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by

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SUFFICIENT CONDITIONS FOR MEMBERSHIP IN THE CLASSES $\mathbb{A}$ AND $\mathbb{A}_{\aleph_0}$

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INTRODUCTION

Let H be a separable, infinite dimensional, complex Hilbert space and let L(H) denote the algebra of all bounded linear operators on H.

This paper deals with some aspects of the theory of dual algebras. Roughly speaking, we shall give some new criteria for membership in the class $\mathbb{A}(H)$ and some others for membership in the class $\mathbb{A}_{\aleph_0}$. These classes of operators (whose definitions are reviewed below) were introduced in [2] and studied in several papers during the past three years (see [3] for an in- depth development of the theory of dual algebras and a bibliography of pertinent articles).

As it was shown in [1] , the class $\mathbb{A}(H)$ figures in the invariant subspace problem for contractions T such that the spectrum of T contains the unit circle $\mathbb{T}$.

On the other hand, much is known about the structure of an operator T in $\mathbb{A}_{\aleph_0}$. Such T is a "universal dilation", is reflexive and has a huge invariant subspace lattice (see [3], Chapters (X, V, VI, IX).

Various criteria for membership in the class $\mathbb{A}_{\aleph_0}$ are known but a few criteria for membership in $\mathbb{A}(H)$ were been formulated.

The content of the paper is the following. In the first section we recall some notations and basic definitions from the theory of dual algebras.

In the second section, a certain growth condition on the resolvent of a contraction is reviewed. It will be shown that this condition ensures the membership in the class $A(H)$. This result improves similar ones obtained in [1] and [5].

In the third section, a new criteria that a dual algebra have property $(A_{\aleph_0})$ will be given. A corollary of this result, which extends Lemma 7.6 from [3] will be presented.

1. Preliminaries

The notation and terminology employed herein agree with that in [3] . Nevertheless , for completeness, we begin by reviewing a few pertinent definitions.

If $T \in L(H)$, the spectrum of T will be denoted by $\sigma(T)$, and the essential spectrum of T by $\sigma_e(T)$. It is well known that L(H) is the dual space of the Banach space (τc) of trace-class operators on H equipped with the trace-norm $\| \ \|_\tau$. This duality is implemented by the bilinear functional

$$\langle T, L\rangle = tr(TL), \ T \in L(H), \ L \in (\tau c)$$

A subalgebra A of L(H) that contains 1_H and is closed in the weak*-topology on L(H) is called a dual algebra. It follows from general principles (cf [4]) that if A is a dual algebra, then A can be identified with the dual space of $Q_A = (\tau c)/{}^\perp A$, where ${}^\perp A$ is the preannihilator in (τc) of A, under the pairing

$$\langle T, [L]\rangle = tr(TL), \quad T \in A \ , \ [L] \in Q_A$$

(Throughout the paper we write $[L]_A$,or simply $[L]$ where no confusion will result, for the coset in Q_A containing the operator $L \in (\tau c)$. It is also easy to see (cf [4]) that the weak* topology that accrues to A by virtue of being the dual space of Q_A is identical with the relative weak* topology that A inherits as a subspace of L(H).

If x and y are vectors in H, then the rank-one operator $x\otimes y$, defined as usual, by $(x\otimes y)(z)=(z,y)x$, z H, belongs to ($\mathcal{C}$) and satisfies $tr(x\otimes y)=(x,y)$. Thus if A is a dual algebra, $[x\otimes y]\in Q_A$. Let $A\subset L(H)$ be a dual algebra and let n be any cardinal number such that $1\leq n\leq\aleph_0$. Then A is said to have property (A_n) provided every nxn system of simultaneous equations of the form

$$[L_{ij}]=[x_i\otimes y_j] \qquad 0\leq i,j<n$$

(where $\{[L_{ij}]\}$ are arbitrary but fixed elements from Q_A) has a solution $\{x_i\}_{0\leq i<n}$, $\{y_i\}_{0\leq i<n}$ consisting of a pair of sequences of vectors from H.

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$, and let $\mathbb{T}=\partial\mathbb{D}$.

A set $\Lambda\subset\mathbb{D}$ is said to be dominating for $\mathbb{T}$ if almost every point of $\mathbb{T}$ is a nontangential limit of a sequence of points from Λ.

The spaces $L^p=L^p(\mathbb{T})$ and $H^p=H^p(\mathbb{T})$, $1\leq p\leq\infty$, are the usual function spaces.

If T is an absolutely continuous contraction in L(H) (i.e. a contraction whose maximal unitary direct summand is either absolutely continuous or acts on the space (0)), we denote by A_T the dual algebra generated by T and we write Q_T for the predual Q_{A_T}.

For such T, as is well-known (cf. [9]), the Sz.-Nagy-Foias functional calculus ϕ_T is a weak* continuous, norm decreasing, algebra homomorphism of H^∞ onto a weak* dense subalgebra of A_T. The class $\mathbb{A}=\mathbb{A}(H)$ is the set of all absolutely continuous contractions T for which ϕ_T is an isometry. If $T\in\mathbb{A}(H)$, then one knows (cf. [4])

that Φ_T is a weak* homomorphism between H^∞ and A_T and that there exists a linear isometry φ_T of Q_T onto L^1/H^1_0 (the predual of H) such that $\Phi = \varphi^*$.

For any cardinal number n satisfying $1 \leq n \leq \aleph_0$, the class $\mathbb{A}_n$ consists of all T in $\mathbb{A}$ for which the dual algebra A_T has property (A_n) (see [3] for various pnoperties of these classes).

2. The class $\mathbb{A}(H)$

In this section, we shall give a sufficient condition for membership in the class $\mathbb{A}(H)$, which improves similar results obtained in [1] and [5] .

The class $\mathbb{A}(H)$ is closely related to the invariant subspace problem for contractions T such that the spectrum $\sigma(T)$ of T contains the unit circle $\mathbb{T}$. Indeed, it was shown in [1] that if T is a contraction in L(H) such that $\sigma(T) \supset \mathbb{T}$, then either T has a nontrivial hyperinvariant subspace or T belongs to $\mathbb{A}$. On the other hand, it is easy to see that operators in class $\mathbb{A}_1$ have nontrivial invariant subspaces.

In [2] , it was conjectured that $\mathbb{A} = \mathbb{A}_1$, and if this is true, then an easy corollary is that every contraction $T \in L(H)$ such that $\sigma(T) \supset \mathbb{T}$ has a nontrivial invariant subspace.Thus it is of some importance to find sufficient conditions that a contraction belongs to $\mathbb{A}$.

If T is a contraction in L(H) and $0 < \theta < 1$, then we put

$$\zeta_\theta(T)=(\mathbb{D}\cap\sigma(T))\cup\{\lambda\in\mathbb{D}\setminus\sigma(T):\theta\|(T-\lambda)^{-1}\|>\frac{1}{1-|\lambda|}\}$$

These sets were introduced in [1], where it was shown that if T is completely nonunitary and if all these sets are dominating for $\mathbb{T}$, then $T\in\mathbb{A}$.

More recently, it was shown in [5] that if there exists some θ, satisfying $0<\theta<1/2$, such that $\zeta_\theta(T)$ is dominating for $\mathbb{T}$, then $T\in\mathbb{A}$. Our improvement of these results is the following:

Theorem 2.1

Suppose T is an absolutely continuous contraction in L(H) such that for some θ satisfying $0<\theta<1$, the set $\zeta_\theta(T)$ is dominating for $\mathbb{T}$. Then $T\in\mathbb{A}$.

Proof.

Fix a nonconstant f in H^∞ such that $\|f\|_\infty=1$ and take a sequence $(\lambda_n)_{n=1}^{\infty}\subset\zeta_\theta(T)$ such that $\lim\limits_{n\to\infty}|f(\lambda_n)|=\|f\|_\infty=1$.

If $\lambda_n\in\sigma(T)\cap\mathbb{D}$ for all $n\geq 1$, then one knows from [6] that $f(\lambda_n)\in\sigma(f(T))$ for all $n\geq1$, hence $1=\lim\limits_n|f(\lambda_n)|\leq\|f(T)\|\leq\|f\|_\infty=1$. Thus we may suppose, by taking a subsequence, that $\lambda_n\in\zeta_\theta(T)\setminus\sigma(T)$ and $f(\lambda_n)\in\mathbb{D}\setminus\sigma(f(T))$, for all $n\geq1$.

It follows from the invariant form of Schwartz's lemma([7], Chapter I) that we have

$$\left|\frac{f(z)-f(\lambda_n)}{1-\overline{f(\lambda_n)}f(z)}\right|\leq\left|\frac{z-\lambda_n}{1-\overline{\lambda}_n z}\right|,$$

for all $n\geq1$ and $z\in\mathbb{D}$.

Hence there exist functions $h_n \in H^\infty$, $\|h_n\|_\infty \leq 1$ such that

$$\frac{f(z)-f(\lambda_n)}{1-\overline{f(\lambda_n)}f(z)} = h_n(z)\ \frac{z-\lambda_n}{1-\overline{\lambda}_n z}\ , \qquad z\in D,\ n\geq 1.$$

Since the functional calculus is a norm-decreasing algebra homomorphism, it follows easy that

$$(f(T)-f(\lambda_n))^{-1}(I-\overline{f(\lambda_n)}f(T))h_n(T)=(T-\lambda_n)^{-1}(I-\overline{\lambda}_n T)$$

By a short calculation (see [9] , p.263), we obtain

$$\begin{aligned}
&\frac{1}{\Theta}<\|(T-\lambda_n)^{-1}\|(1-|\lambda_n|)\leq\\
&\leq \|(T-\lambda_n)^{-1}(I-\overline{\lambda}_n T)\|\leq\\
&\leq \|(f(T)-f(\lambda_n))^{-1}(I-\overline{f(\lambda_n)}f(T))\|\leq\\
&\leq\ \ 1+2(1-|f(\lambda_n)|)\|(f(T)-f(\lambda_n))^{-1}\|
\end{aligned}$$

hence

$$\|(f(T)-f(\lambda_n))^{-1}\|>\frac{1-\Theta}{2\ (1-|f(\lambda_n)|)}\ ,\ \text{for all}\ \ n\geq 1.$$

Thus $\sigma(f(T))\cap\mathbb{T}\neq\emptyset$ and the proof is complete.

Remark. The proof works equally if we replace $L(H)$ by an arbitrary unital Banach algebra $\mathcal{B}$ and $f\longrightarrow f(T)$ by a norm contractive unital homomorphism $\varphi:H^\infty\longrightarrow B$ with the property that for some $0<\Theta<1$, the set:

$$\Gamma_\Theta(\varphi)=(\mathbb{D}\cap\sigma(x))\cup\{\lambda\in\mathbb{D}\setminus\sigma(x);\ \Theta\|(x-\lambda)^{-1}\|>\frac{1}{1-|\lambda|}\}$$

is dominating for $\mathbb{T}$ (Here $x=\varphi(z)$).

Now suppose that T is a given completely non-unitary contraction in $L(H)$, write $D_T=(I-T^*T)^{1/2}$, $D_{T^*}=(I-TT^*)^{1/2}$ and define the subspaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ of H to be the closures of the ranges of D_T and D_{T^*} respectively.

The analytic function Θ_T defined on $\mathbb{D}$ by

$$\Theta_T(\lambda)=(-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T)\big|_{\mathcal{D}_T} \quad , \lambda\in\mathbb{D}$$

satisfies $\|\Theta_T\|_\infty\leq 1$ (cf. [9],p.238) and the contractive analytic function $\{\mathcal{D}_T,\mathcal{D}_{T^*},\Theta_T\}$ is called the characteristic function of T.

Theorem 2.2.

Suppose T is a completely non-unitary contraction in $L(H)$ and $\{\mathcal{D}_T,\mathcal{D}_{T^*},\Theta_T\}$ its characteristic function. If there exists θ, satisfying $0<\theta<1$, such that the following set

$$\zeta_\theta^1(T)=(\mathbb{D}\cap\sigma(T))\cup\{\lambda\in\mathbb{D}\setminus\sigma(T);\theta\|\Theta_T(\lambda)^{-1}\|>1\}$$

is dominating for $\mathbb{T}$, then $T\in\mathbb{A}$.

Proof.

It follows from ([9],p.259) that $\lambda\in\mathbb{D}\setminus\sigma(T)$ if and only if $\Theta_T(\lambda)$ is boundedly invertible, hence the definition of the set $\zeta_\theta^1(T)$ is consistent. Now, if $\lambda\in\zeta_\theta^1(T)\setminus\sigma(T)$ then we have (cf. [9], p.263):

$$\|\Theta_T(\lambda)^{-1}\|=\|(T-\lambda)^{-1}(I-\bar{\lambda}T)\|$$

and the proof goes like in the above theorem.

This last result offers a method to construct various examples of contractions in $\mathbb{A}$, by using the functional model associated with a given contractive analytic function.

3. Sufficient conditions for membership in $A_{\aleph_0}$

In this section, we shall give a new sufficient condition that a given dual algebra A have property $(A_{\aleph_0})$.

The main result is the following:

Theorem 3.1

Suppose $A \subset L(H)$ is a dual algebra and there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of H such that

$$\alpha)\ \sup_{n>N} |(Tx_n, y_n)| = \|T\|, \quad T \in A, \quad N \in \mathbb{N},$$

and

$$\beta)\ \|[x_n \otimes x_m]\|, \|[y_n \otimes y_m]\|, \|[x_n \otimes y_m]\| < \frac{1}{2^{n+m}}$$

Suppose also that $\{[L_{ij}]\}_{i,j \geq 1}$ is a given doubly indexed family of elements of Q_A and $\varepsilon > 0$. Then there exist sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ from H satisfying

$$\langle T, [L_{ij}] \rangle = \langle T, [u_i \otimes v_j] \rangle, \quad T \in A, \quad 1 \leq i, j < \infty$$

If, moreover, $\sum_{i=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty, j \geq 1$, and $\sum_{j=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty, i \geq 1$, then the above sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$

can be chosen to satisfy

$$\| u_i \| < \left(\sum_{j=1}^{\infty} \| [L_{ij}] \|^{1/2} + \varepsilon\right) \qquad , 1 \leq i < \infty$$

and

$$\| v_j \| < \left(\sum_{i=1}^{\infty} \| [L_{ij}] \|^{1/2} + \varepsilon\right) \qquad , 1 \leq j < \infty$$

in particular, A has property $(A_{\aleph_0})$.

Before proving this theorem, we need two lemmas that are similar with ([8], Lemmas 3.9 and 3.10). If $\mathcal{S} \subset H$, we denote by span $(\mathcal{S})$ the set of all finite linear combinations of elements from $\mathcal{S}$.

Lemma 3.2.

Suppose $A \subset L(H)$ is a dual algebra with properties α) and β). Let $N > 0$ and suppose $u_i \in \operatorname{span}\{x_n;\ n \geq 1\}$, $v_j \in \operatorname{span}\{y_n;\ n \geq 1\}$ and $\{[L_{ij}]\}_{1 \leq i,j \leq N} \subset Q_A$.

Assume that

$$\| [u_i \otimes v_j] - [L_{ij}] \| < \varepsilon_{ij} \quad , 1 \leq i,j \leq N$$

Let $1 \leq i_0, j_0 \leq N$ and let $0 < \delta < \varepsilon_{i_0 j_0}$.

Then there exist $u'_{i_0} \in \operatorname{span}\{x_n;\ n \geq 1\}$ and $v'_{j_0} \in \operatorname{span}\{y_n;\ n \geq 1\}$ such that

A) $\| [u'_{i_0} \otimes v'_{j_0}] - [L_{i_0 j_0}] \| < \delta$

B) $\| [u_i \otimes v_{j_0}] - [L_{i j_0}] \| < \varepsilon_{i j_0}$ for each i

C) $\| [u'_{i_0} \otimes v_j] - [L_{i_0 j}] \| < \varepsilon_{i_0 j}$ for each j

D) $\| u'_{i_0} - u_{i_0} \|^2 < \varepsilon_{i_0 j_0}$

E) $\| v'_{j_0} - v_{j_0} \|^2 < \varepsilon_{i_0 j_0}$

Proof

Let $[K]=[L]-[u_{i_o}\otimes v_{j_o}]$. Set $d=\|[K]\|$.

We may assume that $d>0$ since otherwise we can simply take

$u'_{i_o}=u_{i_o}$ and $v'_{j_o}=v_{j_o}$.

Let $0<\rho<\min\limits_{1\le i,j\le N}\left(\frac{\delta}{4};\varepsilon_{ij}-\|[L_{ij}]-[u_i\otimes v_j]\|\right)$

It follows from property β) of A that we may choose $m\in Z^+$ large enough such that

$$1<2^m\rho \quad\text{and}\quad \max_{1\le i,j\le N}\left(\|[x_n\otimes v_j]\|,\ \|[u_i\otimes y_n]\|\right)<\frac{\rho}{2^{n-m}}$$

for all $n>m$.

Since A has also property α), it follows from ([3], Proposition 1.21) that the closed absolutely convex hull of the set $\{[x_n\otimes y_n];\ n>m\}$ equals the closed unit ball in Q_A. We may therefore choose $k\in Z^+$, $\alpha_1,\ldots,\alpha_k\in\mathbb{C}$ such that $\sum\limits_{i=1}^{k}|\alpha|_i\le 1$ and

$$\|d^{-1}[K]-\sum_{i=1}^{k}\alpha_i[x_{m+i}\otimes y_{m+i}]\|<\frac{\delta}{4d}.$$

Choose $\gamma_1,\ldots,\gamma_k\in\mathbb{C}$ such that $\gamma_i^2=\alpha_i d$, $1\le i\le k$.

Set $s=\sum\limits_{i=1}^{k}\gamma_i x_{m+i}$ and $t=\sum\limits_{i=1}^{k}\bar{\gamma}_i y_{m+i}$

We claim that we may take

$$u'_{i_o}=u_{i_o}+s \quad\text{and}\quad v'_{j_o}=v_{j_o}+t.$$

First, observe that

$$\|(u_{i_o}+s)-u_{i_o}\|^2=\|\sum_{i=1}^{k}\gamma_i x_{m+i}\|^2\leq$$

$$\leq\sum_{i=1}^{k}|\gamma_i|^2+\sum_{\substack{i,j=1\\ i\neq j}}^{k}|\gamma_i\gamma_j|\,|(x_{m+i},x_{m+j})|<d+\rho<\varepsilon_{i_oj_o}$$

and likewise $\|(v_{j_o}+t)-v_{j_o}\|^2<\varepsilon_{i_oj_o}$

Next, we show that we may satisfy condition A. We have:

$$\|[(u_{i_o}+s)\otimes(v_{j_o}+t)]-[L_{i_oj_o}]\|\leq$$

$$\leq\|[u_{i_o}\otimes v_{j_o}]-[L_{i_oj_o}]+[s\otimes t]\|+\|[s\otimes v_{j_o}]\|+\|[u_{i_o}\otimes t]\|\leq$$

$$\leq\|[\sum_{i=1}^{k}\gamma_i x_{m+i}\otimes\sum_{j=1}^{k}\gamma_j y_{m+j}]-[K]\|+\|[s\otimes v_{j_o}]\|+\|[u_{i_o}\otimes t]\|\leq$$

$$\leq\|\sum_{i=1}^{k}\gamma_i^2[x_{m+i}\otimes y_{m+i}]-[K]\|+\sum_{\substack{i,j=1\\ i\neq j}}^{k}|\gamma_i\gamma_j|\,\|[x_{m+i}\otimes y_{m+j}]\|+$$

$$+\sum_{i=1}^{k}|\gamma_i|\,\|[x_{m+i}\otimes v_{j_o}]\|+\sum_{i=1}^{k}|\gamma_i|\,\|[u_{i_o}\otimes y_{m+i}]\|<$$

$$<\frac{\delta}{4}+3\rho<\delta$$

Finally, note that

$$\|[(u_{i_o}+s)\otimes v_j]-[L_{i_oj}]\|\leq\|[u_{i_o}\otimes v_j]-[L_{i_oj}]\|+$$

$$+\sum_{i=1}^{k}|\gamma_i|\,\|[x_{m+i}\otimes v_j]\|<\varepsilon_{i_oj}\quad\text{for each } j$$

and similarly

$$\|[u_i\otimes(v_{j_o}+t)]-[L_{ij_o}]\|<\varepsilon_{ij_o}\qquad\text{for each } i.$$

The proof is complete.

By N^2 succesive applications of the preceding lemma we immediately obtain the following result:

Lemma 3.3.

Let $A \subset L(H)$ be a dual algebra which satisfies the hypothesis of Theorem 4.1.

Let $N > 0$, $u_i \in \operatorname{span}\{x_n ; n \geq 1\}$, $v_j \in \operatorname{span}\{y_n : n \geq 1\}$ and $\{[L_{ij}]\}_{1 \leq i,j \leq N} \subset Q_A$.

Assmume that

$$\|[u_i \otimes v_j] - [L_{ij}]\| < \varepsilon_{ij} \qquad 1 \leq i, j \leq N$$

and let $0 < \delta_{ij} < \varepsilon_{ij}$ for $1 \leq i, j \leq N$.

Then there exist $u_i' \in \operatorname{span}\{x_n ; n \geq 1\}$ and $v_j' \in \operatorname{span}\{y_n ; n \geq 1\}$ such that

A') $\|[u_i' \otimes v_j'] - [L_{ij}]\| < \delta_{ij}$ for all i, j

B') $\|u_i' - u_i\| < \sum_{j=1}^{N} (\varepsilon_{ij})^{1/2}$ for all i

C') $\|v_j' - v_j\| < \sum_{i=1}^{N} (\varepsilon_{ij})^{1/2}$ for all j

Proof of Theorem 3.1

First, we suppose that $\{[L_{ij}]\} \subset Q_A$, $i, j \geq 1$ satisfy

$$\sum_{i=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty \quad \text{for all } j \geq 1$$

and

$$\sum_{j=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty \quad \text{for all } i \geq 1.$$

Set $u_i^{(o)}=v_j^{(o)}=0$ for all $i,j\geq 1$.

We choose by induction sequences $\{u_i^{(k)}\}_{k=0}^{\infty}$ and $\{v_j^{(k)}\}_{k=0}^{\infty}$ $(i,j\geq 1)$ in H, such that

a) $\| [u_i^{(k)} \otimes v_j^{(k)}] - [L_{ij}]\| < \dfrac{1}{k^2.2^{2(k+2)}}$ $\quad 1\leq i,j\leq k$

b) $u_i^{(k)} = v_j^{(k)}=0$ for $i>k$ and $j>k$

c) $\| u_i^{(k)}-u_i^{(k-1)}\| < \|[L_{ik}]\|^{1/2} + \dfrac{1}{2^k}$ $\quad$ for $1\leq i<k$

d) $\| u_k^{(k)}-u_k^{(k-1)}\| < \sum_{j=1}^{k} \|[L_{kj}]\|^{1/2} + \dfrac{1}{2^k}$

e) $\| v_j^{(k)}-v_j^{(k-1)}\| < \|[L_{kj}]\|^{1/2} + \dfrac{1}{2^k}$ $\quad$ for $1\leq j<k$

f) $\| v_k^{(k)}-v_k^{(k-1)}\| < \sum_{i=1}^{k} \|[L_{ik}]\|^{1/2} + \dfrac{1}{2^k}$

Suppose for the moment, that these sequences $\{u_i^{(k)}\}$ and $\{v_j^{(k)}\}$ $(i,j\geq 1)$ have been constructed to satisfy the above inequalities.

Let $i\geq 1$.Then:

$$\sum_{k=1}^{\infty}\| u_i^{(k)}-u_i^{(k-1)}\| = \sum_{k=1}^{i-1}\| u_i^{(k)}-u_i^{(k-1)}\| + \| u_i^{(i-1)} - u_i^{(i)}\| +$$

$$\sum_{k=i+1}^{\infty}\| u_i^{(k)}-u_i^{(k-1)}\| < 0+\sum_{j=1}^{i}\|[L_{ij}]\|^{1/2} + \frac{1}{2^i} +$$

$$+\sum_{k=i+1}^{\infty}\left(\|[L_{ik}]\|^{1/2} + \frac{1}{2^k}\right) = \sum_{j=1}^{\infty}\|[L_{ij}]\|^{1/2} + \frac{1}{2^{i-1}}$$

It follows that the sequence $\{u_i^{(k)}\}_{k=0}^{\infty}$ is norm convergent to some $u_i\in H$, with

$$\| u_i \| < \sum_{j=1}^{\infty} \| [L_{ij}] \|^{1/2} + \frac{1}{2^{i-1}}$$

Similarly, for each j, the sequence $\{v_j^{(k)}\}_{k=0}^{\infty}$ converges in norm to some $v_j \in H$ and $\| v_j \| < \sum_{i=1}^{\infty} \| [L_{ij}] \|^{1/2}$ +

Finally, from a), we deduce that

$$[u_i \otimes v_j] = [L_{ij}] \quad i,j \geqslant 1$$

Thus, to complete the proof in this case, it suffices to construct, by induction, the sequences $\{u_i^{(k)}\}_{k=0}^{\infty}$ and $\{v_j^{(k)}\}_{k=0}^{\infty}$.

Let $n \geqslant 0$. Suppose that $\{u_i^{(o)}, \dots, u_i^{(n)}\}$ and $\{v_j^{(o)}, \dots, v_j^{(n)}\}$ have been constructed for every $i,j \geqslant 1$ to satisfy a)-f) for the appropiate values of k, and the further induction hypothesis that each $u_i^{(k)}$ for $0 \leqslant k \leqslant n$ and $i \geqslant 1$ belongs to span$\{x_n : n \geqslant 1\}$ (i.e. is some finite linear combination of vectors in the sequence $\{x_n\}$ and similarly for the $v_j^{(k)}$ relative to the sequence $\{y_n\}$.

By lemma 3.3 (with $N=n+1$), there exist $u_i^{(n+1)} \in$ span$\{x_k, k \geqslant 1\}$ and $v_j^{(n+1)} \in$ span$\{y_k, k \geqslant 1\}$ $(1 \leqslant i, j \leqslant n+1)$, such that

- $\| [u_i^{(n+1)} \otimes v_j^{(n+1)}] - [L_{ij}] \| < \dfrac{1}{(n+1)^2 . 2^{2(n+3)}}$;

- $\| u_i^{(n+1)} - u_i^{(n)} \| < \displaystyle\sum_{j=1}^{n+1} \| [u_i^{(n)} \otimes v_j^{(n)}] - [L_{ij}] \|^{1/2} + \frac{1}{2^{n+2}}$

- $\| v_j^{(n+1)} - v_j^{(n)} \| < \displaystyle\sum_{i=1}^{n+1} \| [u_i^{(n)} \otimes v_j^{(n)}] - [L_{ij}] \|^{1/2} + \frac{1}{2^{n+2}}$

Let $1 \leqslant i \leqslant n+1$. Then we have:

$$\| u_i^{(n+1)} - u_i^{(n)} \| \leq \sum_{j=1}^{n+1} \| [u_i^{(n)} \otimes v_j^{(n)}] - [L_{ij}] \|^{1/2} + \frac{1}{2^{n+2}} \leq$$

$$\leq \sum_{j=1}^{n} \frac{1}{n.2^{n+2}} + \| [L_{in+1}] \|^{1/2} + \frac{1}{2^{n+2}} \leq \| [L_{in+1}] \|^{1/2} + \frac{1}{2^{n+1}}$$

and similarly

$$\| v_j^{(n+1)} - v_j^{(n)} \| < \| [L_{n+1j}] \|^{1/2} + \frac{1}{2^{n+1}} \quad \text{for} \quad 1 \leq j < n+1$$

Since $u_{n+1}^{(n)} = v_{n+1}^{(n)} = 0$, we obtain

$$\| u_{n+1}^{(n+1)} - u_{n+1}^{(n)} \| < \sum_{j=1}^{n+1} \| [u_{n+1}^{(n)} \otimes v_j^{(n)}] - [L_{n+1j}] \|^{1/2} + \frac{1}{2^{n+2}} \leq$$

$$\leq \sum_{j=1}^{n+1} \| [L_{n+1j}] \|^{1/2} + \frac{1}{2^{n+1}}$$

and

$$\| v_{n+1}^{(n+1)} - v_{n+1}^{(n)} \| < \sum_{i=1}^{n+1} \| [L_{in+1}] \|^{1/2} + \frac{1}{2^{n+1}} .$$

Now, set $u_i^{(n+1)} = v_j^{(n+1)} = 0$ for all $i, j > n+1$. Thus we have constructed by induction the required sequences.

If $\{[L_{ij}]\}_{i,j \geq 1} \subset Q_A$ do not satisfy the above conditions, we can proceed like in the proof of ([8], Theorem 3.14) and the proof is complete.

<u>Corollary 4.4.</u>

Suppose $T \in A$ and there exist sequences $\{x_n\}_{n=1}^{\infty}$ and

$\{y_n\}_{n=1}^{\infty}$ in the unit ball of H such that

$$\alpha_1)\quad \sup_{n>N} |(f(T)x_n, y_n)| = \|f\|_\infty \;,\; f\in H^\infty,\; N\in\mathbb{N}$$

and

$$\beta_1)\; \|[x_n \otimes x_m]\|,\; \|[y_n \otimes y_m]\|,\; \|[x_n \otimes y_m]\| < \frac{1}{2^{n+m}}$$

for all n,m N, $n\neq m$.

Then $T\in A_{\aleph_0}$.

Proof. Just apply Theorem 3.1, with $A=A_T$.

Remark. Corollary 3.4 generalizes a similar result obtained in [3] (Lemma 7.6).

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