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by

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ON DUALITY AND STABILITY OF PARAMETRIZED OPTIMIZATION PROBLEMS AND RELATED TOPICS

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Abstract. We present a survey of some of our contributions to the theories of duality and stability of parametrized optimization problems, conjugations of functionals, and generalized convexity. We place the survey in a historical framework, giving an introduction to some of the ideas in these topics and showing also some related developments.

1. In his monograph of 1976, at the end of the introduction to the chapter on duality, Avriel ([3], p.106) wrote: "Little can be said about duality in general nonconvex programming, for this subject is at about the same stage as convex duality was in the early 1950 s. A few works on the subject exist, such as..., but results are not very satisfactory...". Also, in the introduction of their paper of 1980 on duality, Jefferson and Scott ([36],p.519) wrote: "An essential concept in the analysis of mathematical programs is the idea of duality. This has furnished new approaches and interpretational insights and has provided the basis for many powerful algorithms. To date, most of this theory has concentrated on convex mathematical programs, whereas the vast spectrum of nonconvex programs has remained relatively untouched...". On the other hand, in the Abstract of his paper of 1977, Balder [4] wrote: "By an effective extension of the conjugate function concept a general framework for duality-stability relations in nonconvex optimization problems can be studied...".

The aim of the present paper is to present a survey of some of our contributions to the above mentioned topics, i.e., to duality and stability theory of parametrized optimization problems, and to the theory of conjugations of functionals. We shall also mention some of our related results on generalized convexity. We shall place this survey in a historical framework, giving an introduction to some of the ideas in these topics, and showing also some related developments (naturally, from a subjective point of view).

In order to keep the paper within a reasonable size, we shall simplify some parts of the presentation (where it would have been too technical to give more details). Also, we shall omit deliberately many related subjects, e.g. Lagrangian functionals (for their general definition, see [82], definition 1.1), minimax theory, ε-subdifferentials, characterizations of solutions, local optima, etc., and some particular cases, e.g., semi-infinite optimization, differentiable optimization, discrete optimization, etc. We shall concentrate especially on explaining the development of various concepts, methods, and their connections, rather than on stating or proving theorems. Instead of aiming at completeness of bibliography, we shall only give some samples of references. We hope that the present paper will offer a picture of some recent developments and will stimulate further research.

2. Let F be a locally convex space, $G(\neq\emptyset)$ a subset of F and $h:F\to\overline{R}=[-\infty,+\infty]$, and let us consider the (global, scalar) "primal" infimization problem

$$(P) \qquad \alpha=\inf h(G)\left[=\inf_{y\in G} h(y)\right], \tag{2.1}$$

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embedded into a family of parametrized (or "perturbed") infimization problems

$$(P_x) \qquad f(x)=\inf_{y\in F} p(y, x) \qquad (x\in X), \tag{2.2}$$

where the parameter set X is a locally convex space and where the parametrization functional $p:F\times X\to\overline{R}$ satisfies

$$p(y, 0)=\begin{cases} h(y) & \text{if } y\in G \\ +\infty & \text{if } y\in F\setminus G; \end{cases} \tag{2.3}$$

by (2.1)-(2.3), we have

$$\alpha=\inf_{y\in F} p(y, 0)=f(0). \tag{2.4}$$

The functional $f:X\to\overline{R}$ of (2.2), called the optimal value (or, the "primal", or the "marginal") functional, is of great importance, since it shows how the optimal value of problem (P_x) varies when the parameter x varies. Also, the properties of the functionals p, f and $h+\chi_G$ are intimately related, where χ_G denotes the indicator functional of the set G (i.e., $\chi_G(y)=0$ for $y\in G$ and $=+\infty$ for $y\in F\setminus G$), and where $-\infty+\infty=+\infty$.

There also exist some other parametrization schemes. For example, instead of perturbing the objective functional h of (P), by (2.2), one can perturb the constraint set G of (P), embedding (P) into the family of problems

$$(P_x) \qquad f(x)=\inf_{y\in\Gamma(x)} h(y) \qquad (x\in X), \tag{2.5}$$

where X is a locally convex space and $\Gamma:X\to 2^F$ (where 2^F denotes the collection of all subsets of F) is a multifunction, satisfying

$$\Gamma(0)=G. \tag{2.6}$$

However, this scheme turns out to be a particular case of (2.2), by taking the "natural perturbation functional" ([67], [68])

$$p(y, x)=\begin{cases} h(y) & \text{if } y\in\Gamma(x) \\ +\infty & \text{if } y\in F\setminus\Gamma(x). \end{cases} \tag{2.7}$$

Furthermore, one can perturb both h and G, embedding (P) into the family of problems

$$(P_x) \qquad f(x)=\inf_{y\in\Gamma(x)} p(y, x) \qquad (x\in X), \tag{2.8}$$

but this turns out to be a particular case of (2.2), by taking

$$p'(y, x)=\begin{cases} p(y, x) & \text{if } y\in\Gamma(x) \\ +\infty & \text{if } y\in F\setminus\Gamma(x); \end{cases} \tag{2.9}$$

for some further relations between these schemes, see [68] and [77]. In the sequel we shall consider only the embedding scheme (2.2)-(2.4). For a more general "semi-embedding" scheme, see [75], §6.

3. The Lagrangian dual problem to (P) relative to the parametrization (X, p), in the sense of Rockafellar [57] (see also [28], [55], [56] and [37]) is, by definition, the supremization problem

$$(Q) \qquad \beta=\sup \lambda(X^*), \tag{3.1}$$

where X^* is the conjugate space of X, endowed with the weak* topology, and

$$\lambda(w)= \inf_{(y,x)\in F\times X} \{p(y,x)-w(x)\} \qquad (w\in X^*). \tag{3.2}$$

Actually, in the formulation of [57], there is $+w(x)$ instead of $-w(x)$ in λ of (3.2), which yields the same value for β of (3.1), but we shall find it more convenient to use (3.2) above (following e.g. [19]). This definition of a dual problem to (P) has turned out to be very useful, since it has yielded as particular cases (i.e., for suitable choices of (X, p)), the "usual" dual problems to various concrete problems (P) (defined, initially, in a direct way, without perturbations), as well as some "new" dual problems to them (see e.g. [57]). For an example of recovering the "usual" dual problems with this method, let us consider the convex programming problem

$$(P) \qquad \alpha = \inf_{\substack{y \in R^n \\ u(y) \leq 0}} h(y), \tag{3.3}$$

where $u: R^n \to R^m$ and $h: R^n \to \bar{R}$ are convex and $\leq$ is understood in the sense of the natural partial order of R^m; this is the particular case $F = R^n$, $G = \{y \in R^n \mid u(y) \leq 0\}$, of (2.1). Then, taking $X = R^m$ and

$$p(y, x) = \begin{cases} h(y) & \text{if } y \in R^n, x \in R^m, u(y) \leq x \\ +\infty & \text{if } y \in R^n, x \in R^m, u(y) \not\leq x, \end{cases} \tag{3.4}$$

one can compute (see e.g. [57], p.23 or [19], p.64) that λ of (3.2) becomes the "usual" dual objective functional

$$\lambda(w) = \begin{cases} \inf_{y \in F} \{h(y) + w(u(y))\} & \text{if } w \in (R^m)^*, w \geq 0 \\ -\infty & \text{if } w \in (R^m)^*, w \not\geq 0, \end{cases} \tag{3.5}$$

where $w \geq 0$ means that $w(x) \geq 0$ for all $x \in X = R^m$, $x \geq 0$. Note also that f of (2.2) becomes now

$$f(x) = \inf_{\substack{y \in R^n \\ u(y) \leq x}} h(y) \qquad (x \in X = R^m). \tag{3.6}$$

4. A major observation has been that λ and β of (3.2), (3.1), can be expressed with the aid of (Fenchel) conjugation. We recall that the conjugate of a functional $\varphi: X \to \bar{R}$ is the functional $\varphi^*: X^* \to \bar{R}$ defined by

$$\varphi^*(w) = \sup_{x \in X} \{w(x) - \varphi(x)\} \qquad (w \in X^*); \tag{4.1}$$

the second conjugate of φ is, by definition, $\varphi^{**} = (\varphi^*)^*: X \to \bar{R}$, i.e.,

$$\varphi^{**}(x) = \sup_{w \in X^*} \{w(x) - \varphi^*(w)\} \qquad (x \in X). \tag{4.2}$$

It is easy to compute (see e.g. [57], [19]) that

$$\lambda(w) = -p^*(0, w) \qquad (w \in X^*), \tag{4.3}$$

where we use the canonical identification $F^* \times X^* \cong (F \times X)^*$, given by

$$(v, w)(y, x) = v(y) + w(x) \qquad ((v, w) \in F^* \times X^*, (y, x) \in F \times X); \tag{4.4}$$

also, by (3.2), (2.2) and (4.1), we have

$$\lambda = -f^*, \tag{4.5}$$

whence, by (3.1), (4.1) and (4.2), we obtain

$$\beta = \sup_{w \in X^*} \inf_{x \in X} \{f(x) - w(x)\} = \sup(-f^*)(X^*) = f^{**}(0). \tag{4.6}$$

The importance of the above observations lies in the fact that they permit to use the well-developed machinery of Fenchel conjugation to study the dual problems (3.1), (3.2).

For example, by (4.5), λ is always concave and w*-upper semi-continuous (this also follows from (3.2)). Furthermore, since for any $\varphi:X\to\overline{R}$ there holds $\varphi\geq\varphi^{**}$, from (2.4) and (4.6) we obtain

$$\alpha\geq\beta \tag{4.7}$$

(this follows also directly from (2.4), (3.1) and (3.2)). Also, for any $\varphi:X\to\overline{R}$ we have (see [23] and formula (11.13) below)

$$\varphi^{**}=\varphi_{\overline{co}}\ , \tag{4.8}$$

the closed convex hull of φ (i.e., the greatest closed convex functional $\leq\varphi$; we recall that a functional $\psi:X\to\overline{R}$ is said to be "closed", if either ψ is lower semi-continuous, nowhere having the value $-\infty$, or ψ is the constant functional $-\infty$). Hence, by (4.6), we obtain

$$\beta=f_{\overline{co}}(0). \tag{4.9}$$

Formulae (4.5) and (4.9) are called in [57], p.19, "the central theorem about dual problems". Indeed, let us recall that the most useful cases are when weak duality holds (i.e., $\alpha=\beta$), or, when strong duality holds (that is, $\alpha=\beta$ and there exists a "solution" of the dual problem (Q) of (3.1), (3.2), i.e., a functional $w_o\in X^*$ such that $\lambda(w_o)=\sup\lambda(X^*)=\beta$; in other words, we have $\alpha=\beta$, with sup replaced by max in (3.1)), since then often the "solutions" of the primal problem (P) (i.e., the elements $g_o\in G$ such that $h(g_o)=\inf h(G)$) can be found via the dual problem (Q) (see e.g. [57], pp.4-5). Now, from (2.4) and (4.9) we see that weak duality $\alpha=\beta$ is equivalent to the "stability" relation $f(0)=f_{\overline{co}}(0)$; hence, in particular, when p of (2.2), (2.3) is convex and $\alpha=f(0)$ is finite, we have $\alpha=\beta$ if and only if f is lower semi-continuous at 0 (see e.g. [19], p.50, proposition 2.1). Also, it is known (see e.g. [57], theorem 16 or [19], p.50, proposition 2.2) that strong duality holds if and only if the subdifferential $\partial f(0)$ of f at 0 is non-empty, and then $\partial f(0)$ coincides with the set of all solutions w_o of the dual problem (Q). In the sequel, for brevity, we shall concentrate only on relations between weak duality $\alpha=\beta$ and stability, and we shall not mention the corresponding results for strong duality, involving various concepts of subdifferentials (e.g.,"quasi-subdifferential"[34], [93], "pseudo-subdifferential" [69], "semi-subddiferential" [72], [75], etc.) of f at 0.

The equalities (2.4), (4.9) are also useful when (weak or strong) duality does not hold, since they lead to formulae for the evaluation of the "duality gap"

$$\gamma=\alpha-\beta; \tag{4.10}$$

for results of this type, see e.g. [2].

5. As shown e.g. by the above mentioned characterization of weak duality $\alpha=\beta$, the dual problems (3.1), (3.2) are useful especially in the convex case. However, the assumption of convexity is too restrictive; for example, in mathematical economics, one often deals with infimization of quasi-convex functionals. Therefore, it has been necessary to develop more general concepts of dual problems.

For problem (3.3), with $u:R^n\to R^m$ convex and $h:R^n\to R$ quasi-convex, Luenberger [41] has considered the dual problem (3.1), with the dual objective functional $\lambda:(R^m)^*\to\overline{R}$ defined by

$$\lambda(w)=\begin{cases}\inf\limits_{\substack{y\in R^n\\ w(u(y))\le 0}} h(y) & \text{if } w\in(R^m)^*,\ w\ge 0\\ -\infty \text{ if } w\in(R^m)^*,\ w\not\ge 0,\end{cases} \tag{5.1}$$

and has used it to construct a theory which parallels the results on the dual problem (3.1), (3.5) for convex h. The main difference between (3.5) and (5.1) is that in (3.5) the "penalty term" $w(u(y))$ is added to the objective functional h, in order to "compensate" that the constraint set of (3.5) is F (instead of $G=\{y\in R^n|u(y)\le 0\}$), while in (5.1), $w(u(y))$ is used to form new constraint sets

$$\{y\in R^n|w(u(y))\le 0\} \qquad (w\in(R^m)^*,\ w\ge 0), \tag{5.2}$$

for the unchanged objective functional h. Also [41], λ of (5.1) is quasi-concave and upper semi-continuous. The constraint sets (5.2) have been introduced initially by Glover [29] for 0-1 integer programming and have been called (see e.g. [29], [30], [33]) surrogate constraint sets; also, problem (3.1), (5.1) is called a surrogate dual problem to (P) of (3.3). Note that, in the above case, the sets (5.2) are convex and they contain the initial constraint set $\{y\in R^n|u(y)\le 0\}$. More general surrogate dual problems, involving surrogate constraint sets and penalizations in the objective functional, have been introduced in [32].

Greenberg and Pierskalla [33] have observed the following obvious connection between the Lagrangian dual problem (3.1), (3.5) and the surrogate dual problem (3.1), (5.1), to (P) of (3.3): we have

$$\begin{aligned}\alpha\ge\beta_{surr}&=\sup_{\substack{w\in(R^m)^*\\ w\ge 0}}\ \inf_{\substack{y\in R^n\\ w(u(y))\le 0}} h(y)\ge\\ \ge\beta_{Lagr}&=\sup_{\substack{w\in(R^m)^*\\ w\ge 0}}\ \inf_{y\in R^n}\{h(y)+w(u(y))\};\end{aligned} \tag{5.3}$$

hence, for the corresponding duality gaps, we have

$$0\le\gamma_{surr}=\alpha-\beta_{surr}\le\gamma_{Lagr}=\alpha-\beta_{Lagr}, \tag{5.4}$$

and thus, in particular, the equality $\alpha=\beta_{surr}$ holds for a larger class of problems (3.3), than the equality $\alpha=\beta_{Lagr}$. For evaluations of the "surrogate duality gap" γ_{surr} of (5.4), see e.g. [15].

Let us note that "surrogate duality" is useful even for some convex problems for which we have $\alpha\ne\beta_{Lagr}$ (whence also $\alpha=\beta_{surr}$). Indeed, for example, if F is a normed linear space, G a convex subset of F, $y_0\in F\setminus\overline{G}$ (where $\overline{G}$ is the closure of G), and

$$h(y)=\|y_0-y\| \qquad (y\in F) \tag{5.5}$$

(which is a finite, convex, continuous functional), then (P) of (2.1) becomes

$$(P)\qquad \alpha=\inf_{y\in G}\|y_0-y\|=\operatorname{dist}(y_0,G), \tag{5.6}$$

for which it is known that

$$\alpha=\max_{D\in\mathcal{D}_{G,y_0}}\operatorname{dist}(y_0,D)=\max_{D\in\mathcal{D}_{G,y_0}}\inf h(D), \tag{5.7}$$

where $\mathcal{D}_{G,y_o}$ denotes the collection of all support half-spaces $D=\{y\in F \mid w(y)\geq \inf w(G)\}\supset G$ $(0\neq w\in F^*)$, with $y_o\notin D$. Formula (5.7), used in approximation theory, may be regarded as a formula of "surrogate duality" $\alpha=\beta_{surr}$, with surrogate constraint sets $D\in\mathcal{D}_{G,y_o}$ and dual variables w (see e.g.[77] and the references therein).

Surrogate dual problems are also convenient for computations; for some recent results in this direction, see e.g. [18], [61].

6. Due to the importance of Fenchel conjugation for the study of Lagrangian duality, Greenberg and Pierskalla [34] have introduced a new concept of "quasi-conjugation", as a tool for the study of surrogate duality.

For any locally convex space X, and any $\varphi:X\to\bar{R}$ and $\nu\in R$, the first ν-quasi-conjugate of φ is the functional $\varphi^{\gamma}_{\nu}:X^*\to\bar{R}$ defined [34] by

$$\varphi^{\gamma}_{\nu}(w)=\nu-\inf_{\substack{x\in X\\ w(x)\geq\nu}}\varphi(x) \qquad (w\in X^*), \tag{6.1}$$

the second ν-quasi-conjugate of φ is the functional $(\varphi^{\gamma}_{\nu})^{\gamma}_{\nu}:X\to\bar{R}$ defined [34] by

$$(\varphi^{\gamma}_{\nu})^{\gamma}_{\nu}(x)=\nu-\inf_{\substack{w\in X^*\\ w(x)\geq\nu}}\varphi^{\gamma}_{\nu}(w) \qquad (x\in X), \tag{6.2}$$

and the normalized second quasi-conjugate of φ is defined [34] by

$$\varphi^{\gamma\gamma}=\sup_{\nu\in R}\,(\varphi^{\gamma}_{\nu})^{\gamma}_{\nu}:X\to\bar{R}. \tag{6.3}$$

Greenberg and Pierskalla [34] have shown, among other results, that for any $\varphi:X\to\bar{R}$ we have

$$\varphi^*=\sup_{\nu\in R}\varphi^{\gamma}_{\nu}, \qquad \varphi\geq\varphi^{\gamma\gamma}\geq\varphi^{**}, \tag{6.4}$$

where φ^*, φ^{**} are the Fenchel conjugates (4.1), (4.2), and that

$$\beta_{surr}=f^{\gamma\gamma}(0), \tag{6.5}$$

where β_{surr} and f are those of (5.3) and (3.6) respectively.

Let us also mention that a more symmetric approach to "quasi-convex conjugation", and applications to duality in quasi-convex optimization, have been given by Passy and Prisman ([51], [52]); for a different approach to conjugation and quasi-convex duality, see also Flachs [27]. In the sequel we shall consider only the quasi-conjugates (6.1)-(6.3).

7. The construction of a duality theory for quasi-convex infimization has been continued by Crouzeix ([9]-[11]), who has replaced problem (P) of (3.3) by the general primal problem (P) of (2.1), embedded into a parametrized family (2.2) (instead of (3.6)), with the aid of an arbitrary $p:F\times X\to\bar{R}$ satisfying (2.3), and has defined the surrogate dual problem relative to the parametrization (X, p), as the supremization problem (Q) of (3.1), with

$$\lambda(w)=\inf_{\substack{(y,x)\in F\times X\\ w(x)\geq 0}} p(y,x) \qquad (w\in X^*); \tag{7.1}$$

the main difference between the λ's of (3.2) and (7.1) is similar to the one between (3.5) and (5.1), mentioned after (5.1). Taking, in particular, problem (P) of (3.3), embedded into the parametrized family (3.6), it is easy to compute [9] that λ of (7.1) reduces to (5.1); on the other hand, for the best approximation problem (P) of (5.6), in a normed linear space F, taking X=F and $p:F\times F\to\overline{R}$ defined by

$$p(y, x)=\begin{cases}\|y_o-y\| & \text{if } y\in G+x\\ +\infty & \text{if } y\notin G+x,\end{cases} \qquad (7.2)$$

one can compute (see e.g. [74], theorem 3.4, applied to h of (5.5) above), that β of (3.1), with λ of (7.1), becomes the right hand side of (5.7), with max replaced by sup.

Furthermore, Crouzeix [9] has shown that λ and β of (7.1), (3.1) can be expressed with the aid of the Greenberg-Pierskalla quasi-conjugates (similarly to the case of λ and β of (3.2), (3.1) and Fenchel conjugates), namely,

$$\lambda(w)=-p^{\gamma}_{(y_o,0)}(0, w)=-f_o^{\gamma}(w) \qquad (w\in X^*), \qquad (7.3)$$

$$\beta=\sup_{w\in X^*}\ \inf_{\substack{x\in X\\ w(x)\geq 0}} f(x)=\sup(-f_o^{\gamma})(X^*)=f^{\gamma\gamma}(0), \qquad (7.4)$$

where $y_o\in F$ is arbitrary. Hence, λ of (7.1) is always quasi-concave and w*-upper semi-continuous, and the inequalities (5.4) remain valid in this general case (by (6.4)).

8. The next natural question has been whether weak duality $\alpha=\beta$, with β of (7.4), is equivalent to a "stability" property of problem (P) relative to the parametrization (X, p), i.e., whether $\alpha=\beta$ holds if and only if f(0) coincides with the value of a "hull" of f at 0 (following Moreau [47], p.149, by a hull of f is meant the functional - if it exists - which, among a given set of functionals, is the greatest minorant of f). More generally, it is natural to ask

Question A. For any $\varphi:X\to\overline{R}$, is $\varphi^{\gamma\gamma}$ some hull of φ (i.e., does $\varphi^{\gamma\gamma}$ satisfy a relation similar to (4.8), with $\varphi_{\overline{co}}$ replaced by a suitable hull of φ)?

In this direction, Crouzeix [9] has shown that, for any $\varphi:X\to\overline{R}$ we have

$$\varphi_{\overline{q}}\leq\varphi^{\gamma\gamma}\leq\varphi_q(\leq\varphi), \qquad (8.1)$$

where $\varphi_{\overline{q}}$ is the lower semi-continuous quasi-convex hull, and φ_q is the quasi-convex hull, of φ; hence, in particular, if $\varphi_{\overline{q}}=\varphi$ (i.e., if φ is quasi-convex and lower semi-continuous), then $\varphi_{\overline{q}}=\varphi^{\gamma\gamma}=\varphi_q=\varphi$. In the general case, rather than answering question A, formula (8.1) has suggested, in view of applications to duality-stability relations, the following further questions:

Question B. Is there a concept of "conjugation" such that, for any $\varphi:X\to\overline{R}$, $\varphi_{\overline{q}}$ coincides with the "normalized second conjugate" of φ?

Question C. Same as B, with $\varphi_{\overline{q}}$ replaced by φ_q. Any conjugation with this property is called [45] "exact quasi-convex conjugation".

An affirmative answer to question A has been given, independently, by Martínez-Legaz ([42], [43]) and Passy and Prisman [51], who have shown that, for any $\varphi:X\to\overline{R}$, we have

$$\varphi^{\gamma\gamma}=\varphi_{eq}, \qquad (8.2)$$

where φ_{eq} is the "evenly quasi-convex hull" of φ; we recall that a function $\psi:X\to\overline{R}$ is said to be evenly quasi-convex [51], if all level sets

$$S_c(\psi)=\{x\in X\,|\,\psi(x)\leq c\} \qquad (c\in R) \qquad (8.3)$$

of ψ are evenly convex sets, in the sense of Fenchel [26] (i.e., intersections of open half-spaces). From (8.2) it follows that for β of (7.4) we have

$$\beta=f_{eq}(0), \tag{8.4}$$

(which corresponds to (4.9)), and hence weak duality $\alpha=\beta$ is equivalent to the stability relation $f(0)=f_{eq}(0)$ (where f is the optimal value functional (2.2)).

In connection with question B, let us mention that Crouzeix [9] has introduced, for any $\varphi:X\to\overline{R}$ and $\nu\in R$, "another ν-quasi-conjugate" $\varphi_\nu^C:X^*\to\overline{R}$, and has shown that

$$\varphi^*=\sup_{\nu\in R}\varphi_\nu^C\ ,\quad \varphi_{\bar q}=\inf_{\nu\in R}(\varphi_\nu^C)^*\ , \tag{8.5}$$

where $\varphi^*, (\varphi_\nu^C)^*$ denote the Fenchel conjugates (4.1) of φ, φ_ν^C respectively; however, (8.5) expresses $\varphi_{\bar q}$ only as a "mixed" second conjugate, since at involves two different types of conjugates. Furthermore, El Qortobi [20] and Attéia and El Qortobi [1] have introduced, for any $\varphi:X\to\overline{R}$, the projective conjugate $\varphi^{\#}:X^*\to\overline{R}$ and the "projective biconjugate" $\varphi^{\#\#}:X\to\overline{R}$, of φ, and have shown that

$$\varphi_{\bar q}=\varphi^{\#\#}; \tag{8.6}$$

however, they have observed ([20],[1]) that, in general, $\varphi^{\#\#}\neq(\varphi^{\#})^{\#}$. Also, Martínez-Legaz ([42], [43]) has introduced, for any $\varphi:X\to\overline{R}$, the "H-conjugate" f^o of f, with respect to a family H of functions $h:R\to\overline{R}$, as a certain mapping $\varphi^o:X^*\to H$, and the "H-conjugate" ψ^o of any $\psi:X^*\to H$, as a functional $\psi^o:X\to\overline{R}$ (whence $\varphi^{oo}=(\varphi^o)^o:X\to\overline{R}$), and has shown that if H is the family of all non-decreasing functions $h:R\to\overline{R}$, which are continuous from the left, then

$$\varphi^o(w)(\nu)=\varphi_\nu^C(w)\qquad (w\in X^*),\quad \varphi_{\bar q}=\varphi^{oo}, \tag{8.7}$$

with φ_ν^C of (8.5); however, here the values of the H-conjugate φ^o are not in $\overline{R}$, but in the family H of functions $h:R\to\overline{R}$.

A natural solution to question B has been given in [72], defining, for any locally convex space X and any $\varphi:X\to\overline{R}$ and $\nu\in R$, the first ν-semi-conjugate $\varphi_\nu^\Theta:X\to\overline{R}$ of φ, by

$$\varphi_\nu^\Theta(w)=\nu-1-\inf_{\substack{x\in X\\ w(x)>\nu-1}}\varphi(x)\qquad (w\in X^*), \tag{8.8}$$

the second ν-semi-conjugate $(\varphi_\nu^\Theta)_\nu^\Theta:X\to\overline{R}$ of φ, by

$$(\varphi_\nu^\Theta)_\nu^\Theta(x)=\nu-1-\inf_{\substack{w\in X^*\\ w(x)>\nu-1}}\varphi_\nu^\Theta(w)\qquad (x\in X), \tag{8.9}$$

and the normalized second semi-conjugate $\varphi^{\Theta\Theta}:X\to\overline{R}$ of φ, by

$$\varphi^{\Theta\Theta}=\sup_{\nu\in R}(\varphi_\nu^\Theta)_\nu^\Theta; \tag{8.10}$$

indeed, in [72] it has been shown that, for any $\varphi:X\to\overline{R}$, we have

$$\varphi_{\bar q}=\varphi^{\Theta\Theta}. \tag{8.11}$$

Note that definition (8.8) is similar to definition (6.1), with the difference that, instead of the closed half-spaces $\{x\in X|w(x)\geq\nu\}$ of (6.1), the open half-spaces $\{x\in X|w(x)>\nu-1\}$ are used in (8.8), and, instead of the added term ν in (6.1), the term $\nu-1$ is added in (8.8) (in order to ensure $\varphi^*\geq\sup_{\nu\in R}\varphi_\nu^\Theta\geq\varphi^*-1$, corresponding to (6.4)). Let us mention that, instead of closed or open half-spaces, one can also use (closed) hyperplanes, to define a concept of conjugation. Indeed, in [69], for any locally convex space X and any $\varphi:X\to\bar{R}$ and $\nu\in R$, the first ν-pseudo-conjugate $\varphi^{\pi}_{\nu}:X^*\to\bar{R}$ of φ has been defined by

$$\varphi^{\pi}_{\nu}(w)=\nu-\inf_{\substack{x\in X\\ w(x)=\nu}}\varphi(x)\qquad (w\in X^*),\tag{8.12}$$

and $(\varphi^{\pi}_{\nu})^{\pi}_{\nu}$, $\varphi^{\pi\pi}$ have been defined correspondingly (i.e., replacing $\geq\nu$ by $=\nu$ and γ by π, in (6.2) and (6.3)). Then, as has been observed in [69], we have $\varphi^*\geq\varphi^{\gamma}_{\nu}\geq\varphi^{\pi}_{\nu}$, $\varphi^*=\sup_{\nu\in R}\varphi^{\gamma}_{\nu}=\sup_{\nu\in R}\varphi^{\pi}_{\nu}$ and $\varphi\geq\varphi^{\gamma\gamma}\geq\varphi^{\pi\pi}\geq\varphi^{**}$. For further results on pseudo-conjugates and semi-conjugates, see [69], [70] and [72], [75] respectively.

In connection with question C, Martínez-Legaz [45] has introduced, for any $\varphi:R^n\to\bar{R}$, the "H-conjugate" φ^{∇} of φ, with respect to the family H of all lexicographically non-decreasing functions $h:R^n\to\bar{R}$ (i.e., $h\in H$ if for all $x,y\in R^n$ such that x is lexicographically less than y, we have $h(x)\leq h(y)$), as a certain mapping $\varphi^{\nabla}:\mathcal{L}(R^n)\to H$ (where $\mathcal{L}(R^n)$ denotes the set of all linear endomorphisms of R^n), and the "H-conjugate" ψ^{∇} of any $\psi:\mathcal{L}(R^n)\to H$, as a functional $\psi^{\nabla}:R^n\to\bar{R}$ (whence $\varphi^{\nabla\nabla}=(\varphi^{\nabla})^{\nabla}:R^n\to\bar{R}$), and has shown that for any $\varphi:R^n\to\bar{R}$ we have

$$\varphi_q=\varphi^{\nabla\nabla}.\tag{8.13}$$

9. Using the above results, there have been introduced some new dual problems to (P), relative to the parametrization (X, p), and the questions on equivalence of duality and stability for them (corresponding to the one raised, for β of (7.4), before question A above) have been answered.

Namely, one can define (see [69], [11], [72]) the pseudo-dual and semi-dual problems to (P) of (2.1), relative to the parametrization (X, p) of (2.2), (2.3), as the supremization problem (Q) of (3.1), with

$$\lambda(w)=\inf_{\substack{(y,x)\in F\times X\\ w(x)=0}}p(y,x)\qquad (w\in X^*),\tag{9.1}$$

$$\lambda(w)=\inf_{\substack{(y,x)\in F\times X\\ w(x)>-1}}p(y,x)\qquad (w\in X^*),\tag{9.2}$$

respectively. Then, one can show ([69], [11] [72]) that, for the β's of (3.1) corresponding to (9.1), (9.2), we have, respectively,

$$\alpha\geq\beta=\sup_{w\in X^*}\inf_{\substack{x\in X\\ w(x)=0}}f(x)=f^{\pi\pi}(0),\tag{9.3}$$

$$\alpha\geq\beta=\sup_{w\in X^*}\inf_{\substack{x\in X\\ w(x)>-1}}f(x)=f^{\theta\theta}(0)=f_{\bar{q}}(0),\tag{9.4}$$

with f of (2.2); hence, by (2.4) and (9.4), weak duality $\alpha=\beta$ of (3.1), (9.2) is equivalent to the stability relation $f(0)=f_{\bar{q}}(0)$.

Similarly, for $F=R^n$, one can define [45] a "dual" problem to (P) of (2.1), relative to the parametrization $(X=R^m,p)$ of (2.2), (2.3), as the supremization problem (Q) of (3.1), with X^* replaced by $\mathcal{L}(R^m)$ and with

$$\lambda(w)=\inf_{\substack{(y,x)\in R^n\times R^m \\ w(x)\geq_L 0}} p(y,x) \qquad (w\in\mathcal{L}(R^m)), \quad (9.5)$$

where $\geq_L$ means "equal or lexicographically less than". Then, one can show, using (8.13) and [79], formula (2.30) (or [45], proposition 2.4 and corollary 2.5) that, for β of (3.1), with X^* replaced by $\mathcal{L}(R^m)$ and with λ of (9.5), we have

$$\alpha\geq\beta=\sup_{w\in\mathcal{L}(R^m)} \inf_{\substack{x\in R^m \\ w(x)\geq_L 0}} f(x)=f^{\nabla\nabla}(0)=f_q(0), \qquad (9.6)$$

where f is the functional (2.2). Hence, by (2.4) and (9.6), weak duality $\alpha=\beta$ is equivalent to the stability relation $f(0)=f_q(0)$.

10. The similarity of the definitions of quasi-conjugates, pseudo-conjugates and semi-conjugates and of the results on them (including applications to duality and stability in optimization) suggests to try to unify them. To this end, it is useful to observe that the terms ν and $\nu-1$ have been added in (6.1), (8.12), (8.8), (6.2), etc. only in order to compare "nicely" φ^{μ}_{ν} and $\varphi^{\mu\mu}$ ($\mu=\gamma,\pi,\theta$) with the Fenchel conjugates φ^*, φ^{**} (see e.g. (6.4)) and that their omission does not alter the normalized second conjugates $\varphi^{\mu\mu}$ ($\mu=\gamma,\pi,\theta$), nor the dual optimization problems defined with the aid of φ^{μ}_{ν}, but permits to write the conjugates φ^{μ}_{ν} in the unified form

$$\varphi^{\mu}_{\nu}(w)=-\inf \varphi(\Delta^{\mu}_{\nu,w}) \qquad (w\in X^*,\ \mu=\gamma,\pi,\theta), \quad (10.1)$$

where

$$\Delta^{\gamma}_{\nu,w}=\{x'\in X \mid w(x')\geq\nu\} \qquad (w\in X^*), \quad (10.2)$$

$$\Delta^{\pi}_{\nu,w}=\{x'\in X \mid w(x')=\nu\} \qquad (w\in X^*), \quad (10.3)$$

$$\Delta^{\theta}_{\nu,w}=\{x'\in X \mid w(x')>\nu-1\} \qquad (w\in X^*). \quad (10.4)$$

Now, given any family of sets $\Delta_{\nu,w}\subseteq X$ ($\nu\in R$, $w\in X^*$), for any $\varphi:X\to\bar{R}$ and $\nu\in R$ one can define [73] the ν-Δ-conjugate functional $\varphi^{\Delta}_{\nu}:X^*\to\bar{R}$, by

$$\varphi^{\Delta}_{\nu}(w)=-\inf \varphi(\Delta_{\nu,w}) \qquad (w\in X^*); \quad (10.5)$$

in [73], these have been also called "surrogate conjugate functionals". Furthermore, it is convenient to consider the sets $\Delta_{\nu,w}$ as images of a multifunction $\Delta:(\nu,w)\to\Delta_{\nu,w}$ from $R\times X^*$ into 2^X and to assume that Δ induces a multifunction $\Delta:(\nu,x)\to\Delta_{\nu,x}$ from $R\times X$ into 2^{X^*}, defined "in the same way" (for the precise definition of a "universally defined multifunction" Δ, see [73], §1); this is satisfied for $\Delta_{\nu,w}=\Delta^{\mu}_{\nu,w}$ of (10.2)-(10.4), and it permits to define [73] "the second ν-Δ-conjugate" $(\varphi^{\Delta}_{\nu})^{\Delta}_{\nu}$ and "the normalized second Δ-conjugate" $\varphi^{\Delta\Delta}$.

For $\Delta:R\times X^*\to 2^X$ as above, one can define [73] a "surrogate dual" problem to (P) of (2.1), relative to the parametrization (X, p) of (2.2), (2.3), as the supremization problem (Q) of (3.1), with

$$\lambda(w)=\inf_{\substack{(y,x)\in F\times X \\ x\in\Delta_{0,w}}} p(y,x) \qquad (w\in X^*); \quad (10.6)$$

in particular, for $\Delta_{\nu,w}=\Delta^{\mu}_{\nu,w}$ ($\mu=\gamma,\pi,\theta$) above, (10.6) reduces to (7.1), (9.1) and (9.2), respectively. One can show [73] that, for β of (3.1) corresponding to λ of (10.6), we have

$$\alpha \geq \beta = \sup_{w \in X^*} \inf f(\Delta_{o,w}) = f^{\Delta\Delta}(0), \qquad (10.7)$$

with f of (2.2); hence, by (2.4) and (10.7), weak duality $\alpha=\beta$ of (3.1), (10.7) is equivalent to $f(0)=f^{\Delta\Delta}(0)$. The latter equality may be regarded as a stability relation, since $f^{\Delta\Delta}$ coincides [73] with the "Δ-quasi-convex hull" of f. We recall [73] that a subset M of X is said to be Δ-convex, if for each $x \notin M$ there exists $w \in X^*$ such that $M \cap \Delta_{w(x),w} = \emptyset$, $x \in \Delta_{w(x),w}$ (i.e., M can be separated from each $x \notin M$ by a set of the form $X \setminus \Delta_{w(x),w}$, with suitable $w \in X^*$) and that Δ-quasi-convexity of $\psi: X \to \overline{R}$ means that all level sets (8.3) of ψ are Δ-convex sets; in the particular case when $\Delta_{\nu,w}=\Delta^{\gamma}_{\nu,w}$ above, this yields the evenly convex sets M and the evenly quasi-convex functionals ψ, and for $\Delta_{\nu,w}=\Delta^{\theta}_{\nu,w}$ it yields the closed convex sets M, respectively the lower semi-continuous quasi-convex functionals ψ.

11. An important generalization of the Fenchel conjugates (4.1), (4.2) has been given by Moreau [48],[49], replacing the locally convex spaces X, X* by arbitrary sets X, W (without any structure assumed on them) and replacing the natural bilinear coupling functional $n: X \times X^* \to R$, i.e., the functional

$$(x, w) \to n(x, w) = w(x) \qquad (x \in X, w \in X^*) \qquad (11.1)$$

by an arbitrary functional $k: X \times W \to \overline{R} = [-\infty, +\infty]$, called "coupling functional". Since here k may take also the values $\pm\infty$ (which turns out to be useful in various applications), Moreau has extended the usual addition on R to $\overline{R}$, defining the "upper addition" $\dot{+}$ and the "lower addition" $\underset{\cdot}{+}$ on $\overline{R}$, by

$$a \dot{+} b = a \underset{\cdot}{+} b = a+b \qquad (a,b \in R), \qquad (11.2)$$

$$a \dot{+} (+\infty) = +\infty, \quad a \underset{\cdot}{+} (-\infty) = -\infty \qquad (a \in \overline{R}), \qquad (11.3)$$

and has worked out the rules with these operations [49].

The (Fenchel-Moreau) conjugate of a functional $\varphi: X \to \overline{R}$, with respect to a coupling functional $k: X \times W \to \overline{R}$, is the functional $\varphi^{c(k)}: W \to \overline{R}$ defined ([48], [49]) by

$$\varphi^{c(k)}(w) = \sup_{x \in X} \{k(x,w) \underset{\cdot}{+} -\varphi(x)\} \qquad (w \in W), \qquad (11.4)$$

and the second conjugate of φ, with respect to k, is the functional $\varphi^{c(k)c(k)}: X \to \overline{R}$ defined ([48], [49]) by

$$\varphi^{c(k)c(k)}(x) = \sup_{w \in W} \{k(x,w) \underset{\cdot}{+} -\varphi^{c(k)}(w)\} \qquad (x \in X); \qquad (11.5)$$

clearly, for a locally convex space X, $W=X^*$ and $k=n$ of (11.1), $\varphi^{c(n)}=\varphi^*$ and $\varphi^{c(n)c(n)}=\varphi^{**}$ of (4.1) and (4.2), respectively. Concerning Fenchel-Moreau conjugates, see also [4], [7], [17], [21], [59], [60], [89], [91], [92].

In the sequel, we shall assume that X is an arbitrary set, $W \subseteq \overline{R}^X$ (where $\overline{R}^X$ denotes the family of all functionals $w: X \to \overline{R}$), and $k=n$ of (11.1), with X* replaced by W; this n is no longer bilinear, but it is still linear on W, if $W \subseteq R^X$ (endowing $W \subseteq R^X$ with the usual vector operations, defined pointwise on X). As has been observed in [81], remark 2.2 and Addendum (see also [82], §1), this assumption is no restriction of the generality, since it is equivalent to the case of arbitrary sets X,W and an arbitrary coupling functional $\varphi: X \times W \to \overline{R}$, but it will simplify the formulas below.

Let us embed now the infimization problem (P) of (2.1), where G is a subset of an arbitrary set F (assuming no structure on F), into a family of parametrized problems (2.2), where X is an arbitrary parameter set and $p: F\times X\to\overline{R}$ is an arbitrary functional satisfying, for some $x_o\in X$,

$$p(y, x_o)=\begin{cases} h(y) & \text{if } y\in G \\ +\infty & \text{if } y\in F\setminus G. \end{cases} \tag{11.6}$$

Of course, it is now necessary to use the "embedding" condition (11.6) instead of (2.3), since we assume no structure on X (and hence no "zero element" of X). By (2.2) and (11.6), we have

$$f(x_o)=\inf_{y\in F} p(y, x_o)=\alpha. \tag{11.7}$$

Some authors (e.g. [59], [4]) assume, in order to simplify the formulas, the "normalization" $w(x_o)=0$ $(w\in W)$, which is satisfied by $x_o=0$ in a locally convex space X and $W\subseteq X^*$, but we shall not make here this assumption, since it is too restrictive.

One can define ([79], [82]) the <u>Lagrangian dual</u> problem to (P), relative to the parametrization (X, p), as the supremization problem

$$(Q) \qquad \beta=\sup \lambda(W),$$

where (11.8)

$$\lambda(w)=\inf_{(y,x)\in F\times X} \{p(y,x)\dot{+}-w(x)\}\dot{+}w(x_o) \qquad (w\in W). \tag{11.9}$$

Formulae (4.3), (4.5) and (4.6) admit the following extensions ([82],[79]) to this case:

$$\lambda(w)=-p^{c(n)}(0, w)\dot{+}w(x_o) \qquad (w\in W), \tag{11.10}$$

$$\lambda(w)=-f^{c(n)}(w)\dot{+}w(x_o) \qquad (w\in W), \tag{11.11}$$

$$\beta=\sup_{w\in W} \{-f^{c(n)}(w)\dot{+}w(x_o)\}=f^{c(n)c(n)}(x_o), \tag{11.12}$$

where, in (11.10), we use the canonical embedding (4.4) of $R^F\times\overline{R}^X$ into $\overline{R}^{F\times X}$. Thus, again, we have (4.7). Furthermore [79] (see also [49], for R replaced by $\overline{R}$), (4.8) extends to

$$\varphi^{c(n)c(n)}=\varphi_{H(W+R)}, \tag{11.13}$$

the "(W+R)-convex hull" of $\varphi: X\to\overline{R}$ (i.e.[17], the greatest (W+R)-convex functional $\leq\varphi$; we recall that a functional $\psi: X\to\overline{R}$ is said to be "(W+R)-convex" [17], if it is a supremum of a family of functionals of the form $w+d$, where $w\in W$ and $d\in R$). Hence, by (11.12), we obtain

$$\beta=f_{H(W+R)}(x_o)=\sup_{\substack{w\in W,\ d\in R \\ w+d\leq f}} (w(x_o)+d). \tag{11.14}$$

From (11.7) and (11.14) we see that <u>weak duality</u> $\alpha=\beta$ <u>is equivalent to the "stability" relation</u> $f(x_o)=f_{H(W+R)}(x_o)$.

The above generalization can be applied to a large class of problems, encompassing both the continuous case and the case of discrete optimization (since there are no structures assumed on F and X); for example, in a subsequent paper (in preparation), we give applications to combinatorial optimization. Also, as has been shown, independently, by Lindberg [40], Balder [4], and Dolecki and Kurcyusz [17], this generalization provides a unified way of obtaining various known dual problems (Q) defined with the aid of so-called" augmented Lagrangians", just by taking suitable particular sets $W\subset\overline{R}^X$ in (11.8), (11.9). A further advantage of this general approach will be shown in section 12 below.

Let us mention that it has also been a matter of interest whether a given duality theory is "symmetric", i.e., such that, under some mild assumptions, the "dual" to the dual problem (Q) of (3.1) or (11.8) (defined as a suitable infimization problem, usually with the aid of a parametrization of (Q)) coincides with the primal problem (P) of (2.1). For example, the duality theory of Lindberg [40] is symmetric, while those of Balder [4] and Dolecki and Kurcyusz [17] are non-symmetric.

12. For any multifunction $\Delta:R\times X^*\to 2^X$ as in section 10 above, if we take $V=V_\Delta\subset\overline{R}^X$ defined by

$$V=\{-\chi_{\Delta_{w(y),w}}\dot{+}d \mid y\in X,\ w\in X^*,\ d\in R\},\tag{12.1}$$

where χ_M denotes the indicator functional of the set $M\subseteq X$, then, as has been shown in [73], for any $\varphi:X\to\overline{R}$ we have

$$\varphi^{c(n)}(-\chi_{\Delta_{w(y),w}}\dot{+}d)=\varphi^{\Delta}_{w(y)}(w)\dot{+}d \qquad (y\in X,\ w\in X^*, d\in R),\tag{12.2}$$

with φ^{Δ}_{ν} of (10.5); indeed, this follows directly from (11.4) applied to k=n of (11.1), with X* replaced by V of (12.1). Similarly [73], when Δ is universally defined (see section 10 above), we have

$$\varphi^{c(n)c(n)}=\varphi^{\Delta\Delta}.\tag{12.3}$$

Thus, ν-Δ-conjugation (and hence, in particular, quasi-conjugation, pseudo-conjugation, semi-conjugation) is a "particular case" of Fenchel-Moreau conjugation (11.4),with W replaced by $V=V_\Delta$ of (12.1), and with k=n corresponding to this V. For $\Delta_{\nu,w}=\Delta^{Y}_{\nu,w}$ of section 10, a related result (involving "conjugation" in the sense of Lindberg [40], which is equivalent [81] to Fenchel-Moreau conjugation), has been given by Martínez-Legaz [44].

From the above and from the expressions of the dual problems with the aid of conjugate functionals (see e.g.(7.3), (7.4), (10.7) and (11.10)-(11.12)), it follows that the various surrogate dual problems (11.8), (7.1), (10.6), etc. to (P) of (2.1), relative to the parametrization (X, p) of (2.2), (2.3), are "particular cases" of the general Lagrangian dual problem (11.9), for a suitable modification V of W. Of course, this can be deduced also directly from the definitions of these problems, without using conjugates; see e.g. [82], where various other relations between Lagrangian dual problems and surrogate dual problems have been also given (see also section 16 below).

Similarly to (3.2)→(11.9) and (4.1)→(11.4), etc., it is useful to replace the locally convex space X by an arbitrary set X, and X* by any set $W\subseteq\overline{R}^X$ (whence 0 by $w(x_0)$), in surrogate duality and surrogate conjugation (e.g., in (7.1), (9.1), (10.6), (10.5), etc.); see [73],[78],[82].

13. Given two optimization problems (P_1), (P_2) (defined on possibly different spaces) and dual problems (Q_1), (Q_2) (in some sense) to (P_1) and (P_2) respectively, the "primal-dual pairs" of problems $\{(P_1),(Q_1)\}$, $\{(P_2),(Q_2)\}$ are said to be equivalent [75] (denoted $\{(P_1),(Q_1)\}\sim\{(P_2),(Q_2)\}$, if $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$, where α_i and β_i are the optimal values of (P_i) and (Q_i) respectively. The usefulness of this concept consists in the fact that from duality theorems for $\{(P_i),(Q_i)\}$ (i.e., conditions in order that $\alpha_i=\beta_i$) one can deduce duality theorems for $\{(P_{3-i}),(Q_{3-i})\}$ (i=1,2). For example, formulae (11.7)-(11.9) and (11.12) show that the pair $\{(P_{G,h}),(Q^{G,h})\}$,where $(Q^{G,h})$ is the Lagrangian dual pro-

blem (11.8), (11.9) to $(P_{G,h})$, is equivalent to the pair $\{(P_{\{x_o\},f}), (Q^{\{x_o\},f})\}$, with f of (2.2), where $(Q^{\{x_o\},f})$ is the "unperturbational" Lagrangian dual in the sense (16.1) (below) to $(P_{\{x_o\},f})$ $\inf f(\{x_o\})=f(x_o)=\alpha$. The latter has the advantage that the constraint set $\{x_o\}$ is a singleton and, by the above remark, from duality theorems for $\{(P_{\{x_o\},f}), (Q^{\{x_o\},f})\}$ (which, by (11.14), amount to conditions for $f(x_o)=f_{H(W+R)}(x_o)$), one can deduce duality theorems for $\{(P_{G,h}), (Q^{G,h})\}$. This method has been applied in a consequent manner, e.g. in [75], for Lagrangian duality and surrogate duality. Let us also mention that the notation X for the parameter set is used (in contrast with the notations of other authors) in order to emphasize that we apply theorems in X (e.g., separation theorems) in order to draw conclusions on problem $(P_{G,h})$ of (2.1).

The remarks made after formula (4.9) suggest to classify the (equivalence classes of) primal-dual pairs $\{(P), (Q)\}$, by saying [75] that $\{(P_1), (Q_1)\}$ is "better than" $\{(P_2), (Q_2)\}$ (denoted $\{(P_1), (Q_1)\}\gtrsim\{(P_2), (Q_2)\}$), if for the gaps we have $\alpha_1-\beta_1\leq\alpha_2-\beta_2$. For example, for the optimization problem

$$(P)\qquad \alpha=\inf_{\substack{y\in F\\ u(y)=x_1}} h(y), \tag{13.1}$$

where F, X are locally convex spaces, $x_1\in X$, and u is a continuous linear mapping of F into X, one can consider the parametrization functionals (of the form (2.7), with parameter space F for p_1, and X for p_2)

$$p_1(y,x)=\begin{cases} h(y) & \text{if } y\in x+\{y'\in F \mid u(y')=x_1\} \\ +\infty & \text{otherwise,}\end{cases} \tag{13.2}$$

$$p_2(y,x)=\begin{cases} h(y) & \text{if } u(y)=x_1+x \\ +\infty & \text{otherwise,}\end{cases} \tag{13.3}$$

and one can ask, which one of the primal-dual pairs $\{(P), (Q_1)\}$, $\{(P), (Q_2)\}$, corresponding to p_1 and p_2 respectively, by pseudo-duality (3.1), (9.1), is better. In [74], §3, it has been shown that p_1 yields the better pair and, if F is a Banach space and $h(y)=\|y\|$ $(y\in F)$, then $\{(P), (Q_1)\}$ is optimal, since it has the gap 0, while the (pseudo-duality) gap in $\{(P), (Q_2)\}$ is related to the "characteristic" of the subspace $u^*(X^*)$ of F^*, in the sense of Dixmier [16] (where u^* is the adjoint operator).

14. Let us mention now some further developments in the theory of conjugations.

Crouzeix ([9], p.28) has made the following remark: "... the notions of conjugate functions introduced by Greenberg and Pierskalla do not make use of the notion of level sets which plays in quasi-convexity the role played by epigraphs in convexity and therefore we have introduced analogous notions to those of Greenberg and Pierskalla, involving the level sets" (by this, Crouzeix has meant the φ_ν^C's occurring in (8.5) above and a notion of "tangential" corresponding to the "quasi-subdifferential" of [34], [93]); a similar remark has been also made in [10], p.75. Therefore, in [76], [73], there have been given some expressions of $\varphi_\nu^\gamma, \varphi_\nu^\pi, \varphi_\nu^\theta, \varphi_\nu^\Delta, (\varphi_\nu^\gamma)_\nu^\gamma, \varphi^{\gamma\gamma}$ (of (6.1), (8.12), (8.8), (10.5), (6.2), (6.3)) etc., involving the level sets (8.3) (of $\psi=\varphi$) and the "strict" [49] level sets

$$A_c(\varphi)=\{x\in X \mid \varphi(x)<c\} \qquad (c\in R); \tag{14.1}$$

for example,

$$\varphi_{\nu}^{\gamma}(w)=\inf_{\substack{\nu\in R \\ w(x)<\lambda\,(x\in A_{\lambda-\nu}(\varphi))}} \nu=\sup_{\substack{\nu\in R \\ \exists x\in A_{\lambda-\nu}(\varphi),\, w(x)\geq\lambda}} \nu \qquad (w\in X^*). \qquad (14.2)$$

Also, some expressions of the corresponding concepts of "surrogate" subdifferentials ([34], [93], [69], [72], [73]), with the aid of the level sets (14.1), have been given in [75], §3 and [73], §6. Some extensions of these results of [76], [75], [73] (and of some results of [70], [74]) have been obtained by Volle [90].

An axiomatic approach to the theory of Fenchel-Moreau conjugations has been started in [80]. The main result of [80] states that if X, W are two sets and if an operator $c:\varphi\in\overline{R}^X\to\varphi^c\in\overline{R}^W$ satisfies, for every index set I, the conditions

$$(\inf_{i\in I}\varphi_i)^c=\sup_{i\in I}\varphi_i^c \qquad (\{\varphi_i\}_{i\in I}\subseteq\overline{R}^X), \qquad (14.3)$$

$$(\varphi\dotplus d)^c=\varphi^c\dotplus -d \qquad (\varphi\in\overline{R}^X,\ d\in\overline{R}), \qquad (14.4)$$

then there exists a uniquely determined coupling functional $k:X\times W\to\overline{R}$ such that $\varphi^c=\varphi^{c(k)}$ of (11.4), for all $\varphi\in\overline{R}^X$, namely,

$$k(x,w)=(\chi_{\{x\}})^c(w) \qquad (x\in X,\ w\in W); \qquad (14.5)$$

thus, since conversely, $c(k):\overline{R}^X\to\overline{R}^W$ of (11.4) satisfies (14.3), (14.4), one obtains not only an axiomatic characterization of Fenchel-Moreau conjugations, but also that the correspondence $k\to c(k)$, between coupling functionals and conjugations, is one-to-one. Some results on the "dual" $c':\overline{R}^W\to\overline{R}^X$ of an operator $c:\overline{R}^X\to\overline{R}^W$ have been also given in [80]. Furthermore, in [81], there have been shown some relations between "dualities" in the sense of Evers and van Maaren [22], i.e., mappings $\Delta:2^X\to 2^W$ satisfying, for every index set I,

$$\Delta(\bigcup_{i\in I}G_i)=\bigcap_{i\in I}\Delta(G_i) \qquad (\{G_i\}_{i\in I}\subseteq 2^X), \qquad (14.6)$$

polarities $\pi(\rho):2^X\to 2^W$ (where $\rho\subseteq X\times W$ is a binary relation), in the sense of [6], coupling functionals and conjugations. Also, in [83] and [46], there have been studied "dualities" (called "polarities" in [6], [50], [53]) between two complete lattices E and F, i.e., mappings $\Delta:E\to F$ satisfying, for every index set I,

$$\Delta(\inf_{i\in I}x_i)=\sup_{i\in I}\Delta(\{x_i\}) \qquad (\{x_i\}_{i\in I}\subseteq E); \qquad (14.7)$$

in particular, for $E=\overline{R}^X$, $F=\overline{R}^W$, (14.7) reduces to (14.3) (with $c=\Delta$), so one obtains a generalization of Fenchel-Moreau conjugations (for which (14.4) need not hold), considered in a particular case, by Ben-Tal and Ben-Israel [5], and for $E=(2^X,\supseteq)$, $F=(2^W,\supseteq)$, (14.7) reduces to (14.6). Some corresponding problems for conjugations of multifunctions and conjugations of vector-valued functions, are now being investigated.

Finally, let us mention that Deumlich and Elster have introduced and studied (for a survey, see [14]), a concept of "Φ-conjugation" of functionals. Starting from the observation of Fenchel [25] that conjugate functionals (4.1) are closely connected with polarity with respect to a hyperparaboloid, Deumlich and Elster have based their concept of "Φ-conjugation" (which we shall not reproduce here) on polarity with respect to an arbitrary nondegenerate hypersurface Φ of order two. For applications of this Φ-conjugation to duality-stability relations, especially in fractional programming, see [13].

15. Let us also mention, briefly, some related results of [79] on generalized con-

vexity. The approaches of sections 10-12 above have stimulated the study of "convexity" of a subset G of a set X with respect to a family of sets $\mathcal{M} \subseteq 2^X$ (in the sense of [12]) and with respect to a family of functionals $W \subseteq \overline{R}^X$ (in the sense of [24]) and related concepts of "quasi-convexity" and "convexity" of functionals (via level sets and epigraphs, respectively). We recall that a set $G \subseteq X$ is said to be i) convex with respect to $\mathcal{M} \subseteq 2^X$ [12], if for each $x \notin G$ there exists $M \in \mathcal{M}$ such that $G \subseteq M$, $x \in X \setminus M$ (i.e., such that M "separates" G from x); ii) convex with respect to $W \subseteq \overline{R}^X$ [24], if for each $x \notin G$ there exists $w \in W$ such that $\sup w(G) < w(x)$ (i.e., such that w "separates" G from x). For example [79], if X is a locally convex space, $\mathcal{M}$-convexity includes the Δ-convexity mentioned after formula (10.7) above (by taking $\mathcal{M} = \{X \setminus \Delta_{w(x),w} \mid x \in X, w \in X^*\}$), whence also the evenly convex sets and the closed convex sets; moreover, it also includes (see [79]) the convex sets in a linear space X, the closed sets in a topological space X, etc. In [79] it has been shown that the theories of $\mathcal{M}$-convex sets and W-convex sets $G \subseteq X$ (where $\mathcal{M} \subseteq 2^X$, $W \subseteq \overline{R}^X$) are equivalent, and that, correspondingly, so are the theories of $\mathcal{M}$-quasi-convex, W-quasi-convex and W-convex functionals. Furthermore, as has been observed in [83], §5, a number of results of Volle [90] on "$\Delta^*\Delta$-convexity" of sets $G \subseteq X$ and "$\Delta^*\Delta$-quasi-convexity" of functionals $\varphi: X \to \overline{R}$, where $\Delta: 2^X \to 2^W$ is a duality (14.6), can be simply obtained as the particular case $\mathcal{M} = \{\Delta^*(\{w\}) \mid w \in W\}$ of some results of [79] on $\mathcal{M}$-convexity of G and $\mathcal{M}$-quasi-convexity of φ. For some other results on various concepts of generalized convexity, with applications to optimization, see e.g. [58].

16. Let us return now to optimization. The great variety of "usual" dual problems (3.1) and (11.8) to a given primal problem (2.1), has stimulated attempts to unify them, in a general theory of dual optimization problems. An attempt in this direction has been made by Tind and Wolsey [86] (see also [8] and [54]). A different unified theory has been constructed in [78] for surrogate dual problems and in [82] for the most general case, encompassing, among others, the Lagrangian, the surrogate, and the Tind-Wolsey dual problems, as well as the dual problems of Gould [31], Klötzler [39], Hoffmann [35], etc. Here we shall only mention, briefly, some aspects of the theory of [78] and [82], which are related to parametrization.

According to the definition given in [82], by a "dual" problem to the infimization problem (2.1) we mean any supremization problem (11.8), where $W = W^{G,h}$ and $\lambda = \lambda^{G,h}: W \to \overline{R}$. In [82] it is shown that there are some close connections between "unperturbational" dual problems (i.e., defined without assuming any parametrization (X, p) of (P), such as in linear programming or in (3.5), (5.1), (5.7), etc.) and "perturbational" dual problems (i.e., defined with the aid of a parametrization (X, p) of (P)). Namely, from each unperturbational dual problem one can deduce a "perturbational version" of it, with the aid of a certain scheme (part of which is similar to the idea of "universally defined" multifunctions [73]) and, conversely, from each perturbational dual problem, with F, X locally convex spaces and $W \subseteq X^*$, one can regain its "unperturbational version", by taking $X = F$ and p = the perturbation (2.7), with $\Gamma(x) = G + x$, for all $x \in F$. For example, the unperturbational version of the Lagrangian dual (11.9) is (Q) of (11.8) with $W \subseteq \overline{R}^F$ and

$$\lambda(w) = \inf_{y \in F} \{h(y) \dotplus -w(y)\} \dotplus \inf w(G) \qquad (w \in W), \qquad (16.1)$$

which is useful when $G \neq F$ (see e.g. [62]); similar relations hold between the unperturbational and perturbational Tind-Wolsey dual problems to (P) (see [82]) and between the unperturbational and perturbational surrogate dual problems to (P) (see [78], [82]). We recall that if G is a subset of a set F, $h:F\to\overline{R}$, $W=W^{G,h}$ is a set and $\Delta_{G,w}\subseteq F$ ($w\in W$) is a family of ("surrogate constraint") sets, the unperturbational surrogate dual to (P) of (2.1) is defined [78] as the supremization problem (11.8), with

$$\lambda(w)=\inf h(\Delta_{G,w}) \qquad (w\in W); \qquad (16.2)$$

furthermore, if X is a set, $p:F\times X\to\overline{R}$ satisfies (11.6) for some $x_o\in X$, and $\tilde{\Delta}_{(F,x_o),w}\subseteq F\times X$ ($w\in W$) is a family of sets (the most interesting particular case being $\tilde{\Delta}_{(F,x_o),w}=F\times \Delta^o_{\{x_o\},w}$, where $\{\Delta^o_{\{x_o\},w}\}_{w\in W}$ is a family of subsets of X), the perturbational surrogate dual to (P) is defined [78] as (11.8), with

$$\lambda(w)=\inf p(\tilde{\Delta}_{(F,x_o),w}) \qquad (w\in W). \qquad (16.3)$$

These general definitions of surrogate dual problems encompass, as particular cases, the various surrogate dual problems to (P), mentioned above (e.g., quasi-dual, pseudo-dual, etc.). In a series of papers, of which we mention [71], [70], [75] and [78], there have been obtained (perturbational and unperturbational) surrogate duality results, with the aid of "level set methods" (consisting, roughly speaking, in separation, in various senses, of the level sets of h from the surrogate constraint sets), which are more convenient for surrogate dual problems, than the epigraph methods of convex analysis.

Some of the perturbational dual problems (3.1) and (11.8) to (P) correspond to various concepts of conjugations, applied to f of (2.2) (we have seen many examples above), and therefore they are called [78] "perturbational conjugate dual" problems to (P); their relations to general unperturbational and perturbational surrogate dual problems have been investigated in [78].

Further results on dual problems (3.1) and (11.8) defined with the aid of a parametrization (X, p) have been given, for a large class of perturbations p, introduced and studied in [82], namely, for the p's that can be written as

$$p(y, x)=h(y)\dot{+}\pi_G(y, x) \qquad (y\in F,\ x\in X), \qquad (16.4)$$

where $\pi_G:F\times X\to\overline{R}$ is a coupling functional with values $\pi_G(y, x)\in\overline{R}$ not depending on h; these p's, called "h-separated" perturbation functionals [82], encompass various important perturbation functionals as particular cases, for example , (2.7) above (by taking $\pi_G(y, x)=\chi_{\Gamma(x)}(y)$, for all $y\in F$, $x\in X$).

17. Finally, let us mention that, besides the "usual" dual problems (11.8) to (P) of (2.1), there also appear, in a natural way, some "unusual" dual problems

$$(Q) \qquad \beta=\inf \lambda(W), \qquad (17.1)$$

where $W=W^{G,h}$ and $\lambda=\lambda^{G,h}:W\to\overline{R}$. Some general classes of such unusual dual problems, namely, surrogate duality generalizing the reverse convex duality of [66] and surrogate and Lagrangian duality generalizing the convex supremization duality of [63], [65], [64], [87], [88], [38], have been studied recently in [85] and [84] respectively. For these classes there exist certain parametrization theories (see [85] and, respectively, [87], [38]), and for the Lagrangian convex supremization duality, there exist ([87], [38]) duality-stability relations involving the Fenchel conjugates and the subdifferentials at 0 of the "partial functionals"

$$p_y(x)=p(y, x) \qquad (y\in F,\ x\in X), \tag{17.2}$$

corresponding to those known for convex infimization duality (see e.g. [57], [19]). However, until the present, for the above classes of unusual dual problems, there do not exist stability results involving conjugates or subdifferentials of the optimal value functional (2.2), respectively

$$f(x)=\sup_{y\in F} p(y, x) \qquad (x\in X). \tag{17.3}$$

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