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On complex vector bundles on projective threefolds

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Introduction

One of the first questions which arises in the study of (complex) vector bundles on complex projective manifolds is to determine those topological bundles which admit algebraic, or equivalently holomorphic, structures.

In the case of curves, every topological vector bundles has an algebraic structure. Schwarzenberger [15] has shown for surfaces that a topological vector bundle admits an algebraic structure if and only if its first Chern class is algebraic.

From the papers of Atiyah-Rees [3], Horrocks [10] and Vogelaar [18] it follows that every topological vector bundle on the projective space $\mathbb{P}^3$ has an algebraic structure. The same conclusion holds on a homogeneous rational threefold, [5] and [6]. What about the general case of (complex, smooth) projective threefolds?

One can expect that in this case a topological vector bundle admits an algebraic structure if and only if its Chern classes are algebraic. The "only if" part is clear, and for the proof of the "if" part only rank 2 and 3 vector bundles should be taken into account.

First, the topological classification is needed. It is discus[sed] in the second section of the present paper for any compact connected three-dimensional complex manifolds. The rank 3 bundles are parametrized, via Chern classes, by cohomology classe[s] (c_1,c_2,c_3) subject only to an integrality condition; the rank 2 vector bundles are determined, roughly speaking, by the Chern classes (c_1,c_2) and by two other "finite" invariants $\beta(c_1,c_2)$

and $\alpha(c_1,c_2,\beta(c_1,c_2))$ (see Theorem 1 for the precise statement).

In the third section we answer into affirmative the above question in stable rank, i.e. ≥ 3, by proving the following:

Theorem 2. A topological vector bundle of rank 3 on a projective 3-fold admits an algebraic structure if and only if its Chern classes are algebraic.

The remaining case of rank 2 vector bundles is treated in the fourth section, where we prove a weaker result:

Theorem 3. For every rank 2 topological vector bundle on a projective 3-fold X with algebraic Chern classes (c_1,c_2), there exists an algebraic vector bundle with the same Chern classes.

Moreover, if $c_1=c_1(K_X)$, then one can find even a K_X-symplectic such bundle, with a prescribed arbitrary value in $\mathbb{Z}_2$ for the Atiyah-Rees invariant.

Recall that if E is a holomorphic K_X-symplectic bundle, then the (analytical) Atiyah-Rees invariant [3] is

$$\alpha(E) = h^0(E) + h^2(E) \pmod 2 .$$

In the Theorems 2 and 3 above one constructs algebraic bundles by the method of extensions [16], in the case of rank 3 bundles by a variant given in [5]. The fact that 1-dimensional cycles are smoothable and that they are very movable, due to the residual intersection device, plays an important role in constructions. Non-reduced curves (of smooth support) are intensively used. Accordingly, a key point in the proof of Theorem 3 is played by a method of "tripling" curves, due to Forster and the first author [4]: it was very interesting to us to see how the triple structures are connected with the integrality condition $(c_1+c_1(X))c_2 \equiv 0 \pmod 2$ and how they relate the Athiyah-Rees invariant with θ-characteristics.

In a final section we combine the previous results in order to prove the next:

Theorem 4. Every topological vector bundle on a projective rational threefold admits algebraic structures.

In that case an essential step in the proof is the fact that the analytical Atiyah-Rees invariant for K_X-symplectic bundles corresponds to the topological ones [3]. We were not able to prove the result for rank 2 vector bundles in its full generality, mainly due to the fact that we failed to understand the intricate question whether the second topological invariant β has an analytic counterpart.

The first section contains preliminaries concerning extensions, double and triple structures,cycles of dimension 1 and a statement derived from the Atiyah-Hirzebruch spectral sequence in K-theory.

We adopt the following conventions. A 3-fold means in the sequel a complex,connected,smooth,projective manifold of dimension 3. We abbreviate l.c.i. for locally complete intersection. We simply say bundle for a topological complex vector bundle, and if the context is clear,for an algebraic vector bundle.Sometimes we denote by the same letter an integer cohomology class as well as its image by the reduction $\mathbb{Z} \longrightarrow \mathbb{Z}_2$.

1.Preliminaries

1.1 Let us firstly recall the construction of rank 2 bundles by extensions ([16],[14]Ch.I.§5). Denote by X a 3-fold,Y a l.c.i. curve in X and by L_1, L_2 line bundles on X. Assume that $\det(N_Y) \cong L_1^{-1} \otimes L_2$ and that $H^2(X, L_1 \otimes L_2^{-1})=0$. Then there exists a bundle E of rank 2, given by an extension

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow I_Y L_2 \longrightarrow 0.$$

One has $\det(E) \cong L_1 \otimes L_2$ and the next formulae for the Chern classes:

$c_1(E)=c_1(L_1)+c_1(L_2)$

$c_2(E)=c_1(L_1)c_1(L_2)+[Y]$.

We shall consider for our purposes L_1 to be sufficiently positive and L_2 to be sufficiently negative. Then the condition $H^2(X,L_1\otimes L_2^{-1})=0$ is satisfied, and, we notice in this case the equality

$h^0(E)+h^2(E)=h^0(L_1)+h^1(L_2|Y)$,

a relation which will be needed in the computation of the Atiyah-Rees invariant. Due to the condition on det , the difficulty in performing this construction lies in finding the right curves: we shall consider convenient disjoint unions of double and triple structures.

In order to obtain bundles of rank 3 we need the following variant of the above construction, [5]. Let $Y\subset X$ be as above, let F be a rank 2 bundle on X and L a line bundle on X. Assume that $H^2(X,F)=0$ and that det $N_Y\otimes F$, regarded as a rank 2 bundle on Y , has a nowhere vanishing global section. Then there exists a bundle E of rank 3 on X, given by an extension

$$0\longrightarrow F\longrightarrow E\longrightarrow I_Y\longrightarrow 0.$$

The Chern classes of E are given by the formulae:

$c_1(E)=c_1(F)$; $c_2(E)=c_2(F)+[Y]$,

$c_3(E)=(c_1(F)+c_1(X))[Y]-2\chi(O_Y)$.

1.2 Ferrand's method of "doubling" a curve is as follows, [7]. Again X is a 3-fold and $Y\subset X$ is a l.c.i. curve. Assume that one has an epimorphism $N_Y^*=I_Y/I_Y^2\xrightarrow{\varepsilon}L\longrightarrow 0$, where L is a line bundle on Y. Consider the closed subscheme Z of X given by the ideal

$$I_Z=\ker(I_Y\longrightarrow I_Y/I_Y^2\xrightarrow{\varepsilon}L).$$

Then Z is a l.c.i. ,it has the same support as Y, and

$$(\det N_Z)|Y\cong(\det N_Y)\otimes L^{-1}.$$

At the level of cohomology classes one has $[Z]=2[Y]$. The exact sequence $0 \longrightarrow L \longrightarrow O_Z \longrightarrow O_Y \longrightarrow 0$ gives rise to the exact sequence $H^1(Y,L) \longrightarrow \mathrm{Pic}\, Z \longrightarrow \mathrm{Pic}\, Y \longrightarrow 1$.

Notice also that ,if L is sufficiently positive,then there exist epimorphisms ε . In that case, Pic $Z \cong$ Pic Y.

Next we briefly recall the "tripling" method [4]. One starts with a doubled curve as above. The inclusions of ideals yield the exact sequence

$$0 \longrightarrow I_Y^2/I_Y I_Z \longrightarrow I_Z/I_Y I_Z \longrightarrow I_Y/I_Y^2 \longrightarrow I_Y/I_Z \longrightarrow 0,$$

and the isomorphisms

$$I_Y/I_Z \cong L\ ,\ I_Y^2/I_Y I_Z \cong L^2,\ I_Z/I_Y I_Z \cong N_Z^*|Y.$$

One finds an exact sequence of bundles on Y:

$$0 \longrightarrow L^2 \xrightarrow{\ \ i\ \ } N_Z^*|Y \longrightarrow (\det N_Y^*) \otimes L^{-1} \longrightarrow 0.$$

Suppose that it splits and let τ be a retract of i. Take the subscheme T of X associated to the ideal

$$I_T = \ker(I_Z \longrightarrow I_Z/I_Y I_Z \xrightarrow{\ \tau\ } L^2).$$

Then T is a l.c.i. curve with the same support as Y, $[T]=3[Y]$ and $\det N_T|Y \cong (\det N_Y) \otimes L^{-2}$. The exact sequences

$$0 \longrightarrow L \longrightarrow O_Z \longrightarrow O_Y \longrightarrow 0,\ 0 \longrightarrow L^2 \longrightarrow O_T \longrightarrow O_Z \longrightarrow 0$$

allow to connect Pic T to Pic Y. They also are needed in the computation of the Atiyah-Rees invariant.

The obstruction to the above splitting lies in $H^1(Y, \det N_Y \otimes L^3)$, hence, if L is sufficiently positive, the above construction may be performed.

As we have seen, by doubling and tripling procedures one can vary the cohomology classes of curves. More important is to remark that, due to the degree of freedom allowed by the choice of the line bundle L, one can also vary the determinant of the normal bundle.

1.3. We shall need the residual intersection device and the smoothness of 1-dimensional cycles, see ([9], §7) for an account of these topics. We explain here for later use the meaning of the statement that, on 3-folds, the (smooth) 1-cycles are very movable.

We rely on the following form of Bertini's theorem: if L is a line bundle on a quasi-projective manifold X and V is a finite dimensional subspace of $H^0(X,L)$ which generates L, then for s general in V the scheme of zeroes of s is non-singular, and connected if dim $X \geq 2$.

Let Y be a non-singular curve in X, not necessarily connected. Take $k \gg 0$, so that $I_Y \otimes L^k$ is finitely generated. Then the general section s in $H^0(X, I_Y \otimes L^k)$ defines a non-singular, connected surface H_s passing through Y (one uses Bertini's theorem on $X \setminus Y$ and the fact that the image in $H^0(Y, N_Y^* \otimes L^k|Y)$ of the general section has no zeroes). Now, if one considers another (general) section t in $H^0(X, I_Y \otimes L^k)$, then H_t is still non-singular and connected, and the ideal-theoretically equality $H_s \cap H_t = Y \cup Z$ holds, where Z is a second non-singular curve (we choose t with the additional condition that it smoothly vanishes in the quotient bundle of $N_Y^* \otimes L^k|Y$ by the sub-bundle generated by s; the points of $Y \cap Z$ are exactly these zeroes). Moreover, we can choose s and t such that H_s and H_t should avoid a given finite subset of $X \setminus Y$ and Z should avoid a given finite subset of Y. In particular Z may avoid a given curve which has no common irreducible component with Y.

Let denote $h = c_1(L)$. One has $[H_s] = [H_t] = k.h$ and consequently $k^2h^2 = [Y] + [Z]$.

Notice also that if H′ and H′′ are general in $|H^0(X,L)|$, then $H' \cap H''$ is a smooth connected curve and $h^2 = [H' \cap H'']$. Here one can again choose a curve as representative of h^2, which should be disjoint of a given curve.

Let assume now that ξ is a 1-dimensional cycle on X. By a result of Hironaka and Kleiman, ξ is smoothable in its rational equivalence class. In particular, when passing to the associated cohomology classes in $H^4(X,Z)$, one gets

$$[\xi] = \sum_{i=1}^{k} n_i[Y_i],\ n_i \in \mathbb{Z},$$

where Y_i are smooth and connected curves. As we have already seen one can write $[Y_1]=k_1^2h^2-[Z_1]$ with $k_1 \gg 0$ and Z_1 disjoint of $Y_2,Y_3,\dots,Y_k$. By repeating this procedure for $Y_2,\dots,Y_k$ we can finally assume that ξ may be expressed as above, with new curves and coefficients, such that the curves Y_i are pairwisely disjoint. This fact, and some informations on the parity of n_i, is what we shall use in the sequel.

1.4 Let Y be a finite polyhedron and let d_{2k+1} denote the only possible non-trivial coboundaries in the Atiyah-Hirzebruch spectral sequence for K(Y), associated to the simplicial decomposition of Y. The following assertion is valid ([2] or [8], th.I in Ch.VIII):

Fix an integer $q \geq 1$. If $\alpha \in H^{2q}(Y,Z)$ safisfies $d_{2k+1}\alpha = 0$ for all $k \geq 1$, then there exists a class ξ in K(Y), such that ξ is trivial on the skeleton Y^{2q-1} and $c_q(\xi) = (q-1)!\alpha$.

§2. Toward the topological classification

2.1 Proposition Let Y be a finite polyhedron of dimension less or equal to 7 and assume $H^7(Y,\mathbb{Z})$ has not 2-torsion. The cohomology classes (c_1,c_2,c_3) are the Chern classes of a rank 3 vector bundle on Y iff

$$c_3 \equiv (c_1 + Sq^2)c_2 \quad \text{in } H^6(Y,\mathbb{Z}_2).$$

Proof. The necessity of the condition follows by pulling-back a formula of Wu, see [13], which asserts that the Stiefel-Whitney classes of the tautological bundle on BU(3) satisfy

$$W_6 = (W_2 + Sq^2)W_4.$$

Conversely, assume that the congruence condition holds for a triple (c_1,c_2,c_3). Every class $\xi \in K(Y)$ decomposes into $\xi = [E] + k[\mathbb{1}]$, where E is a rank 3 bundle on Y, $k \in \mathbb{Z}$ and $\mathbb{1}$ is the trivial line bundle, see [11]. Consequently, it suffices to find a $\xi \in K(Y)$, with $c_i(\xi)=c_i$, $i=1,2,3$.

First we choose a line bundle L with $c_1(L)=c_1$, and we put $\xi_1=[L]$, so that $c_1(\xi_1)=c_1$, $c_2(\xi_1)=0$ and $c_3(\xi_1)=0$. The only possible non-trivial coboundary in the Atiyah-Hirzebruch spectral sequence, starting at $E_r^{4,q}$, is $d_3 : H^4(Y,\mathbb{Z}) \longrightarrow H^7(Y,\mathbb{Z})$. As the values of d_3 lie in the 2-torsion part of $H^7(Y,\mathbb{Z})$, one gets $d_3=0$. Therefore there exists by 1.4 a class $\xi_2 \in K(Y)$, such that $c_1(\xi_2)=0$, $c_2(\xi_2)=c_2$ and let denote $c_3(\xi_2)=\tau$. Similarly one finds $\xi_3 \in K(Y)$ with $c_1(\xi_3)=0$, $c_2(\xi_3)=0$ and $c_3(\xi_3)=2\sigma$, where σ is a given, but arbitrary, class in $H^6(Y,\mathbb{Z})$.

Take $\xi = \xi_1 + \xi_2 + \xi_3$. Then $c_1(\xi)=c_1$, $c_2(\xi)=c_2$, $c_3(\xi)=c_1c_2+\tau+ +2\sigma$. Since $\tau \equiv Sq^2c_2$ and $c_3 \equiv c_1c_2+Sq^2c_2$ in $H^6(X,\mathbb{Z}_2)$, one can find a cohomology class σ, such that $c_3(\xi)=c_3$.

2.2 Theorem 1. Let X be a compact connected 3-dimensional complex manifold.

(1) The cohomology classes (c_1,c_2,c_3) are the Chern classes of a rank 3 topological vector bundle on X, iff $(c_1+c_1(X))c_2 \equiv c_3 \pmod 2$. Moreover, the triple (c_1,c_2,c_3) determines up to isomorphism the bundle.

(2) The pair (c_1,c_2) corresponds to the Chern classes of a rank 2 topological vector bundle on X iff $(c_1+c_1(X))c_2 \equiv 0 \pmod 2$. Moreover,

$$\mathrm{Vect}^2_{top}(X) \cong \{(c_1,c_2,\beta(c_1,c_2),\alpha(c_1,c_2,\beta(c_1,c_2)))\},$$

where (c_1,c_2) are as above, β is an arbitrary element in the quotient

$H^5(X,\mathbb{Z}_2)/(c_1+c_1(X))H^3(X,\mathbb{Z})+ c_2H^1(X,\mathbb{Z})$,

and α is an arbitrary element in some quotient of

$H^6(X,\mathbb{Z}_2)/(c_1+c_1(X))H^4(X,\mathbb{Z})$.

(Notice the abuse of notation in writing the denominators.)

For (1) see [6], theorem 1 or the above proposition. The statement (2) follows by analysing the first two steps of the Postnikov tower of the map $BU(2) \longrightarrow BU$, as described in [17]. For the first step $E_7 \longrightarrow BU$ one uses [12], namely formula 1.5 written for Chern classes, and further one applies the above propositin to the space $Y=SX$, the suspension of X. By remarking that the cup product $H^2 \times H^4 \longrightarrow H^6$ is trivial on SX, one finds that the set of homotopy classes of liftings $X \longrightarrow E_7$ of a given class $\xi \in [X,BU]$, with $c_3(\xi)=0$, is an affine space over $H^5(X,\mathbb{Z}_2)/\Delta(\xi)$, where $\Delta(\xi)$ is the range of the map

$$H^1(X,\mathbb{Z}) \times H^3(X,\mathbb{Z}) \longrightarrow H^5(X,\mathbb{Z}_2),$$

defined by $(\tau_1,\tau_3) \longmapsto c_2(\xi)\tau_1+(c_1(\xi)+Sq^2)\tau_3$. Further $[X,BU(2)] \cong [X,E_8]$ and the enumeration of the liftings relative to $E_8 \longrightarrow E_7$ is done in [17], p.101. All these facts imply (2).

It turns out that the right interpretation of the parametrization in (2) is given by actions of the groups $H^6(X,\mathbb{Z}_2)$ and $H^5(X,\mathbb{Z}_2)$ on the fibres in the decomposition

$$[X,BU(2)] \cong [X,E_8] \longrightarrow [X,E_7] \longrightarrow [X,BU].$$

The actions are transitive and those corresponding to the second arrow have precise isotropy groups. We don't know the isotropy groups for the first arrow, or equivalently, whether the parameter space for α is exactly

$H^6(X,\mathbb{Z}_2)/(c_1+c_1(X))H^4(X,\mathbb{Z})$.

The question is related, via K_X-symplectic K-theory [3], to the problem of lifting (up to homotopy) the determinants of automorphisms of the bundles, from the skeleton X^5 to X (a problem which "lies" in $H^1(X,\mathbb{Z})$, in fact in $H^1(X,\mathbb{Z}_2)$).

Let us assume in addition that $H_1(X,\mathbb{Z})=0$. Then one has no β in the parametrization of rank 2 bundles, and, if $c_1 \not\equiv c_1(X) \pmod 2$, then α disappears too. In the K_X-symplectic case, i.e. when $c_1 \equiv c_1(X) \pmod 2$, the space of values for α is precisely $H^6(X,\mathbb{Z}_2)=\mathbb{Z}_2$, see [6] or the above remark. As we have already mentioned in the introduction, in this case the analytical invariant α distinguishes the topological type, more precisely, if E_1, E_2 are holomorphic K_X-symplectic bundles with $\alpha(E_1)\neq\alpha(E_2)$, then E_1 is not topologically isomorphic with E_2, see [3].

Concluding, we remark that the statements (1) and (2) in the theorem are valid (replacing $c_1(X)$ by Sq^2) for an arbitrary finite polyhedron X of dimension less or equal to 6 and without 2-torsion in $H^6(X,\mathbb{Z})$.

§3. The proof of Theorem 2

3.1 We follow the general framework of [5]. We have to prove that if c_1, c_2, c_3 are algebraic cohomology classes, satisfying $(c_1+c_1(X))c_2 \equiv c_3 \pmod 2$, then there exists a rank 3 algebraic bundle with these Chern classes.

It is useful to write down the transformation formulae for the Chern classes of a rank 3 bundle G, tensorized with a line bundle P:

$$c_1(G\otimes P)=c_1(G)+3c_1(P),$$
$$c_2(G\otimes P)=c_2(G)+2c_1(G)c_1(P)+3c_1(P)^2,$$
$$c_3(G\otimes P)=c_3(G)+c_2(G)c_1(P)+c_1(G)c_1(P)^2+c_1(P)^3.$$

We can tensorize the topological bundle of Chern classes (c_1,c_2,c_3) with a sufficiently positive algebraic line bundle, such that c_1 would be the Chern class of an ample line bundle. By tensorizing again with a sufficiently high power of this line bundle, and by using 1.3 , we can assume in addition that $c_2=\sum k_i[Y_i]$, where $k_i>0$ and Y_i are pairwise disjoint smooth curves. The ample line bundle L appearing in the sequel satisfies $c_1(L)=c_1$. Put $h=c_1$. Let consider general elements H_1 and H_2 in $|H^0(X,L)|$ and let denote $C=H_1\cap H_2$, so that $[C]=h^2$. By the exact sequence

(1) $$0 \longrightarrow O(-H_1-H_2) \longrightarrow O(-H_1)\oplus O(-H_2) \longrightarrow I_C \longrightarrow 0,$$

we get $N_C\cong O(H_1)\oplus O(H_2)|Y\cong 2L|Y$. Accordingly, for every $k\geq 0$ one finds epimorphisms

$$I_C/I_C^2 \longrightarrow \det N_C\otimes L^{2k-1}|C \longrightarrow 0,$$

and consequently one can use Ferrand's method. Since $H^1(C,\det N_C\otimes L^{2k-1}|C)=0$ for large values of k (but independent of the realization C of h^2), one gets l.c.i. curves C´ supported by C, so that $[C']=2[C]$ and $\det N_{C'}\cong L^{1-2k}|C'$. By using this fact and the method of Serre one can find integers, and we fix such a value, say k_0, so that the next statement holds: for an arbitrary disjoint union Y of such double structures (with $k=k_0$) there exists an extension

(2) $$0 \longrightarrow L^{k_0} \longrightarrow F \longrightarrow I_Y L^{1-k_0} \longrightarrow 0,$$

with F locally free. We shall consider only curves Y which are disjoint of the curves Y_i which appear in the expression of c_2. One has

$c_1(F)=c_1(L)=h=c_1$,

$c_2(F)=(k_0(1-k_0)+2d)h^2$,

where d is the number of double curves considered in Y. The integer $d\geq 1$ is arbitrary and it will be conveniently chosen at the end of the proof.

<u>3.2</u> Start with an F as above. We seek for integers $t\gg 0$ and curves Z such that there exist extensions of the form

(3) $$0 \longrightarrow F\otimes L^{3t} \longrightarrow E\otimes L^{2t} \longrightarrow I_Z \longrightarrow 0,$$

with E locally free of rank 3, and such that the Chern classes of E are as prescribed, c_1, c_2, c_3. According to the conditions in Serre's construction one firstly has to verify that $H^2(X, F\otimes L^{3t})=0$ and that $\det N_Z \otimes F \otimes L^{3t}$ has a section without zeroes on Z. There is a rank t_0, such that for every $t \geq t_0$ one has $H^2(X,F\otimes L^{3t})=0$ for every F as above (use (1), the exact sequences $0 \longrightarrow L^{k_0-1} \otimes \det N_C \longrightarrow O_{C'} \longrightarrow O_C \longrightarrow 0$, and (2)).

If we can find an extension (3) with E locally free, then $c_1(E)= =c_1(F)=c_1$. For the second Chern class one finds

$$c_2(E\otimes L^{2t})=c_2(F\otimes L^{3t})+[Z].$$

As we claim $c_2(E)=c_2=\sum k_i[Y_i]$, one derives that Z would have the associated cohomology class

$$[Z] = \sum k_i[Y_i] + (t(3t+1)-k_0(k_0-1)-2d)h^2.$$

Take $t \geq t_0$ such that $t(3t+1)-k_0(k_0-1)-2d > 1$. Let us explain how such curves Z may be chosen so that they satisfy for large values of t the condition on $\det N_Z \otimes F \otimes L^{3t}$, keeping in mind that we also look for c_3.

Before doing this, we do the following obsevation. Let C be a non-singular curve, $C=H_1 \cap H_2$, as above. If P' is a line bundle and if the degree of P' exceeds a suitable integer, which depends only on $c_1(X)$ and h, then there are epimorphisms $I_C/I_C^2 \longrightarrow P' \longrightarrow 0$. Let fix such a degree. An epimorphism as above determines a double structure C', and let us define $\mathcal{E} = \chi(O_{C'})$. For the sake of simplicity we call such a curve an $\mathcal{E}$-curve. If epimorphisms $I_C/I_C^2 \longrightarrow P'' \longrightarrow 0$ will be taken into account, with deg P''=degP'+1, then it will be obtained curves C' with $\chi(O_{C'})= \mathcal{E}+1$. They are simply called $(\mathcal{E}+1)$-curves.

Let us come back to the construction of Z. We can assume that $\sum k_i[Y_i] = [T]$, where T is a l.c.i. curve supported by the union of Y_i (one chooses smooth surfaces S_i which contain Y_i, correspondingly, one replaces Y_i by its (k_i-1)-infinitesimal neighbourhood in S_i and one takes the union). Fix such a T.

Take an arbitrary decomposition

(4) $\quad t(3t+1)-k_o(k_o-1)-2d=2\alpha+2\beta \quad , \quad \alpha \geqslant 0, \quad \beta \geq 0.$

The desired curve Z will be a disjoint union of T, α copies of curves of type $\mathcal{E}$ and β copies of type $\mathcal{E}+1$. Moreover, we may assume that Z is disjoint of the curve Y which appears in (2). There is an integer t_1 such that for every $t \geqslant t_1$ the bundle $\det N_Z \otimes F \otimes L^{3t}$ possesses a nonvanishing section: for this purpose restrict (2) to any component of Z and so on...

All in all, we can find an integer t_2, so that for $t \geqslant t_2$ there exist extensions (3) with E locally free and $c_1(E)=c_1$, $c_2(E)=c_2$, for every curve Z as described above. The integer t_2 depends only on $c_1(X)$, $h=c_1$, k_o, the decomposition $c_2=\sum u_i[Y_i]=[T]$ and deg P'.

<u>3.3</u> Let us take an E as above. By the variation of α and β in (4) we would like to obtain $c_3(E)=c_3$. One has:

$$c_3(E\otimes L^{2t})=(c_1(F\otimes L^{3t})+c_1(X))[Z]-2\chi(O_Z),$$

hence

$$8h^3t^3+4h^3t^2+2hc_2t+c_3(E)=$$
$$=(6ht+h+c_1(X))(c_2+2(\alpha+\beta)h^2)-2(\chi(O_T)+\alpha\mathcal{E}+\beta(\mathcal{E}+1)).$$

By requiring $c_3(E)=c_3$ one finds an equation in α and β, which must be solved. Due to the integrality condition $(h+c_1(X))c_2\equiv c_3 \pmod 2$, we can divide by 2 and write this equation like

$$P(t)=\alpha A(t)+\beta(A(t)-1),$$

where $A(t)=6h^3t+h^3+h^2c_1(X)-\mathcal{E}$ and $P(t)=4h^3t^3+$ lower terms (which depend only on $c_1(X), c_1, c_2, c_3$ and the fixed curve T).

From now on, we reverse the sense of the argument. Thus, $A(t)$ and $A(t)-1$ are relative prime numbers, hence for $t \gg 0$ it is possible to find a decomposition

$$P(t)=\alpha_t A(t)+\beta_t(A(t)-1),$$

where α_t, β_t are positive integers. Since

$$P(t)=(\alpha_t+\beta_t)A(t)-B(t),$$

we asymptotically find $\alpha_t+\beta_t \sim \frac{2}{3}t^2$. Thus

$t(3t+1)-k_o(k_o-1)-2(\alpha+\beta) \sim \frac{5}{3}t^2$ for $t \gg 0$. Consequently, there are integers $t \geqslant t_2$, sufficiently large such that $P(t)$ can be expressed as above and the number $t(3t+1)-k_o(k_o-1)-2(\alpha_t+\beta_t)$ should be positive. Take d to be half of this number. One chooses Y, one constructs F, then Z associated to α_t and β_t and finally one obtains the desired bundle E.

§4. The proof of Theorem 3

4.1 Let consider a topological bundle F of rank 2, with algebraic Chern classes c_1 and c_2. We like to construct an algebraic bundle E of rank 2 and with the same Chern classes. We begin as in 3.1. By succesive convenient twistings of E with algebraic line bundles, we may assume that $c_1=c_1(L)$, where L is an ample line bundle, and $c_2=\sum n_i[Y_i]$, with $n_i \geqslant 3$ and Y_i disjoint connected smooth curves. Let denote $h=c_1(L)$. Let $Y_{i_1},\ldots,Y_{i_r}$ be those components whose coefficient n_i is odd. We choose an even integer $m \gg 0$, so that $m^2h^2=[Y_{i_1}]+\ldots+[Y_{i_r}]+[U]$, where U is smooth and disjoint of the other components, see 1.3. We also choose an even integer $n \gg 0$, such that $n^2h^2=[U]+[Z]$, with $[Z]$ smooth and disjoint of all Y_i. Accordingly,

$[Y_{i_1}]+\ldots+[Y_{i_r}]=(m^2-n^2)h^2+[Z]$. Further we tensorize once more F with L^q. One gets $c_2(F\otimes L^q)=c_2(F)+q(q+1)h^2$. If we take $q\gg 0$, then the new bundle $F\otimes L^q$ has (with new coefficients and new components Y_i) $c_2=\sum 2n_i[Y_i]+3[Z]$, where $n_i>0$, Y_i and Z are smooth connected and pairwisely disjoint. The new c_1 is still the first Chern class of an ample line bundle; we keep the same notation L for it, and h for $c_1(L)$. After these normalizations of the initial topological data, we can look for extensions. Let us choose, as in § 3, a l.c.i. curve Y supported by the union of the curves Y_i and satisfying $[Y]=\sum n_i[Y_i]$, therefore $c_2=2[Y]+3[Z]$.

We seek for integers k and for l.c.i. curves T, which give rise to extensions

$$0\longrightarrow L^k\longrightarrow E\longrightarrow I_TL^{1-k}\longrightarrow 0,$$

with E locally free. One has $c_1(E)=c_1(L)=h=c_1$ and $c_2(E)=$ $=k(1-k)h^2+[T]$. If we shall show that for $k\gg 0$, one can choose T so that there exist extensions with $k(1-k)h^2+[T]=c_2$, then the proof will be over. We shall take T to be a disjoint union of a convenient double structure Y' on Y, of a triple structure Z'' on Z and of $\frac{1}{2}k(k-1)$ double structures C' supported on pairwise disjoint curves of the form $C=H'\cap H''$, where H' and H'' are general elements of $|H^0(X,L)|$.

First of all, notice that $H^2(X,L^{2k-1})=0$ for $k\gg 0$. We have to accomplish Serre's condition: $\det N_T\cong L^{1-2k}|T$. For an arbitrary l.c.i. curve S (read Y or the previously described curves C) one finds, for large values of k, epimorphisms

$$I_S/I_S^2\longrightarrow \det N_S\otimes L^{2k-1}\longrightarrow 0,$$

hence double structures S' with $\det N_{S'}|S\cong L^{1-2k}|S$. Since, for $k\gg 0$, the restriction map $\mathrm{Pic}\,S'\longrightarrow \mathrm{Pic}\,S$ is bijective one obtains exactly the desired relation $\det N_{S'}\cong L^{1-2k}|S'$. It remains to prove that, again for $k \gg 0$, there are triple structures Z'' on Z, with $\det N_{Z''} \cong L^{1-2k} | Z''$. Because $\det N_Z \cong \omega_Z \otimes (\omega_X|Z)^*$, $[Z] \equiv c_2 \pmod 2$ and $\deg \omega_Z \equiv 0 \pmod 2$, we get that the degree of $\det N_Z \otimes L^{2k-1}|Z$ has the same parity with $(c_1+c_1(X))c_2$, therefore it is even by the integrality condition. Consequently, for every k, one finds a line bundle P_k on Z, so that $P_k^2 \cong \det N_Z \otimes L^{2k-1} | Z$. There exist an integer k_o with the following property: if P is a line bundle on Z and $\deg P \geq k_o$, then there is an epimorphism $I_Z/I_Z^2 \longrightarrow P \longrightarrow 0$ and a retract $I_{Z'}/I_Z I_{Z'} \xrightarrow{\tau} P^2$ for the inclusion $0 \longrightarrow P^2 \xrightarrow{\quad i \quad} I_{Z'}/I_Z I_{Z'}$, where Z' is the associated double structure. This retract defines a triple structure Z'' on Z. By enlarging if necessary k_o, it follows the isomorphisms $\mathrm{Pic}\, Z'' \xrightarrow{\sim} \mathrm{Pic}\, Z' \xrightarrow{\sim} \mathrm{Pic}\, Z$ and consequently $\det N_{Z''} \cong L^{1-2k}|Z''$. We take $P=P_k$ for $k \gg 0$ and thus we get the needed curve Z''.

4.2 Let F be a rank 2 topological bundle with Chern classes c_1, c_2, and in addition, $c_1 \equiv c_1(X) \pmod 2$. As described above, we may assume after a normalization of F that $c_1 = c_1(L)$, with L ample on X and $c_2 = \sum 2n_i[Y_i] + 3[Z]$, where $n_i > 0$ and Y_i, Z are connected, pairwise disjoint smooth curves. The condition $c_1 + c_1(X) \equiv 0 \pmod 2$ implies, by the exponential exact sequence, the existence of an algebraic line bundle A on X, with the property $L \otimes K_X^{-1} \cong A^2$.

We have already seen that extensions like

$$0 \longrightarrow L^k \longrightarrow E \longrightarrow I_T L^{1-k} \longrightarrow 0$$

give rise to algebraic bundles E of rank 2, with prescribed Chern classes c_1 and c_2. Moreover, $\det E \cong L$, hence for $E_{norm} := E \otimes A^{-1}$ one gets $\det(E_{norm}) \cong K_X$. We shall show how to vary the extension parameters in order to obtain for the

$\alpha(E_{norm})=h^0(E\otimes A^{-1})+h^2(E\otimes A^{-1})$ (mod 2)

both values 0 and 1. This will conclude the proof of the K_X-symplectic part of Theorem 3. For, let us come back to the construction presented in 4.1. The triple structure Z'' on Z involves an integer $k \gg 0$ and an epimorphism $I_Z/I_Z^2 \longrightarrow P \longrightarrow 0$, denoting by P a line bundle on Z with $P^2 \cong \det N_Z \otimes L^{2k-1}|Z$. By using the extension, the exact sequences

$$0 \longrightarrow P \longrightarrow O_{Z'} \longrightarrow O_Z \longrightarrow 0,\quad 0 \longrightarrow P^2 \longrightarrow O_{Z''} \longrightarrow O_{Z'} \longrightarrow 0$$

and taking care in choosing a sufficiently high value of k (which depends only on the given data Y,Z and A) in order to have vanishing terms in computations, one gets

$$h^0(E\otimes A^{-1})+h^2(E\otimes A^{-1})=h^0(L^k\otimes A^{-1})+h^1(L^{1-k}\otimes A^{-1}|T)=$$
$$= h^0(L^k\otimes A^{-1})+h^1(L^{1-k}\otimes A^{-1}|Y') +\tfrac{1}{2}k(k-1)h^1(L^{1-k}\otimes A^{-1}|C')+$$
$$+ h^1(L^{1-k}\otimes A^{-1}|Z)+h^1(P\otimes(L^{1-k}\otimes A^{-1})|Z).$$

Let denote $Q=P\otimes(L^{1-k}\otimes A^{-1})|Z$. We have $Q^2 \cong \det N_Z \otimes (L\otimes A^{-2})|Z \cong$ $\cong \det N_Z \otimes K_X|Z \cong K_Z$, hence Q is a θ-characteristic on Z. Now we modify the curve T by changing the triple structure on Z, more precisely we start with an epimorphism $I_Z/I_Z^2 \longrightarrow P\otimes R \longrightarrow 0$, where $R^2 \cong O_Z$. If $\widetilde{T}$ is the new curve and $\widetilde{E}$ is the new bundle, then $\alpha(E_{norm})-\alpha(\widetilde{E}_{norm})= h^1(Z,Q)-h^1(Z,Q\otimes R)$. Because h^1 (mod 2) takes both values 0 and 1 when the θ-characteristic changes, see [1] for the precise result, we can modify the value $\alpha(E_{norm})$, and the proof is complete.

§5. The case of rational manifolds

We prove here Theorem 4. Let X be a complex projective, smooth rational threefold. Since X is simply connected, $H^1(X,\mathbb{Z})=H^5(X,\mathbb{Z})=0$ and $H^2(X,\mathbb{Z})$ has no torsion. By ([6], th.1, or § 2 above) the topological classification of rank 2 bundles is given by the

Chern classes and, when $c_1+c_1(X)\equiv 0 \pmod 2$ (in which case we can normalize so that $c_1=c_1(K_X)$), by the Atiyah-Rees invariant α. Moreover, the analytical invariant α distinguishes between the topological types.

By using these facts and Theorems 2 and 3, it remains to prove that every cohomology class in $H^2(X,\mathbb{Z})$ or $H^4(X,\mathbb{Z})$ is algebraic. For $H^2(X,\mathbb{Z})$ this is clear by the exponential exact sequence. Because we have not found a reference for the analogous statement for $H^4(X,\mathbb{Z})$, we sketch below a proof of it. As it is well known, there exists a rational threefold Z together with two morphisms $Z \longrightarrow X$, $Z \longrightarrow \mathbb{P}^3$, which are both compositions of monoidal transformations with smooth center. Thus we are reduced to prove the following fact. If $X \xrightarrow{f} Y$ is a monoidal transformation of 3-folds with smooth center, then every cohomology class in $H^4(X,\mathbb{Z})$ is algebraic if and only if the same property is true on Y.

For the proof, if X has the property, then Y has it too, as $f_*f^*=\mathrm{id}$. both in cohomology or in the term A^1 of Chow ring. Assume that Y has the property and let ξ be an element of $H^4(X,\mathbb{Z})$. Then $\xi - f^*f_*\xi$ is of the form $i_*(\eta)$, where $\eta\in H^2(E,\mathbb{Z})$, E denoting the exceptional divisor and $i:E \longrightarrow X$ the inclusion. But η is algebraic in virtue of $H^2(E,\mathcal{O})=0$, which concludes the proof.

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