the union of those irreducible components of X not contained in A. Then $\tilde{X}$ is a Stein space and $\tilde{A}=\tilde{X}\cap A$ is a rare analytic subset of $\tilde{X}$. Since $X=\tilde{X}\cup A$ and $\tilde{A}=\tilde{X}\cap A$ we obtain from Lemma 3.3. $H_r(X,A;\mathbb{Z})\cong H_r(\tilde{X},\tilde{A};Z)$ and by b) this concludes the proof.

Proof of Theorem 1

First we remark that given a compact subset $K\subset Y$ we can produce a real analytic (strongly) plurisubharmonic exhaustion function $\varphi: X\to \mathbb{R}$ such that $K\subset\{x\in X/\varphi(x)\leq 1\}\subset Y$. φ can be chosen as a convergent series $\sum|f_j|^2$ where $f_j\in\Gamma(X,\mathcal{O})$ (see e.g. [6]). From this remark it follows that the pair (X,Y) can be exhausted with pairs of compact sets of the form $(X_{\beta_\nu},X_{\alpha_\nu})$ corresponding to triples $(\varphi_\nu,\beta_\nu,\alpha_\nu)$ where $\varphi_\nu: X\to\mathbb{R}$ are (possibly different) real analytic plurisubharmonic exhaustion functions on X and $\alpha_\nu<\beta_\nu$ are real numbers. Taking inductive limit and using Theorem 2 we get $H_r(X,Y;G)=o$ for $r>n$ and any abelian group G. The universal coefficient theorem shows now that $H_n(X,Y;\mathbb{Z})$ is torsion free and the proof of Theorem 1 is complete.

Remark Theorem 1 can be generalized to non-degenerate spaces. Recall that a complex space X is said to be non-degenerate (cf. [1]) if there exists an analytic subset $A\subset X$ (the degeneracy set of X), a Stein space $\tilde{X}$ and a proper holomorphic map $p: X\to\tilde{X}$ such that : i) $\dim_x A>o$ for any $x\in A$ ii) p induces a biholomorphism $X\setminus A\simeq\tilde{X}\setminus\tilde{A}$ (where $\tilde{A}=p(A)$) and $p_*\mathcal{O}_X\simeq\mathcal{O}_{\tilde{X}}$ iii) $\tilde{A}$ is discrete.

In particular such a space is holomorph-convex. If $\tilde{A}$ is finite (equivalently A is compact) X is called 1-convex and A its exceptional set. If X is a non-degenerate space and $Y\subset X$ is an open subset containing A we call (X,Y) a Runge pair if Y is holomorph-convex and the restriction map $\Gamma(X,\mathcal{O})\longrightarrow\Gamma(Y,\mathcal{O})$ has dense image. From Theorem 1 it follows immediately :

Theorem 1' Let X be a non-degenerate n-dimensional complex space and $Y \subset X$ an open subset such that (X,Y) is a Runge pair. Then :

$H_r(X,Y;\mathbb{Z})=0$ for $r>n$ and

$H_n(X,Y;Z)$ has no torsion.

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