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Introduction

The study of inductive limits of the form $\varinjlim(C(X_n)\otimes F_n, \Phi_n)$ requires the analysis of the homomorphisms $\Phi : C(X)\otimes F_1 \to C(Y)\otimes F_2$ where X,Y are compact spaces and F_1, F_2 are finite dimensional C^*-algebras. These problems were raised by E.G.Effros in [2].

Recent papers of C.Pasnicu ([5] and [6]), M.Dădârlat [1] and K.Thomsen [8] are dealing with particular cases when Y satisfies certain topological conditions and/or F_i are factors. Relevant notions which appear are compatible homomorphisms with a covering and homogeneous homomorphisms.

In this note we show how the structure of the homomorphisms Φ depends up to some cohomological obstructions on homomorphisms $C(X) \to C(Y)\otimes M_k$ .This approach enable us to generalize the results in [8] and also to classify the arbitrary unital *-homomorphisms

$$\Phi : C(S^1)\otimes(M_{k_1}\oplus\ldots\oplus M_{k_p}) \to C(S^1)\otimes(M_{n_1}\oplus\ldots\oplus M_{n_q})$$

within unitary equivalence and homotopy.

§1. General results

Let X,Y be compact connected spaces. We attempt to describe the $*$-homomorphisms $\Phi : C(X)\otimes F_1 \to C(Y)\otimes F_2$,where F_1, F_2 are finite dimensional C^*-algebras.

For the begining we consider the unital case. Of course,the direct sum decomposition of F_2 into factors determines a corresponding direct sum decomposition of Φ .Therefore it suffices to consider homomorphisms $\Phi : C(X)\otimes(M_{k_1}\oplus\ldots\oplus M_{k_p}) \to C(Y)\otimes M_n$. Here we denote by M_k the C^*-algebra of kxk complex matrices and by I_k its unit.

We recall that given an unital $*$-homomorphism $\alpha : M_{k_1}\oplus\ldots\oplus M_{k_p} \to M_n$,there are positive integers $m_1,\ldots,m_p$ with $\sum_i m_i k_i = n$ and an unitary $u\in U(n)$ such that

$$\alpha(a_1\oplus\ldots\oplus a_p) = Ad(u)(a_1\otimes I_{m_1}\oplus\ldots\oplus a_p\otimes I_{m_p})$$

for every $a_1\oplus\ldots\oplus a_p \in M_{k_1}\oplus\ldots\oplus M_{k_p}$ .Here we consider the inclusion $\bigoplus_i M_{k_i}\otimes M_{m_i} \subset M_n$.

For $f\in C(X)$ and $a\in F_1 = M_{k_1}\oplus\ldots\oplus M_{k_p}$ we have

$$(f\otimes I)(1\otimes a) = (1\otimes a)(f\otimes I) = f\otimes a,$$

therefore $\Phi(C(X)\otimes I)$ lies in the relative commutant of $\Phi(1\otimes F_1)$ in $C(Y)\otimes M_n$.

Let's suppose for the moment that $\Phi(1\otimes F_1)\subset 1\otimes M_n$ i.e.

$$(1) \qquad \Phi(1\otimes a) = 1\otimes(\bigoplus_i a_i\otimes I_{m_i}) \quad \forall a\in F_1 \;, \quad a=(a_1,\ldots,a_p),$$

for suitable m_i. In this case

$$\Phi(1\otimes F_1) = 1\otimes(\bigoplus_i M_{k_i}\otimes I_{m_i}).$$

We have $(\bigoplus_i M_{k_i}\otimes I_{m_i})'\cap M_n = \bigoplus_i (I_{k_i}\otimes M_{m_i})$ and therefore

$$\Phi(f\otimes I) = \bigoplus_i (\Phi_i'(f)\otimes I_{k_i}),$$

for some unital $*$-homomorphisms $\Phi_i' : C(X)\to C(Y)\otimes M_{m_i}$, $i=1,\ldots,p$.

We obtain

$$\Phi = \bigoplus_i (\Phi_i'\otimes id_{M_{k_i}}).$$

DEFINITION

Let A,B be unital C^{*}-algebras and denote by Hom(A,B) the set of the unital $*$-homomorphisms $\phi: A \to B$. $\phi_1, \phi_2 \in$ Hom(A,B) are said to be inner equivalent if there is an unitary $u \in U(B)$ such that $\phi_2 = Ad(u) \circ \phi_1$. We denote by Hom(A,B)/inn the set of the classes of inner equivalent homomorphisms.

PROPOSITION 1.

There exists a bijection

$$Hom(M_{k_1} \oplus \ldots \oplus M_{k_p}, C(Y) \otimes M_n)/inn \xrightarrow{\delta} \{(E_1, \ldots, E_p) \in Vect(Y)^p \mid \bigoplus_{i=1}^{p} k_i E_i = \underline{n}\}.$$

Here Vect(Y) denotes the set of the isomorphism classes of complex vector bundles over Y, $kE = E \oplus \ldots \oplus E$ (k-times) and $\underline{n}$ is the class of the trivial vector bundle of rank n.

Proof. The proof is based on some elementary manipulations with cohomological sets (see [1] for details).

We remark that $Hom(F_1, C(Y) \otimes M_n) \simeq C(Y, Hom(F_1, M_n))$ and the space $Hom(F_1, M_n)$ can be identified with the disjoint sum

$$\coprod_{n = \sum_i h_i k_i} U(n) / \bigoplus_i U(h_i),$$

where $U(n)/\bigoplus_i U(h_i)$ is the homogeneous space which appears in the fibering

$$\bigoplus_i U(h_i) \xrightarrow{j} U(n) \xrightarrow{\pi} U(n)/\bigoplus_i U(h_i),$$

$j(u_1 \oplus \ldots \oplus u_p) = \bigoplus_i u_i \otimes I_{k_i} \in U(n)$.

For a topological space L we denote by L^c the sheaf of the germs of continuous functions $Y \to L$.

Since $\bigoplus_i U(h_i)$ is a closed subgroup in the Lie group $U(n)$, we have the short exact sequence

$$1 \longrightarrow (\bigoplus_i U(h_i))^c \longrightarrow U(n)^c \longrightarrow (U(n)/\bigoplus_i U(h_i))^c \longrightarrow 1$$

and therefore the exact sequence of pointed cohomological sets

$$1 \to H^0(Y, (\bigoplus_i U(h_i))^c) \to H^0(Y, U(n)^c) \to H^0(Y, (U(n)/\bigoplus_i U(h_i))^c) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} H^1(Y, (\bigoplus_i U(h_i))^c) \xrightarrow{j} H^1(Y, U(n)^c).$$

But $H^0(Y,L^c) = C(Y,L)$ and $H^1(Y,(\bigoplus_i U(h_i))^c) = \bigoplus_i \mathrm{Vect}_{h_i}(Y)$, where $\mathrm{Vect}_h(Y)$ denotes the set of the isomorphism classes of complex vector bundles of rank h over Y.

If we take $\eta : \coprod_{n=\sum h_i k_i} U(n) \to \mathrm{Hom}(F_1, M_n)$, $\eta(v)(a) = \mathrm{Ad}(v)(\bigoplus_i a_i \otimes \otimes I_{h_i})$, we have a bijection $\eta_* : C(Y, \coprod_{n=\sum h_i k_i} (U(n)/\bigoplus_i U(h_i))) \longrightarrow$
$\longrightarrow C(Y,\mathrm{Hom}(F_1,M_n)) \simeq \mathrm{Hom}(F_1, C(Y)\otimes M_n)$.

After these identification we obtain the exact sequence

$$\coprod_{n=\sum h_i k_i} C(Y,U(n)) \to \mathrm{Hom}(F_1, C(Y)\otimes M_n) \xrightarrow{\delta}$$
$$\xrightarrow{\delta} \coprod_{n=\sum h_i k_i} (\bigoplus_i \mathrm{Vect}_{h_i}(Y)) \xrightarrow{j} \coprod_{n=\sum h_i k_i} \mathrm{Vect}_n(Y).$$

The map j is given by $j(E_1 \oplus \ldots \oplus E_p) = k_1 E_1 \oplus \ldots \oplus k_p E_p$.

Moreover, two homomorphisms $\phi_1, \phi_2 \in \mathrm{Hom}(F_1, C(Y)\otimes M_n)$ are conjugated by an unitary in $C(Y)\otimes M_n$ iff $\delta(\phi_1) = \delta(\phi_2)$.

COROLLARY

Assume that every complex vector bundle over Y of rank $\leqslant n$ is trivial. Then for every $\phi \in \mathrm{Hom}(C(X)\otimes F_1, C(Y)\otimes M_n)$ there are $\phi_i^! \in \mathrm{Hom}(C(X), C(Y)\otimes M_{m_i})$, $i=1,\ldots,p$, such that $n=\sum m_i k_i$ and

$$\phi = \mathrm{Ad}(u)\circ(\bigoplus_i \phi_i^! \otimes \mathrm{id}_{M_{k_i}})$$

for some $u \in C(Y,U(n))$.

Proof. By our assumption it follows that any two homomorphism in $\mathrm{Hom}(F_1, C(Y)\otimes M_n)$ are inner equivalent. Consequently we find an unitary $u \in C(Y)\otimes M_n$ such that $(\mathrm{Ad}(u^*)\circ\phi)(1\otimes F_1) \subset 1 \otimes M_n$.

REMARKS

Let us note some other interesting situations. For instance, if $p=1$ i.e. $F_1 = M_k$ and Y is a finite CW-complex of dimension$\leqslant 2n/k$ without torsion in K-theory, then the same reduction is possible since the second set in the Proposition 1 reduces to a single element (see [1]).

Another possible way to study the homomorphisms

$\phi \in \mathrm{Hom}(C(X)\otimes F_1, C(Y)\otimes M_n)$ is to look for C(Y)-module automorphisms $\alpha \in \mathrm{Aut}_{C(Y)}(C(Y)\otimes M_n)$ such that $(\alpha\circ\phi)(1\otimes F_1) \subset 1\otimes M_n$. Any nonzero homomorphism $\alpha' \in \mathrm{Hom}(M_n, C(Y)\otimes M_n)$ extends to an unique C(Y)-module automorphism $\alpha \in \mathrm{Aut}_{C(Y)}(C(Y)\otimes M_n)$. Note also that $\delta(\alpha\circ\phi|_{1\otimes F_1}) = \delta(\alpha')\cdot\delta(\phi|_{1\otimes F_1}) = (\delta(\alpha')\otimes E_1, \dots, \delta(\alpha')\otimes E_p)$ and that $\delta(\alpha') \in T_n\mathrm{Vect}_1(Y) = \{ L \in \mathrm{Vect}_1(Y) \mid nL = \underline{n} \}$. Based on previous arguments we can obtain a bijection

$$\mathrm{Hom}(F_1, C(Y)\otimes M_n)/\mathrm{Aut}_{C(Y)}(C(Y)\otimes M_n) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \{(E_1,\dots,E_p) \in \mathrm{Vect}(Y)^p \mid k_1E_1 \oplus \dots \oplus k_pE_p = \underline{n}\} / T_n\mathrm{Vect}_1(Y) .$$

EXAMPLE

Let k even and m odd. For every $\phi \in \mathrm{Hom}(C(\mathbb{P}^2\mathbb{R})\otimes M_k, C(\mathbb{P}^2\mathbb{R})\otimes M_{km})$ there are $\alpha \in \mathrm{Aut}_{C(\mathbb{P}^2\mathbb{R})}(C(\mathbb{P}^2\mathbb{R})\otimes M_{km})$ and $\phi' \in \mathrm{Hom}(C(\mathbb{P}^2\mathbb{R}), C(\mathbb{P}^2\mathbb{R})\otimes M_m)$ such that $\alpha\circ\phi = \phi'\otimes \mathrm{id}_{M_k}$. The same assertion is not true if we require the automorphism α to be inner.

Proof. $\mathrm{Vect}_1(\mathbb{P}^2\mathbb{R})$ acts freely on $\mathrm{Vect}(\mathbb{P}^2\mathbb{R}) = \{ E \in \mathrm{Vect}(\mathbb{P}^2\mathbb{R}) \mid kE = \underline{mk} \}$.

DEFINITION

$\phi: C(X) \to C(Y)\otimes M_m$ is called homogeneous of degree t if the number of distinct characters which appear in the decomposition of $\phi_y: C(X) \to M_m$, $\phi_y(f) = \phi(f)(y)$ is equal to t, each with multiplicity m/t.

PROPOSITION 2.

Let's suppose that every $\phi_i^!$ is homogeneous of degree t_i $(m_i = t_i s_i)$. Then for every i there are a t_i-fold covering space $\pi_i: Z_i \to Y$, a continuous map $p_i: Z_i \to G_{s_i}(\mathbb{C}^{m_i})$ with $\sum_{z\in\pi_i^{-1}(y)} p_i(z) = I_{m_i}$ $\forall y \in Y$ and a continuous map $\varphi_i: Z_i \to X$ one to one on the fibers $\pi_i^{-1}(y)$ such that

$$\phi_i^!(f)(y) = \sum_{z\in\pi_i^{-1}(y)} f(\varphi_i(z))p_i(z), \quad i=1,\dots,p$$

and therefore, if $\mathrm{Hom}(F_1, C(Y)\otimes M_n)/\mathrm{inn}$ has a single element, we have

$$\phi(f\otimes a)(y) = u(y)(\bigoplus_i a_i \otimes (\sum_{z\in\pi_i^{-1}(y)} f(\varphi_i(z))p_i(z)))u(y)^*.$$

Proof. (see [1]).

REMARKS.

If $H^1(Y,S_{t_i})=0$ (S_t the symmetric group) and $H^1(Y,U(k))=0$ for every $k\leq n$, it follows that $Z_i=Y\amalg\ldots\amalg Y$ (t_i-times) and we find maps $\varphi_i^j:Y\to X, i=1,\ldots,p, j=1,\ldots,t_i$ with $\varphi_i^j(y)\neq\varphi_i^h(y), j\neq h$ and $p_i^j:Y\to G_{s_i}(\mathbb{C}^{m_i})$ with $\sum_j p_i^j=I_{m_i}$ such that

$$\phi_i^!(f)(y) = \sum_{j=1}^{t_i} f(\varphi_i^j(y))p_i^j(y).$$

It can be shown ([1]) that there exists $u_i\in C(Y,U(m_i))$ such that $p_i^j(y)=u_i(y)p_i^j(y_o)u_i(y)^*, i=1,\ldots,p, j=1,\ldots,t_i$ for some $y_o\in Y$ fixed. In this case

$$\phi_i^!(f)(y) = u_i(y)\begin{pmatrix} f(\varphi_i^1(y))\otimes I_{s_i} & & 0 \\ & \ddots & \\ 0 & & f(\varphi_i^{t_i}(y))\otimes I_{s_i}\end{pmatrix}u_i(y)^*,$$

$f\in C(X), y\in Y$ (cf. [8]).

REMARK.

If $\phi:C(X)\otimes(M_{k_1}\oplus\ldots\oplus M_{k_p})\longrightarrow C(Y)\otimes M_n$ is a (not necessarily unital) $*$-homomorphism, let $e=\phi(1\otimes I)$. Assume that every complex vector bundle over Y of rank $\leq n$ is trivial. In the natural way we associate to e a vector bundle E over Y. Let $m=\mathrm{rank}(E)\leq n$. Taking in account that E is trivial and moreover that $H^1(Y,U(m)) = H^1(Y,U(n-m)) = 0$, it follows that there exists $v\in C(Y,U(n))$ such that

$$e(y) = v(y)\begin{pmatrix} \overbrace{1 \;\ddots\; 1}^{m} & 0 \\ 0 & 0 \;\ddots\; 0\end{pmatrix}v(y)^*.$$

Therefore $\mathrm{Ad}(v^*)\circ\phi$ takes values in $C(Y)\otimes M_m$ and further we apply the results from the unital case.

§2. The case $X = Y = S^1$

It is known that every complex vector bundle over S^1 is trivial. Also, the covering spaces of S^1 are wellknown and we can obtain more precise formulae for $\phi_i^!$ and ϕ. (The notations are the same as in §1).

Let's denote by χ the unitary which generates the C^*-algebra $C(S^1)$ ($\chi(z)=z$) and put $u_i := \phi_i^!(\chi)$, $\phi_i^! : C(S^1) \longrightarrow C(S^1)\otimes M_{m_i}$. Taking in account that $\pi_1(U(m_i)) \simeq \mathbb{Z}$ is generated by

$$v(z) = \begin{pmatrix} z & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix},$$

it follows that u_i is homotopic with some v_i,

$$v_i(z) = \begin{pmatrix} z^{d_i} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, d_i \in \mathbb{Z}$$

and therefore $\phi_i^!$ is homotopic with some $\Psi_{d_i} : C(S^1) \to C(S^1)\otimes M_{m_i}$,

$$\Psi_{d_i}(f)(z) = \begin{pmatrix} f(z^{d_i}) & & & 0 \\ & f(1) & & \\ & & \ddots & \\ 0 & & & f(1) \end{pmatrix}.$$

It is natural to consider $*$-homomorphisms $\Psi : C(S^1)\otimes F_1 \longrightarrow C(S^1)\otimes M_n$ given by

$$\Psi(f_1 \oplus \ldots \oplus f_p)(z) = u(z)(\bigoplus_i \Psi_{d_i}(f_i))u(z)^*,$$

where $f_i \in C(S^1)\otimes M_{k_i}$, $u \in C(S^1, U(n))$, $d_i \in \mathbb{Z}$ and the dimension of the block $\Psi_{d_i}(f_i)$ is $m_i k_i$. (If some m_i is zero, the corresponding block does not appear). We have

<u>THEOREM</u>

Given an unital $*$-homomorphism

$$\phi : C(S^1)\otimes(M_{k_1} \oplus \ldots \oplus M_{k_p}) \longrightarrow C(S^1)\otimes(M_{n_1} \oplus \ldots \oplus M_{n_q}),$$

there are a $q \times p$ matrix (m_{ij}), $m_{ij} \in \mathbb{Z}_+$, a $q \times p$ matrix (d_{ij}), $d_{ij} \in \mathbb{Z}$ such that $\sum_j m_{ij}k_j = n_i$, $i=1,\ldots,q$ and $m_{ij}=0$ implies $d_{ij}=0$, and an unitary $u \in C(S^1)\otimes(M_{n_1}\oplus\ldots\oplus M_{n_q})$ such that Φ is homotopic with the (canonical) $*$-homomorphism Ψ given by

$$(*)\quad \Psi(f_1\oplus\ldots\oplus f_p)(z) = u(z)\Big(\bigoplus_i\big(\bigoplus_j \Psi_{d_{ij}}(f_j)\big)\Big)u(z)^*,$$

where $f_j \in C(S^1)\otimes M_{k_j}$ and $\dim \Psi_{d_{ij}}(f_j)=k_j m_{ij}$.

The matrices (m_{ij}) and (d_{ij}) are unique with these properties.

REMARK.

The composition of two canonical homomorphisms is homotopic with a canonical homomorphism.The proof is based on the fact that the generator of $\pi_1(U(n))$ is $\begin{pmatrix} z & & 0 \\ & 1 & \\ 0 & & \ddots\ 1 \end{pmatrix}$.

Considering homotopy classes of canonical homomorphisms $C(S^1)\otimes F_1 \longrightarrow C(S^1)\otimes F_2$ which are uniquely determined by two matrices (m_{ij})(some multiplicities) and (d_{ij})(some degrees) with $m_{ij}=0$ implies $d_{ij}=0$, one can calculate the shape of certain inductive limits of C^*-algebras of the form $C(S^1)\otimes F$. This problem was considered by E.G.Effros and J.Kaminker (see [3]).

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