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In [2] Nikolskii and Vasyunin analyze the interrelation between the functional models for contractions of de Branges-Rovnyak and Sz-Nagy-Foiaş. We present here a slightly different approach, using the relative structure of subspaces of a Hilbert space and avoiding computations as far as possible. Thus, with the exception of proposition 3, this note is mainly a summary of the main results of [2], but presented in an alternate manner, which appeals to simple geometric intuitions. It may be useful for better understanding of other questions relating to this subject (for instance, the scalar case investigated in [5]).

§1. Complementary subspaces.

Let $\mathcal{E}$, K be Hilbert spaces, $\mathcal{E}$ contractively embedded in K. This means that there exists a Hilbert space H and a contraction $T: H \longrightarrow K$, such that $TH = \mathcal{E}$, and $\|Th\|_{\mathcal{E}} = \|P_{(\ker T)^\perp} h\|_H$ (thus, if $\ker T = \{0\}$, then $\|Th\|_{\mathcal{E}} = \|h\|_H$). It is well known then that there exists a Hilbert space $\mathcal{H}$ and two isometric embeddings $i: H \hookrightarrow \mathcal{H}$, $j: K \hookrightarrow \mathcal{H}$, such that $T = j^* i$. If H and T are given, then $\mathcal{H}$, i, j are uniquely determined (up to unitary equivalence) by the minimality condition $\mathcal{H} = iH \vee jK$. There are several ways of constructing $\mathcal{H}$; for instance, start with $H \oplus K$ endowed with the semi scalar product given by the positive operator

$$\begin{pmatrix} I & T^* \\ T & I \end{pmatrix}$$

T then "measures the angle" between the embeddings of H and K; it corresponds to the projection from iH on jK, and $j\mathcal{E}$ is the image of this projection. In this section we fix $\mathcal{H}$ and identify, for simplicity, H and K with their embeddings. Then, if P is the projection onto K, we have $\mathcal{E} = PH$.

The "complementary subspace" $\mathcal{E}'$ is a contractively embedded subspace of K, uniquely defined by the conditions (see):

(i) $\|x + x'\|^2 \leq \|x\|^2_{\mathcal{E}} + \|x'\|^2_{\mathcal{E}'}$ for any $x \in \mathcal{E}$, $x' \in \mathcal{E}'$.

(ii) each $h \in K$ has a unique decomposition $h = x + x'$, $x \in \mathcal{E}$, $x' \in \mathcal{E}'$, with $\|h\|^2 = \|x\|^2_{\mathcal{E}} + \|x'\|^2_{\mathcal{E}'}$.

We have then, with the notations above:

Proposition 1. Let $H' = H^{\perp}$. Then $\mathcal{E}' = PH'$, with $\|Ph'\|_{\mathcal{E}'} = \|h'\|$.

Proof. We may suppose $\ker T = \{0\}$, that is, $\|Ph\|_{\mathcal{E}} = \|h\|$. If $x \in \mathcal{E}$, $x' \in \mathcal{E}'$, then $x = Ph$, $x' = Ph'$, with $h \in H$, $h' \in H'$. We have

(1) $$\|x + x'\|^2 = \|P(h+h')\|^2 \leq \|h + h'\|^2 = \|h\|^2 + \|h'\|^2 = \|x\|_{\mathcal{E}}^2 + \|x'\|_{\mathcal{E}'}^2$$

We have proved (i). For (ii), let $k \in K$; then $k = h + h'$, $h \in H$, $h' \in H'$. Put $x = Ph$, $x' = Ph'$. Then $k = x + x'$, and

$$\|k\|^2 = \|h\|^2 + \|h'\|^2 = \|x\|_{\mathcal{E}}^2 + \|x'\|_{\mathcal{E}'}^2$$

For any other decomposition $k = x + x'$, we may write (1), where equality implies $h + h' \in K$, whence $k = h + h'$ and h and h' must be the projections of k onto H and H' respectively. ■

The proposition shows that complementarity of contractively embedded subspaces is orthogonality in a larger space projected onto K. Note that for this construction $\ker(P|_{H'}) = \{0\}$, although we could have had $\ker(P|_H) \neq \{0\}$.

§ 2. The canonical model.

Suppose E, E_* are two Hilbert spaces, $\Theta(z)$ is a contractive analytic function in $\mathbb{D}$ with values in $\mathcal{L}(E, E_*)$. It is well known ([3]) that $\Theta(z)$ has nontangential strong limits on $\mathbb{T}$, and the formula

$$(\Theta f)(z) = \Theta(z) f(z)$$

yields a contraction in $\mathcal{L}(L^2(E), L^2(E_*))$, which maps $H^2(E)$ into $H^2(E_*)$. Being a contraction operator, Θ may be interpreted as measuring the angle between two embeddings of $L^2(E)$ and $L^2(E_*)$ into a larger space (as in §1). We then obtain the canonical model for contractions ([3]), which we will describe below in the "coordinate free" manner of Vasyunin ([4],[6]). Thus, we have a Hilbert space $\mathcal{H}$ and two isometric embeddings $\pi : L^2(E) \to \mathcal{H}$, $\pi_* : L^2(E_*) \to \mathcal{H}$, such that $\mathcal{H} = \pi(L^2(E)) \vee \pi_*(L^2(E_*))$ and $\pi_*^* \pi = \Theta$. The analiticity of Θ is equivalent to the condition $\pi(H^2(E)) \perp \pi_*(H^2_-(E_*))$. Since Θ commutes with multiplication by z, it is possible to define a unitary operator U on $\mathcal{H}$, such that $\pi^* U \pi$ is multiplication by z on $L^2(E)$, while $\pi_*^* U \pi_*$ is multiplication by z on $L^2(E_*)$. $\pi(L^2(E))$ and $\pi_*(L^2(E_*))$ are then reducing subspaces for U.

Define

$$\mathcal{K} = \mathcal{H} \ominus \left(\pi(H^2(E)) \oplus \pi_*(H^2_-(E_*)) \right)$$

Then $\mathcal{K}$ is a semi-invariant subspace for U, and $T_\Theta = P_{\mathcal{K}} U|_{\mathcal{K}}$ is a copletely non-unitary contraction, whose characteristic function ([3]) is the pure part of Θ.

For comparison, in the actual construction of Sz.Nagy and Foias, we define $\Delta(z) = (I - \Theta^*(z)\Theta(z))^{1/2}$. Then

$$\mathcal{H} = L^2(E_*) \oplus \overline{\Delta L^2(E)}$$

$$\Pi f = \Theta f \oplus \Delta f \quad , \quad \Pi_* g = g \oplus 0$$

and it follows that

$$\mathcal{K} = [H^2(E_*) \oplus \overline{\Delta L^2(E)}] \ominus \{\Theta u \oplus \Delta u : u \in H^2(E)\}$$

In the sequel, if H is a subspace either in $\mathcal{H}$ or in $L^2(E)$ or $L^2(E_*)$, we will always denote by P_H the orthogonal projection onto H. Also, we will denote:

$$P_* = \Pi_* \Pi_*^* \quad , \quad P = \Pi \Pi^*$$

$$P_{*+} = \Pi_* P_{H^2(E_*)} \Pi_*^* \quad , \quad P_{*-} = P_* - P_{*+}$$

$$P_+ = \Pi P_{H^2(E)} \Pi^* \quad , \quad P_- = P - P_+$$

By S we will denote multiplication by z either in $H^2(E)$ or $H^2(E_*)$ (it will be clear from the context where) ; S^* is its adjoint.

§3. The "premodel" space of de Branges and Rovnyak.

From our point of view, the construction of de Branges and Rovnyak is a reconstruction of the above objects from their "shadows" on $H^2(E_*)$ (that is, projections onto $\Pi_*(H^2(E_*))$). It is then natural to break $\mathcal{K}$ into two parts, namely

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_{00}$$

where

$$\mathcal{K}_{00} = \mathcal{K} \cap [\Pi_*(H^2(E_*))]^\perp$$

$$\mathcal{K}_0 = \mathcal{K} \ominus \mathcal{K}_{00}$$

Thus, $\mathcal{K}_{00} = \Pi_*(L^2(E_*))^\perp \cap \Pi(H^2(E))^\perp$

Also, $\mathcal{K}_{00}$ is invariant to U^*, and $U^*|_{\mathcal{K}_{00}}$ is an isometry. (obviously $\mathcal{K}_{00}$ is then invariant to T_Θ^*, and $T_\Theta^*|_{\mathcal{K}_{00}}$ is an isometry). Actually, it can easily be show n that $\mathcal{K}_{00}$ is the largest subspace of $\mathcal{K}$ on which T_Θ^* is an isometry.

But the space we will consider in this section is $\mathcal{K}_0$. Since $\mathcal{K}_{00} = [\Pi_*(L^2(E_*)) \vee \Pi(H^2(E))]^\perp$, we have

$$\mathcal{K}_0 = [\Pi_*(L^2(E_*)) \vee \Pi(H^2(E))] \cap \Pi(H^2(E))^\perp \cap \Pi_*(H^2_-(E_*))^\perp =$$

$$= [\Pi_*(H^2(E_*)) \vee \Pi(H^2(E))] \ominus \Pi(H^2(E))$$

(for the last equality we have used the fact that $\Pi(H^2(E)) \subset \Pi_*(H^2_-(E_*))^\perp$).

We shall denote $\mathcal{H}_+ = \widetilde{\Pi}_*(H^2(E_*)) \vee \widetilde{\Pi}(H^2(E))$. Thus $\mathcal{H}_+$ is invariant to U and

(2) $\mathcal{K}_o = \mathcal{H}_+ \ominus \widetilde{\Pi}(H^2(E))$

Now, we denote by $\mathcal{M}(\Theta)$ the contractively embedded subspace of $H^2(E_*)$ defined (see §1) by $\Theta|_{H^2(E_*)}$. Relation (2) and proposition 1 show that $\widetilde{\Pi}_*^*(\mathcal{K}_o)$ is the complementary space of $\mathcal{M}(\Theta)$. This is the "premodel" space ; it will be denoted by $\mathcal{H}(\Theta)$. Also, by proposition 1, $\widetilde{\Pi}_*^*$ is unitary operator from $\mathcal{K}_o$ to $\mathcal{H}(\Theta)$.

Proposition 2. (i) $S^*\mathcal{H}(\Theta) \hookrightarrow \mathcal{H}(\Theta)$ (that is, $\mathcal{H}(\Theta)$ is invariant to S^* and S^* acts contractively on it).

(ii) For any $f \in \mathcal{H}(\Theta)$, we have the "inequality for difference quotients" (   ) :

$$\| S^* f \|_{\mathcal{H}(\Theta)} \leq \| f \|^2_{\mathcal{H}(\Theta)} - \| f(0) \|^2$$

Proof. Denote by $U_+ = U|_{\mathcal{H}_+}$. Then $\mathcal{K}_o$ is invariant to U_+^*, $S = \widetilde{\Pi}_*^* U_+ \widetilde{\Pi}_*$, $S^* = \widetilde{\Pi}_*^* U_+^* \widetilde{\Pi}_*$. If $f \in \mathcal{H}(\Theta)$, then $f = \widetilde{\Pi}_*^* h$, $h \in \mathcal{K}_o$; therefore

(3) $S^* f = \widetilde{\Pi}_*^* U_+^* \widetilde{\Pi}_* \widetilde{\Pi}_*^* h = \widetilde{\Pi}_*^* U_+^* h - \widetilde{\Pi}_*^* U_+^* (1 - P_*) h = \widetilde{\Pi}_*^* U_+^* h$

since $I - P_*|_{\mathcal{H}_+}$ is the projection onto $\mathcal{H}_+ \ominus \widetilde{\Pi}_*(H^2(E_*))$, and this last space is invariant to U_+^*. Since $\mathcal{K}_o$ is invariant to U_+^*, it follows that $\mathcal{H}(\Theta)$ is invariant to S^*.

Denote $\mathcal{N} = \ker U_+^*$. Then $U_+^*|_{\mathcal{H}_+ \ominus \mathcal{N}}$ is an isometry. For any $h \in \mathcal{K}_o$, we have

$$\| h \|^2 = \| (I - P_{\mathcal{N}}) h \|^2 + \| P_{\mathcal{N}} h \|^2 = \| U_+^* (I - P_{\mathcal{N}}) h \|^2 + \| P_{\mathcal{N}} h \|^2 = \| U_+^* h \| + \| P_{\mathcal{N}} h \|^2$$

If $f \in \mathcal{H}(\Theta)$, $f = \widetilde{\Pi}_*^* h$, then by (3) we have

(4) $\| S^* f \|^2_{\mathcal{H}(\Theta)} = \| U_+^* h \|^2 = \| h \|^2 - \| P_{\mathcal{N}} h \|^2 = \| f \|^2_{\mathcal{H}(\Theta)} - \| P_{\mathcal{N}} h \|^2$

But $\mathcal{N} = \mathcal{H}_+ \ominus U_+ \mathcal{H}_+$. If $\mathcal{N}_o = \widetilde{\Pi}_*($ constants in $H^2(E_*))$, then $\mathcal{N}_o \perp U_+(\widetilde{\Pi}_*(H^2(E_*))$, and also $\mathcal{N}_o \perp U_+(\widetilde{\Pi}(H^2(E))$, since $\widetilde{\Pi}_* U_+(\widetilde{\Pi}(H^2(E)) = U_+ \widetilde{\Pi}_*(\widetilde{\Pi}(H^2(E))$. Therefore $\mathcal{N}_o \subset \mathcal{N}$, yand

$$\| S^* f \|^2_{\mathcal{H}(\Theta)} \leq \| f \|^2_{\mathcal{H}(\Theta)} - \| P_{\mathcal{N}_o} h \|^2 = \| f \|^2_{\mathcal{H}(\Theta)} - \| f(0) \|^2$$

since

$$P_{\mathcal{N}_o} h = P_{\mathcal{N}_o} P_* h = \widetilde{\Pi}_* f(0)$$

If the inequality for difference quotients is actually an equality for any $f \in \mathcal{H}(\Theta)$, $\mathcal{H}(\Theta)$ is said to satisfy the identity for difference quotients. Let us investigate when this happens. By relation (4), we should have $P_{\mathcal{K}} h = P_{\mathcal{K}_0} h$ for any $h \in \mathcal{K}_0$, or, equivalently,

$$\mathcal{N} \ominus \mathcal{N}_0 \perp \mathcal{K}_0 \quad \text{, that is,} \quad \mathcal{N} \ominus \mathcal{N}_0 \subset \tilde{\pi}(H^2(E))$$

Denote by $\mathcal{L}$ the wandering subspace of the pure isometry $U^*|_{\mathcal{K}_{00}}$. From the definition of $\mathcal{K}_{00}$ and the relation

$$\mathcal{N} \ominus \mathcal{N}_0 = \mathcal{H}_+ \ominus [\tilde{\pi}_*(H^2(E_*)) \vee U_+(\tilde{\pi}(H^2(E)))]$$

it follows that

$$\mathcal{N} \ominus \mathcal{N}_0 = U\mathcal{L}$$

Thus, the condition $\mathcal{N} \ominus \mathcal{N}_0 \perp \mathcal{K}_0$ becomes $U\mathcal{L} \subset \tilde{\pi}(H^2(E))$. Since, in any case, $U\mathcal{L} \perp U\tilde{\pi}(H^2(E))$, it follows that $U\mathcal{L} = \tilde{\pi}(E')$, where E' is a subspace of E and we have identified constant functions in $H^2(E)$ with their values. Consequently, $\mathcal{K}_{00} = \tilde{\pi}(H^2_-(E'))$, where $E' \subset E$. We may also rephrase this condition in terms of Θ; we have thus proved

Proposition 3. The following are equivalent:

(i) $\mathcal{H}(\Theta)$ satisfies the identity for difference quotients;

(ii) $\mathcal{K}_{00} = \tilde{\pi}(H^2_-(E'))$, with $E' \subset E$;

(iii) There exists a decomposition $E = E' \oplus E''$, such that $\Theta H^2(E') = 0$, while $\overline{\Delta H^2(E'')} = \overline{\Delta L^2(E'')}$.

(the equivalence of (ii) and (iii) is immediately seen for instance on the model of Sz.Nagy and Foiaş).

The most important case in which the conditions in proposition 3 are satisfied is $\mathcal{K}_{00} = \{0\}$, or $\overline{\Delta H^2(E)} = \overline{\Delta L^2(E)}$. This happens if and only if T^*_Θ contains no isometric part (see [2] for other equivalent conditions). Then the premodel space already coincides with the model ($\mathcal{K}_0 = \mathcal{K}$). In this case, by (3), we obtain that T^*_Θ is unitarily equivalent to $S^*|_{\mathcal{H}(\Theta)}$; this last operator acts by the formula

$$(S^*f)(z) = \frac{f(z) - f(0)}{z}$$

Moreover, the identity for difference quotients yields then the relation

$$\| D_{T^*_\Theta} f \|^2_{\mathcal{H}(\Theta)} = \| f(0) \|^2$$

§ 4. The model space.

In the general case, we will describe the elements of $\mathcal{K}$ by their projections onto $\hat{\pi}_*(L^2(E_*))$ and $\hat{\pi}(L^2(E))$. Let k be an element of $\mathcal{K}_0$; define $f = \hat{\pi}_*^* k \in H^2(E_*)$, $g = \hat{\pi}^* k \in H^2_-(E)$. We have to determine which pairs $\{f, g\} \in H^2(E_*) \oplus H^2_-(E)$ occur in this way and then we have to recapture from $\{f, g\}$ the norm of k. An asymmetry appears between the roles of f and g, since we want to rely eventually on $\mathcal{H}(\Theta)$, which is included in $H^2(E_*)$.

Suppose $k = k_0 + k_{00}$, $k_0 \in \mathcal{K}_0$, $k_{00} \in \mathcal{K}_{00}$. Since $P_* k = P_* k_0$, it follows that f must belong to $\mathcal{H}(\Theta)$. Also, any $f \in \mathcal{H}(\Theta)$ determines uniquely $k_0 \in \mathcal{K}_0$.

To recapture g, we have to "take out" vectors from $\mathcal{K}_{\infty}$, which is orthogonal to $\hat{\pi}_*(H^2(E_*))$ and therefore is not related to $\mathcal{H}(\Theta)$. This will be done by applying powers of U.

For any $n \geq 0$, we have

$$(5) \qquad U^{n+1}(\tilde{\pi}_* f) - P_* P_+ U^{n+1}(\tilde{\pi} g) = U^{n+1} P_* k - P_* P_+ U^{n+1} P k =$$

$$= P_* U^{n+1} k - P_* P_+ U^{n+1} k = P_* (I - P_+) U^{n+1} k$$

But, since $k \in \tilde{\pi}_*(H^2_-(E_*))^\perp$, $U^{n+1} k \in \tilde{\pi}_*(H^2_-(E_*))^\perp$, and $(I-P_+)U^{n+1} k \in \mathcal{K}$. Since $P_* \mathcal{K} = P_* \mathcal{K}_0$, it follows that $\tilde{\pi}_*^*(I-P_+)U^{n+1} k \in \mathcal{H}(\Theta)$. This last statement may be rewritten, using (5), in terms of the functions f and g:

$$z^{n+1} f - \Theta P_{H^2(E)} z^{n+1} g \in \mathcal{H}(\Theta)$$

Also,

$$\| k \|^2 = \| U^{n+1} k \|^2 = \| P_+ U^{n+1} k \|^2 + \| (I-P_+) U^{n+1} k \|^2 =$$

$$= \| P_+ U^{n+1} k \|^2 + \| P_{\mathcal{K}_0} (I-P_+) U^{n+1} k \|^2 + \| P_{\mathcal{K}_{00}} U^{n+1} k \|^2$$

But, for any $h \in \mathcal{H}$, $P_{\mathcal{K}_{00}} U^{n+1} h \to 0$ for $n \to \infty$ (this may be checked separately for vectors in $\tilde{\pi}_*(L^2(E_*))$ and in $\tilde{\pi}(L^2(E))$. It follows that

$$\| k \|^2 = \lim_{n \to \infty} \left(\| P_+ U^{n+1} k \|^2 + \| P_{\mathcal{K}_0} (I - P_+) U^{n+1} k \|^2 \right) =$$

$$= \lim_{n \to \infty} \left(\| P_{H^2(E)} z^{n+1} g \|^2 + \| z^{n+1} f - \Theta P_{H^2(E)} z^{n+1} g \|^2_{\mathcal{H}(\Theta)} \right)$$

On the basis of this computations, we define the "model" space:

$$\mathcal{D}(\Theta) = \{ \{f, g\} : f \in \mathcal{H}(\Theta), g \in H^2_-(\Theta) ; z^{n+1} f - \Theta P_{H^2(E)} z^{n+1} g \in \mathcal{H}(\Theta) \text{ for any } n \geq 0$$
$$\text{and } \| z^{n+1} f - \Theta P_{H^2(E)} z^{n+1} g \|_{\mathcal{H}(\Theta)} \text{ is bounded in } n \}$$

$$\| \{f, g\} \|^2_{\mathcal{D}(\Theta)} = \sup_{n \geq 0} \left(\| P_{H^2(E)} z^{n+1} g \|^2 + \| z^{n+1} f - \Theta P_{H^2(E)} z^{n+1} g \|^2_{\mathcal{H}(\Theta)} \right)$$

For the moment, $\mathcal{D}(\Theta)$ is only a normed linear space. We have shown above that the operator $\mathcal{U} : \mathcal{K} \to \mathcal{D}(\Theta)$, defined by

$$\mathcal{U}(k) = \{ \tilde{\pi}_*^* k, \tilde{\pi}^* k \}$$

is an isometry.

<u>Proposition 4</u>. $\mathcal{D}(\Theta)$ is complete and $\mathcal{U}$ is unitary.

<u>Proof</u>. We will show that $\mathcal{U}$ is onto. Let $\{f, g\}$ belong to $\mathcal{D}(\Theta)$. Let $h_0 \in \mathcal{K}_0$, such that $\tilde{\pi}_*^* h_0 = f$. Then $\mathcal{U}(h_0) \in \mathcal{D}(\Theta)$, and $\{f, g\} - \mathcal{U}(h_0) = \{0, g_0\}$, where $g_0 = g - \tilde{\pi}^* h_0$; we know that $\Theta P_{H^2(E)} z^{n+1} g_0 \in \mathcal{H}(\Theta)$ for any $n \geq 0$, and that $\| \Theta P_{H^2(E)} z^{n+1} g_0 \|_{\mathcal{H}(\Theta)}$ is bounded in n. That means that $P_* P_+ U^{n+1} \tilde{\pi} g_0 = P_* k_n$, where $k_n \in \mathcal{K}_0$, $\| k_n \|$ are bounded in n. Therefore $P_+ U^{n+1} \tilde{\pi} g_0 - k_n$ is orthogonal to $\tilde{\pi}_*(L^2(E_*))$. It follows that $\ell_n = U^{*n+1} P_+ U^{n+1} \tilde{\pi} g_0 - U^{*n+1} k_n$ is orthogonal to $\tilde{\pi}_*(L^2(E_*))$ and also to $\tilde{\pi}(H^2(E))$; that is, $\ell_n \in \mathcal{K}_{00}$. Now, $U^{*n+1} P_+ U^{n+1} \tilde{\pi} g_0 \to \tilde{\pi} g_0$ for $n \to \infty$, while the sequence

$\{U^{*n+1}k_n\}$ has a subequence converging weakly, say, to $y \in \mathcal{H}$. Since $k_n \perp \tilde{\pi}(H^2(E))$, $U^{*n+1}k_n \perp U^{*n+1}\tilde{\pi}(H^2(E))$, and therefore $y \perp \tilde{\pi}(L^2(E))$. If we define $k = \tilde{\pi}g - y$, then $k \in \mathcal{K}_{>0}$ (since it is the weak limit of some ℓ_n-s), and $\tilde{\pi}^* k = g_0$, $\tilde{\pi}_*^* k = 0$. We have thus

$$\mathcal{U}(h_0 + k) = \{f, g\}$$

and the proposition is proved.

We may now write the action of the "model operator" $\mathcal{M}_\Theta = \mathcal{U}\, T_\Theta\, \mathcal{U}^*$. Using the fact that

$$T_\Theta = (1 - P_+ - P_{*-})U\big|_{(1-P_+-P_{*-})\mathcal{H}}$$

straightforward computations lead to the formula

$$\mathcal{M}_\Theta(\{f, g\}) = \{zf + \Theta(z)(zg)(0),\ zg - (zg)(0)\} \tag{6}$$

Actually, the standard form of the de Branges-Rovnyak model is written with $g \in H^2(E)$; this amounts in changing, in the definition of $\mathcal{D}(\Theta)$, g by $Jg = \bar{z}g(\bar{z})$ (and correspondingly in formula (6)).

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