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by

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On the Jacobian criterion of Nagata-Grothendieck

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A criterion of formal smoothness for a local, complete, noetherian algebra over a field k of characteristic exponent p s.t. $[k:k^p]<\infty$ is proved in EGA, 0_{IV}, 22.7.3; it generalizes the Jacobian criterion of Nagata [N]. If p=1, it becomes a criterion of regularity and was generalized e.g. in [M1], [M2], [S1]. If $p\neq 1$ and k is perfect, Matsumura gave a generalization of the afore Jacobian criterion of Nagata in [M2], th.15. The aim of this note is to give the following generalization of the above criterion Nagata-Grothendieck of formal smoothness and to prove it as a regularity criterion (see theorem 2), via [R1] (see lemma 3). The idea and the first version of this proof were done in [R3] for n=0; polynomial over formal power series, i.e. $n\geqslant 1$ are considered in [BR2].

Theorem 1. *Let k be a field of characteristic $p>0$ with $[k:k^p]<\infty$, let R be a formally smooth, local, complete, noetherian k-algebra, let $B=R[Y]$ with $Y=(Y_1,\dots,Y_n)$ indeterminates, $I\subseteq P$ ideals of S with B prime and $A=S/I$.*

i) *The following conditions are equivalent:*

a) *A_P is a formally smooth k-algebra in the P-adic topology;*

b) *there exist the derivations $D_1,\dots,D_m\in \mathrm{Der}_k(S,S)$ and the elements $f_1,\dots,f_m\in I$ s.t. $IS_P=(f_1,\dots,f_m)S_P$ and $\det(D_if_j)\notin P$.*

ii) *If, moreover, there exists a field K s.t. $k\subseteq K\subseteq R$ and R is a formally smooth k-algebra and the residue field L of R is finite over K, then a) and b) are also equivalent to:*

c) *There exists a subfield H of K such that $kK^p\subseteq H$, $[K:H]<\infty$, there exist the derivations $D_1,\dots,D_m\in \mathrm{Der}_H(S,S)$ and the elements $f_1,\dots,f_m\in I$ such that $IS_P=(f_1,\dots,f_m)S_P$ and $\det(D_if_j)\notin P$.*

In order to prove the theorem, we need the following lemmata:

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Lemma 1.([N], [N1], [S1]). Let B be a noetherian ring and $I \subseteq P$ ideal of B with P prime such that B_P is a regular ring.

i) If there exist $D_1,\dots,D_m \in Der(B,B)$ and $f_1,\dots,f_m \in I$ such that $IB_P=(f_1,\dots,f_m)B_P$ and $\det(D_i f_j) \notin P$, then $(B/I)_P$ is a regular ring and ht $IB_P=m$.

ii) If $(B/I)_P$ is a regular ring and there exist $D_1,\dots,D_r \in Der(B,B)$ and $f_1,\dots,f_r \in P$ such that $PB_P=(f_1,\dots,f_r)B_P$ and $\det(D_i f_j) \notin P$, then there exist $D'_1,\dots,D'_m$ among $D_1,\dots,D_r$ and $f'_1,\dots,f'_m \in I$ such that $IB_P=(f'_1,\dots,f'_m)B_P$ and $\det(D'_i f'_j) \notin P$.

Lemma 2. Let K be a field of characteristic $p \neq 0$ and $\{k_i; i \in I\}$ be a family of subfields of K, directed downwards such that $\bigcap_i k_i = K^p$.

i) Then $\bigcap_i k_i((X_1,\dots,X_m))(Y_1,\dots,Y_n)=K^p((X_1,\dots,X_m))(Y_1,\dots,Y_n)$.

ii) Let K' be a subfield of K containing K^p, L a finite field extension of $K_1=K((X_1,\dots,X_m))(Y_1,\dots,Y_n)$ and L' a finite field extension of $K_2=K((X_1,\dots,X_{m'}))(Y_1,\dots,Y_{n'})$. Then there exists a family $\{k_i; i\in I\}$ of subfields of K, directed donwwards, such that $\bigcap_i k_i=K^p$, $[K:k_i]<\infty$, $[K':(K'\cap k_i)]<\infty$ for every $i \in I$ and there exists $t \in I$ such that $\dim_L \Omega_{L/K_t}=\dim_{K_1}\Omega_{K_1/K_t}$ and $\dim_{L'}\Omega_{L'/K'_t}=\dim_{K_2}\Omega_{K_2/K_t}$ where $K_t=k_t((X_1^p,\dots,X_m^p))(Y_1^p,\dots,Y_n^p)$ and $K'_t=k_t((X_1^p,\dots,X_{m'}^p))(Y_1^p,\dots,Y_{n'}^p)$.

Proof. i) It is an easy generalization of a well-known property (EGA, 0_{IV},22.8.8; [M1], p.229; [BR], 3.25).

ii) Consider a p-basis of k' over K^p and extend it to a p-basis B of K. Take the family $\{k_i; i\in I\}$ of all subfields k_i such that i is a finite subset of B and $k_i=K^p(B\setminus i)$. Then go on as in [M1], 40C, th.9, p.285 or in [BR], 3.24.

Lemma 3. Under the hypotheses of theorem 1, denote : $k'=k^{1/p}$, $R'=R\otimes_k k'$, $S'=S\otimes_k k'=R'[Y]$, $I'=I\otimes_k k'=IS'$, $A'=S'/I'=A\otimes_k k'$, $K'=K\otimes_k k'$ and P' the unique prime ideal of S' such that $P'\cap S=P$. Then the conditions a),b),c) are respectively equivalent to the following statements:

a') $A'_{P'}$ is a regular ring;

b') there exist $D'_1,\dots,D'_m\in Der(S',S')$ and $f_1,\dots,f_m\in I$ such that $I'S'_{P'}=(f_1,\dots,f_m)S'_{P'}$ and $\det(D'_if_j)\notin P'$;

c') there exists a subfield H' of K' such that $K'^p\subseteq H'$, $[K':H']<\infty$, $[K:(K\cap H')]<\infty$ and there exists $D_1,\dots,D'_m\in Der_{H'}(S',S')$ and $f_1,\dots,f_m\in I$ such that $I'S'_{P'}=(f_1,\dots,f_m)S'_{P'}$ and $\det(D'_if_j)\notin P'$.

Proof. We have $A'_{P'}\approx A_P\otimes_k k'$. Hence a) and a') are equivalent by [R1]. It suffices to prove that c) and c') are equivalent, since the equivalence of b) and b') has an easier similar proof. R is a formally smooth k-algebra; hence, by [R1], the ring R' is regular and K' is a field.

Let c) be fulfilled. Then $H'=H\otimes_k k'$ is a subfield of K' and $K'^p\subseteq H'$. If $D:S\longrightarrow S$ is a H-derivation, we denote by $D':S'\longrightarrow S'$ the unique H'-derivation which extends D by $D'(s\otimes c)=D(s)\otimes c$ for every $s\in S$ and $c\in k'$. Since $IS_P=(f_1,\dots,f_n)S_P$, it follows that $I'S'_{P'}=(f_1,\dots,f_m)S'_P$, and $\det(D'_i(f_j\otimes 1))=\det(D_if_j)\notin P'$. We have $[K:(H'\cap K)]\le[K:H]=[K':H']$ too.

Let c') be fulfilled. Let $u:S\longrightarrow S'$ be the canonical morphism $u(s)=s\otimes 1$ and let $t=[k':k]$. Then the S-module S' is isomorphic to S^t and the canonical projections $v_i:S'\longrightarrow S$ are morphisms of S-modules. Hence $D_{ij}=v_jD_iu:S\longrightarrow S$ are H-derivations and $D'_if_j=D_{i1}f_j+\dots+D_{it}f_j$. (For b') $\Longrightarrow$ b) observe that every derivation $D:S'\longrightarrow S'$ vanishes on k, since $k'\subseteq S'$). Since $\det(D'_if_j)\notin P'$, it results that there exist $D_1,\dots,D_m$ among D_{ij} with $i=1,\dots,m$ and $j=1,\dots,t$, such that $\det(D_if_j)\notin P'\cap S=P$.

Lemma 3 reduces the proof of theorem 1 to prove:

Theorem 2. Let R be a local, noetherian, complete, regular ring which contains a field of characteristic $p\neq 0$, $S=R[Y]$ with $Y=(Y_1,\dots,Y_n)$ indeterminates, $I\subseteq P$ ideals in S with P prime and $S=A/I$.

i) The following conditions are equivalent:

a) A_P is a regular ring;

b) there exist $D_1,\dots,D_m\in Der(S,S)$ and the elements $f_1,\dots,f_m\in I$ such that $IS_P=(f_1,\dots,f_m)S_P$ and $\det(D_if_j)\notin P$.

ii) Let $K'\subseteq K$ be subfields in R such that $K^p\subseteq K'$ and R is a formally smooth K-algebra in the M-adic topology, where M is the maximal ideal of R, and the residue field $L=R/M$ is a finite extension of K (e.g. if K is acoefficient field of R). Then a) and b) are also equivalent to:

c) there exists a subfield H of K such that $K^p\subseteq H$, $[K:H]<\infty$ and $[K':(H\cap K')]<\infty$, there exist the derivations $D_1,\dots,D_m\in Der_H(S,S)$ and the elements $f_1,\dots,f_m\in I$ such that $IS_P=(f_1,\dots,f_m)S_P$ and $\det(D_if_j)\notin P$.

Proof. b)$\Longrightarrow$a) follows by lemma 1; a)$\Longrightarrow$b) is a consequence of [M2], th.15. Under assumptions of ii), it is obvious that c)$\Longrightarrow$b). In fact, the proof of a)$\Longrightarrow$c) bellow gives, as a particular case, a new proof for a)$\Longrightarrow$b).

Let a) be fulfilled. $K\longrightarrow R$ is formally smooth, hence $\Omega_{R/K}$ is a formally projective R-module (in the M-adic topology). For any subfield H of K with $[K:H]<\infty$, we have an exact sequence $\Omega_{K/H}\otimes_K R\xrightarrow{g}\Omega_{R/H}\longrightarrow\Omega_{R/K}\longrightarrow 0$, and g is formally left invertible in the M-adic topology by EGA, 0_{IV}, 20.4.9. Then for any $i\in\mathbb{N}$, the sequence

$$0\longrightarrow\Omega_{K/H}\otimes_K(R/M^i)\longrightarrow\Omega_{R/H}\otimes_R(R/M^i)\longrightarrow\Omega_{R/K}\otimes_R(R/M^i)\rightarrow 0$$

is exact. Hence $\Omega_{R/H}$ is a formally projective R-module. Let $r=\dim R$ and $x_1,\dots,x_r$ a system of parameters of R which will be chosen bellow. Let $X_1,\dots,X_r$ be indeterminates, $R'=K[[X_1,\dots,X_r]]$,

$R''=H[[X_1^p,\dots,X_r^p]]$ and $u:R'\longrightarrow R$ the finite morphism of K-algebras defined by $u(X_i)=x_i$ and $R''\longrightarrow R'$ be the inclusion. Every H-derivation of R in a complete R-module vanishes on R", then $\Omega_{R/R''}\approx\widehat{\Omega}_{R/H}$. Hence both are free R-modules of finite type. In the exact sequence

$$\Omega_{R/R''}\otimes_R S\longrightarrow\Omega_{S/R''}\longrightarrow\Omega_{S/R}\longrightarrow 0,$$

the first morphism is left invertible, hence $\Omega_{S/R''}$ is a free S-modul of finite type. Put $S'=R'[Y_1,\dots,Y_n]$, $S''=R''[Y_1^p,\dots,Y_n^p]$ and let v: $:S'\longrightarrow S$ be the extension of u by $v(Y_i)=Y_i$. Then $\Omega_{S/R''}\approx\Omega_{S/S''}$.

Let $J_H=\max\operatorname{rank}(D_if_j \bmod P)$, where $D_1,\dots,D_t\in Der_H(S,S)=Der_{S''}(S,S)$ and $f_1,\dots,f_h\in P$ and $t,h\in\mathbb{N}$. This maximum is realized using a minimal system of generators of PS_P. Since S_P is regular, it results that

(1) $$J_H\leq \text{ht } P.$$

By lemma 1,ii), in order to prove a) $\Longrightarrow$ c), it suffices to find a subfield H as in c) such that (1) become equality. Then we can take I=P in c).

Denote $J'_H=\max\operatorname{rank}(D_ig_j)$, when $D_1,\dots,D_m\in Der_{S''}(S,A)$ and $g_1,\dots$ $\dots,g_m$ is a minimal system of generators of PS_P, i.e. m=ht P.

$\Omega_{S/S''}$ is a free S-module of finite type, hence the sequence

$$0\longrightarrow Der_{S''}(S,P)\longrightarrow Der_{S''}(S,S)\longrightarrow Der_{S''}(S,A)\longrightarrow 0$$

is exact. It follows that $J_H=J'_H$. Then it suffices to find a subfield H as in c) such that

(2) $$J'_H\geq \text{ht } P.$$

The canonical exact sequence

$$0\longrightarrow Der_{S''}(A,A)\longrightarrow Der_{S''}(S,A)\xrightarrow{\delta^o} Hom_A(P/P^2,A)$$

shows that $J'_H=\operatorname{rank}\operatorname{Im}\delta^o$, by the definition of δ^o and the choice of $g_1,\dots$ $\dots,g_m$. Hence this sequence gives that $J'_H=\operatorname{rank}Der_{S''}(S,A)-\operatorname{rank}Der_{S''}(A,A)$. Let $T=Q(S)$, $T'=Q(S')$, $T''=Q(S'')$ and $U=Q(A)$ be the fields of quotients. Then

$$\text{rank } Der_{S''}(S,A)=\dim_T \Omega_{T/T''}.$$

Put $Q=P\cap R$, $B=R/Q$ and $r'=\dim A'$, hence $ht\, Q=r-r'$. Let $x_{r'+1},\dots,$ $\dots,x_r\in Q$ such that $x_{r'+1},\dots,x_r$ is a system of parameters in R_Q. We can extend it to a system of parameters $x_1,\dots,x_r$ of R and construct the morphism $u:R'\longrightarrow R$ with this system. Let $B'=K[[X_1,\dots,X_{r'}]]$ and $u':B'\longrightarrow B$ the morphism of K-algebras defined by $u'(X_i)=\hat{x}_i$, with $i=1,\dots,r'$; u' is injective and finite. Put $z_i=v(Y_i) \bmod P\in A$. From $z_1,\dots,z_n$ choose a maximal set, say $z_1,\dots,z_{n'}$, algebraically independent over the field of quotient of B'. Put $A'=B'[Y_1,\dots,Y_{n'}]$ and let $v':A'\longrightarrow A$ be the morphism which extends u' by $v'(Y_i)=z_i$, with $i=1,\dots,n'$. The field U is finite over the field of quotients U' of A'. Put $A''=H[[X_1^p,\dots,X_{r'}^p]][Y_1^p,\dots,Y_{n'}^p]\subseteq A'$; A'' is contained in the image of S'' in A, by the choice of $x_1,\dots,x_r$. Hence $\text{rank } Der_{D''}(A,A)\leq \text{rank } Der_{A''}(A,A)$. Lemma 2 gives a subfield H of K such that $[K:H]<\infty$, $K^p\subseteq H$, $[K':(H\cap K')]<\infty$ and $\dim_{T'}\Omega_{T'/T''}=\dim_T\Omega_{T/T''}$, randk $Der_{A''}(A,A)=\dim_U\Omega_{U/U''}=\dim_{U'}\Omega_{U'/U''}$ where U'' is the field of quotients of A''. But $\dim_{T'}\Omega_{T'/T''}=r+n+q$ and $\dim_{U'}\Omega_{U'/U''}=r'+n'+q$, where q is the length of a p-basis of K over H. Hence $J_R'=r-r'+n-n'=htQ+n-n'$. But $ht\,P=ht\,Q+n-n'$ ([M], th.23 or [BR], 10.18). Hence $J_H'=ht\,Q+n-n'=ht\,P$.

Theorem 1 gives the following generalizations of the results of [R1] (see also [BR], 13.12).

Corollary 1. Let k be a field of characteristic $p>0$ such that $[k:k^p]<\infty$, R be a formally smooth, local, noetherian, complete k-algebra and $S=R[Y_1,\dots,Y_n]$. Then every prime ideal P of S, such that the field S_P/PS_P is separable over k, is not k-differential.

Corollary 2. Let k be a field of characteristic $p>0$ such that $[k:k^p]<\infty$, A be a local noetherian k-algebra with residue field K such that $\dim_K \Upsilon_{K/k} < \infty$ (where $\Upsilon_{K/k}$ is the module of imperfection of K over k). Suppose, moreover, that A is a Nagata ring (=universally japanese), $\Omega_{A/k}$ is a formally projective A-module and there exists a prime ideal P of A such that A_P is a formally smooth k-algebra (in the P-adic topology). Then A is also a formally smooth k-algebra (in the adic topology of the amximal ideal of A).

Remarks. i) Let R be a noetherian complete local ring.

i1) Suppose R contains $\mathbb{Q}$. Then the theorems 1 and 2 coincide and are true too taking as K a coefficient field of R, see e.g. [M2].

i2) Suppose char.R=0 and char.L=p>0. Then the theorems 1 and 2 are true for ideals P of S such that $p\notin P$, see [S1].

The afore proof works equally for i1) and i2) and becomes much simpler (lemma 2 is not necessary).

ii) The localization of formal smoothness (the general case was proved firstly in [A], another proof appearing in [BR1]) was proved in [S2] as a consequence of the Jacobian criterion of Nagata. Using the same idea theorem 2 furnishes new, simpler unitary proofs of the following fundamental theorems : a noetherian complete local ring is a reg-ring and has the property Reg-2; the Nagata rings (i.e. universally japanese) and the reg-rings are stable under extensions of finite type (see [BR3] bellow).

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Appendix

We give here the principal result from [R1] and ~~a sketch of~~ its proof, viewing its role in reducing th.1 to th.2.

THEOREM(EGA, 0_{IV}, 22.5.8 and [R1]) Let k be a field of characteristic exponent p and A a noetherian local k-algebra. Then the following assertions are equivalent

a) A is a formally smooth k-algebra;

b) A is a geometrically regular k-algebra;

b') for any finite field extention k' of k with $k'^p \subseteq k$; the ring $A \otimes_k k'$ is regular;

c) the ring $A \otimes_k k^{1/p}$ is local, noetherian, regular.

Proof. Let M be the maximal ideal of A, F=A/M, $L=k^{1/p}$, $B=A\otimes_k L$ and N the maximal ideal of B. Then $B=\varinjlim A\otimes_k k'$, where k' runs over all the finite extension of k in L; hence B is a local domain.

b') implies c) Let K be the field of quotients of A. In $K^{1/p}$ consider the ring $C=A^{1/p}$; then C is isomorphic to A, hence C is regular. For any finite extension k' of k in L, the canonical morphism v: $:A\otimes_k k' \longrightarrow C$ is flat. The canonical morphism $g:B \longrightarrow C$ is the inductive limit of all such v's; hence g is flat. Then B is noetherian and regular ([R], VIII,1.16).

(Remark. Under the hypotheses of the theorem, the ring B is always noetherian, see [R4].)

c) implies a) B is a smooth L-algebra with respect to k.Hence the canonical morphism $h: \Omega_{L/k}\otimes_L B \longrightarrow \Omega_{B/k}$ is left invertible. But $\Omega_{L/k}\approx\Omega_L$ and $\Omega_{B/k}\approx\Omega_B$, since $L^p=k$ and $k\subseteq B^p$. Hence the canonical morphism $h': \Omega_L \otimes_L B \longrightarrow \Omega_B$ is left invertible. Let P be the prime field of k. Then B is a smooth L-algebra with respect to P(EGA, 0_{IV}, 2o). But B is regular, hence a formally smooth P-algebra (EGA, 0_{IV}, 19.6.1). Hence B is a formally smooth L-algebra in the N-adic topology. Since $B^p \subseteq A$, the N-adic and MB-adic topologies coincides on B. Now conclude by the following

LEMMA (on the lifting of formal smoothness). Let $k \longrightarrow A$ and $k \longrightarrow L$ be morphisms of rings with A and $B=A\otimes_k L$ noetherian rings. Let P be an ideal of k and M an ideal of A with $PA \subseteq M$. Suppose that $Tor_1^k(A,L)=0$ and (at least) one of the following conditions is fulfilled: 1) M is maximal and $MB \neq B$; 2) $L=k/P$. If $L \longrightarrow B$ is formally smooth in the MB-adic topology, then $k \longrightarrow A$ is also formally smooth in the M-adic topology.

Proof. Let C be a k-algebra of polynomials, $u: C \longrightarrow A$ a surjective morphism of k-algebras and $I=\ker u$. Let M' be a finitely generated ideal of C with $u(M')=M$. On C we take the M'-adic topology. We shall prove that the canonical morphism $\delta: I/I^2 \longrightarrow \Omega_{C/R}\otimes_C A$ is formally left invertible; it will follow that $k \longrightarrow A$ is formally smooth (EGA, 0_{IV}, 2o.4). The ring C/I is noetherian; hence the induced topology of I/I^2 is M-adic (e.g. [BR], 4.11). Put $C'=C\otimes_k L$, $u'=u\otimes 1: C' \longrightarrow B$ and $I'=I\otimes_k L$. Since $Tor_1^k(A,L)=0$, then I' is an ideal of C' and $B=C'/I'$. Hence the induced topology of I'/I'^2 is MB-adic (e.g. [BR], 4.11). Since $L \longrightarrow B$ is formally smooth in the MB-adic topology, the canonical morphism $\delta': I'/I'^2 \longrightarrow \Omega_{C'/L}\otimes_{C'} B$ is formally left invertible in the same topology. Hence $g=\delta' \bmod MB$ is left invertible. We have $\delta'=\delta\otimes_A B$. Let 1) be fulfilled and put $F=A/M$. Then g is injective and $g=(\delta\otimes F)\otimes_F B/MB$. Hence $\delta\otimes F$ is injective; then δ is formally left invertible. Let 2) be fulfilled. Then $B=A/PA$ and $g=\delta\otimes(A/M)$, hence δ is formally left invertible.

(Remark Case 2) furnishes a simple proof for the lifting part in EGA, 0_{IV}, 19.7.1).

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Stability of geometric regularity as consequence of Nagata's Jacobian Criterion

A.Brezuleanu, N.Radu.

The localization of formal smoothness (i.e.stability of geometric regularity by localization), proved in [A2] using André|Nagata' Jacobian Criterion (see 1.1). This idea works also to prove in a simpler and unitary way the following classical results: the complete noetherian local rings are quasi-excellent (see 1.2); the reg-rings, resp.Nagata rings, are stable by extensions of finite type (see 1.5, resp.1.8, as corollaries of 1.3). Lemma 1.3 needs the Jacobian Criterion for polynomials over formal power series with coefficients in a field K or in a Cohen p-ring W(in this case for prime ideals which do not contain p).

o.1 LEMMA ([M1], th 7 or [S2I] ,th.II 3) Let k be a ring and B a noetherian k-algebra, let P be a prime ideal in B s.t. B_P is a regular ring and its residual field k(P) is perfect. Let $T=(T_1,..,T_n)$ be indeterminates and $C=B[T]$. Suppose there exist $D_1,..,D_m \in Der_k(B,B)$ and $f_1,...,f_m \in P$ s.t. $PB_P=(f_1,..,f_m)B_P$ and $\det(D_if_j)\notin P$. Then for any prime ideal Q in C with $Q\cap B=P$, there exist $f_1,..,f_m, f_{m+1},..,f_{m+s}\in Q$ and $D'_1,..,D'_m, D'_{m+1},...,D'_{m+s} \in Der_k(C,C)$ s.t. $QC_Q=(f_1,..,f_{m+s})C_Q$ and $\det (D'_if_j;\ i,j=1,...,m+s)\notin Q$.

o.2.LEMMA (cf [M1], th 9(d)) Let R be a B-algebra ~~and I⊆P ideals~~ and $I\subseteq P$ ideals of R with P prime and I finitely generated, s.t. R_P and $\overline{R}_{\overline{P}}$, where $\overline{R}=R/I$ and $\overline{P}=P/I$, are noetherian regular rings. Suppose there exist $D_1,..,D_n \in Der_B(R,R)$ and $x_1,..,x_n \in P$, s.t. $PR_P=(x_1,...,...,x_n)R_P$ and $\det(D_ix_j)\notin P$(hence n=h(P). Then there exist $D'_1,...,D'_r\in Der_B(\overline{R},\overline{R})$ and $y_1,..,y_r\in\overline{P}$ s.t. $\overline{P}\overline{R}_{\overline{P}}=(y_1,...,y_r)\overline{R}_{\overline{P}}$ and $\det(D'_iy_j)\notin\overline{P}$ too.

o.3 PROPOSITION ([N1], [M1] ,th 15, [S2II], th.I 2.4). Let R be a local complete noetherian regular ring of residue field L, $Y=(Y_1,..,Y_n)$ indeterminates, $S=R[Y]$,$I\subseteq P$ ideals in S with P prime and A=S/I.

If char $L=o$ and K is a coefficient field in R, then the following conditions are equivalent:

a) A_P is a regular ring;

b) (resp. c) there exist $D_1,\dots,D_m \in Der(S,S)$ (resp. $\in Der_K(S,S)$) and $f_1,\dots,f_m \in I$ s.t. $IS_P=(f_1,\dots,f_m)S_P$ and $\det(D_i f_j) \notin P$.

We indicate a simplified proof for a) implies c). By o.1 we can suppose $n=o$. Hence $S=R=K[[X_1,\dots,X_r]]$ with X_i indeterminates. Then $(\Omega_{R/K})^{\wedge}$ is a free R-module of rank r. Put $J(P)=\max \operatorname{rank}(D_i f_j \bmod P)$, where $D_1,\dots,D_t \in Der_K(R,R)=Hom_R((\Omega_{R/K})^{\wedge}, R)$ and $f_1,\dots,f_t \in P$. Hence $J(P) \leq ht\, P$ and it suffices to show that $J(P) \geq ht\, P$. In the proof of [BR 3], th 2, above take $S=R$, hence $A=B$, $A'=B'$ and replace S'' by K. They obtain, as there that $J(P)=J'(P)=htP+\lambda'-\operatorname{rank} Der_K(A,A)$, where $\lambda'=\dim A$. The morphisms $K \longrightarrow A' \longrightarrow A$ give the exact sequence $0 \to D_{A'}(A,A) \longrightarrow Der_K(A,A) \longrightarrow Der_K(A',A)$. But A is separable over A', hence $Der_{A'}(A,A)=0$. Then $\operatorname{rank} Der_K(A,A) \leq \operatorname{rank} Der_K(A',A)=\operatorname{rank} Hom_{A'}((\Omega_{A'/K})^{\wedge}, A)=\lambda'$. Hence $J'(P) \geq ht\, P$.

o.4 Let R,L,S,I,P,A be as in o.3. Suppose char $L=p\neq o$, char $R=0$ and let W be a Cohen p-ring included in R.

i) ([S2 II], th. I 2.4) If $p\notin P$, then the conditions a), b), c) from o.3 with K replaced by W, are equivalent.

ii) If $p\in P$, then a) does not imply b)

Indeed, wither $R=W[[X_2,\dots,X_r]]$ or $R=W[[X_1,\dots,X_r]]/(g)$ with $\dim R=r$ (see e.g. BR, 5.6). By o.2, it suffices to consider the first case.

i) Let a) be true. Since $p\notin P$, we can suppose $n=o$, by o.1. The R-module $(\Omega_{R/W})^{\wedge}$ is free of rank $r-1$. As in the proof of o.3, we obtain $J'(P)=ht\, P+\lambda'-1-\operatorname{rank} Der_W(A,A)$, since $\operatorname{rank} Der_W(R,A)=r-1$. Then go on as in 0.3.

ii) The ideal $I=pS$ is prime and differential. Consider firstly $I=P$; then a) is true, but b) is false. The ring S/I is regular. Let $I\subseteq P$ then a) is true, but b) is false, by the first case and a known lemma (see e.g. [BR3], lemma 1,ii)).

o.5 Remarks i) Let B be a noetherian A-algebra, complete in a J-adic topology. Let $H \subseteq J$ be an A-differential ideal of B. If $\Omega_{B/A}$ is a formally projective B-module (in the J-adic topology), then the canonical morphism $Der_A(B,B) \longrightarrow Der_A(B/H, B/H)$ is surjective. (In order to prove this, use e.g. the formal smoothness of the B-algebra $S_B(\Omega_{B/A})$).

ii) Let be the situation from o.4 ii), with $pS \neq P$. Then $S/pS = L[[X]][Y]$ and $P/pS \neq 0$. By the Jacobian Criterion of Nagata (see e.g. [BR 3], th 2), there exists a cofinite subfield H of L, and $D'_1, \ldots, D'_m \in Der_H(S/pS, S/pS)$, and $f_1, \ldots, f_m \in P$ s.t. $P(S/pS)_P = (f_1, \ldots, f_m)(S/pS)_P$ and $\det(D'_i \hat{f}_j) \notin P/pS$. If $b_1, \ldots, b_h$ is a p-basis of L over H, then $Der_H(S/pS, S/pS)$ is a free S/pS-module with the basis $\partial/\partial b_1, \ldots, \partial/\partial b_h, \partial/\partial X_1, \ldots, \partial/\partial X_r, \partial/\partial Y_1, \ldots, \partial/\partial Y_n$. Hence, using i) for $\mathbb{Z} \longrightarrow W$, we find $D_1, \ldots, D_m \in Der(S,S)$ s.t. $PS_P = (p, f_1, \ldots, f_m)S_P$ and $\det(D_i f_j) \notin P$. However condition c) from o.3 is false.

1.0 Let $u: A \longrightarrow B$ be a local morphism of noetherian local rings. If u isregular, then u is formally smooth. The converse, under the hypothesis that A is a reg-ring-i.e the localisation of formal smoothness- was conjectured in EGA, IV, 7.56. It became theorem 1.1 bellow, "one of the most brilliant theorems about complete local rings" ([Ni] p.155).

1.1 THEOREM A-R-S (M.André, N.Radu, H.Seydi). If a local morphism $u: A \longrightarrow B$ is flat, the special fibre of u is geometrically regular and the completion morphism $v: A \longrightarrow \hat{A}$ is regular, then all the fibres of u are geometrically regular, i.e. u is regular.

A.Grothendieck proved 1.1 when the residue field of B is finite over the residue field k of A. In [BR1] ,1.1 was proved when k is finite over k^p, where p is the characteristic exponent of k,hence for char k=o too; also the general case was reduced to A and B formal power series rings. Using the homology of algebras from [A1], M.André proved 1.1 in general ([A2]).In 1976 N.Radu gave another proof,using the good separability of the module of differentials ([BR 2]). In [S3] see also [S1] , H.Seydi proved 1.1 using only the Jacobian Criterion of Nagata. In [S3] the new idea is to reduce the proof of theorem 1.1 to the case when A and B are complete, A is a domain and it suffices to show that the fibre of u in zero is regular (<u>not geometrically regular !</u>) via [N1] . Namely, suppose A contains a field. By Cohen structure theorem and EGA,0_{IV}, 19.7.2; there exists a formally smooth local morphism $v:R\longrightarrow S$ of complete regular local rings and $P\in$ Spec R s.t. A=R/P and $u=v\otimes_R A$. The field A_P is a regular ring; hence, by [N1] there exist $f_1,\ldots,f_m\in P$ and $D_1,\ldots,D_m\in Der_{k'}(R,R)$ s.t $\det(D_i f_j)\notin P$. Then, fortunately, $(\Omega_{R/k'})^{\wedge}$ is an R-module of finite type and consequently EGA, 0_{IV}, 20.7.18 is valid and allows to extend D_i to some derivations $D_i'\in Der_{k'}(S,S)$. Applying [N1] for $f_1,..,f_m$ and $D_1',..,D_m'$ it follows that the fibre of v in P(which is the fibre of u in zero) is regular.

If A does not contain a field, then (see [BR], p.172), instead of the above reduction, we can reduce 1.1 to the case when A and B are complete and <u>regular</u> and we have to prove that the fibre of u in zero is <u>geometrically</u> regular. Since A does not contain a field, then char A=0. Hence the ring $B\otimes_A Q(A)$, which is regular, is also geometrically regular over Q(A). (In [S3],the unequal characteristic case is solved replacing the residue field of A with a Cohen p-ring W; the proof works as in the case " k contains a field", but needs a variant of [N1] over W, which holds only if $p\notin P$, by o.4 ii).)

(given in [S2 II] th I.2.4 [illegible]

Theorem 1.1 on the localization of formal smoothness is essential in the proof of a lot of significant results, e.g.: localization theorems for geometric ℙ; lifting theorem for reg-rings or for other ℙ-rings in the semi-local case; lifting theorems for excellent rings or for other ℙ-lent rings; excellentness of the rings with Artin approximation property (for bibliography see e.g. [BR]).

1.2 In EGA, 0_{IV}, 22.3.3 (~~given in S2 II, th.I 2.4, see 0.4~~) it was proved that <u>if</u> B <u>is a complete noetherian local ring and</u> Q <u>is a prime ideal of</u> B, <u>then the completion morphism</u> $B_Q \longrightarrow (B_Q)^\wedge$ <u>is regular</u>, i.e. B <u>is a reg-ring</u>. We prove it by [N1] as follows. It suffices to show that for B a domain, the formal fibre of B in zero is geometrically regular. If char B=0, the proof is simple (see e.g. [M], 30D, step III or [BR], 9.5).

Let char $B\neq 0$ and K a finite field extension of Q(B). There is a finite B-algebra E with $B\subseteq E\subseteq K$ and Q(E)=K, hence E is a local domain. Let Q' be a prime ideal of E, lying over Q. Then $(E_{Q'})^\wedge = E_{Q'}\otimes_{B_Q}(B_Q)^\wedge$. Hence $K\longrightarrow K\otimes_B(B_Q)^\wedge$ coincides with $K\longrightarrow K\otimes_E(E_{Q'})^\wedge$. Replacing B, Q with E, Q', it suffices to show that the formal fibre $Q(B)\longrightarrow Q(B)\otimes_B(B_Q)$ of B_Q in zero is regular. Let k be a field of coefficients in B; by the Cohen structure theorem, there are the undeterminates $X=(X_1,\dots,X_n)$, $R=k[[X]]$ and a surjective morphism of k-algebras $h:R\longrightarrow B$. P=Ker h is a prime ideal, since B is a domain. If P=0, then $(B_Q)^\wedge$ is regular. Let $P\neq 0$. $B_P=k(P)$ is regular, hence by [N1] there exist: (i) $f_1,\dots,f_m\in P$ s.t. $PR_P=(f_1,\dots,f_m)R_P$; (ii) $D_1,\dots,D_m\in Der(R,R)$ s.t. (iii) det $(D_if_j)\notin P$. The trace Q' of Q in R includes P. Denote $S=(R_{Q'})^\wedge$ and $v:R_{Q'}\longrightarrow S$, $u:B_Q\longrightarrow(B_Q)^\wedge$ the completion morphisms. Then S is regular and $(B_Q)^\wedge=S/PS$. By fractions and Q'-adic completion, every $D_i:R\longrightarrow R$ extends to a derivation $D_i':S\longrightarrow S$. Let $P'\in$ Spec S lying pver P. Then $PS_{P'}=(f_1,\dots,f_m)S_{P'}$ by (ii) and $\det(D_i'f_j)\notin P'$ by (iii). Hence $(S/PS)_{P'}$ is a regular ring, by [N1]. Hence the fibre of u in O is regular (The case char B=0 can be solved as char $B\neq 0$, replacing [N1] ... by o.3 and [BR3], th 2 as in 1.3)

1.2.1 In [N2] it was proved that every complete noetherian local ring B has the property Reg-2. Namely, if A is a finite B-algebra, then Reg A is open by [N2], ;then apply [N2], Also using [M1], th.15, it results directly that Reg A is open for every B-algebra A of finite type.

1.3 LEMMA Let A,M be a complete noetherian local ring, $Y=(Y_1,\ldots,Y_r)$ indeterminates $B=A[Y]$, Q a maximal ideal of B and $C=B_Q$. Then the completion morphism $u:C\longrightarrow\hat{C}$ is regular.

Proof. Let P be a prime ideal of B included in Q. We have to prove that the fibre morphism $v=u\otimes k(P):k(P)\longrightarrow k(P)\otimes_C\hat{C}$ is regular. Let P' be the trace of P in A. Replacing A,B with A/P, B/P'B, the fibre v does not change, A becomes a domain and B remains A[Y]. Let L be a finite field extension of k(P). Let D be a finite B-algebra in L, including B and with Q(D)=L. Then $L\otimes_C\hat{C}=L\otimes_D\hat{D}$ and we have to show that this ring is regular. Let $T=(T_1,\ldots,T_n)$ be indeterminates and J and ideal s.t. $D=C[T]/J'$. Replacing D by a localization of it in a maximal ideal, we can suppose D is local.

If A is equicharacteristic, then A contains a coefficient field K . ~~If A is of unequalcharacteristic, then A contains a coefficient field K.~~ If A is of unequalcharacteristic, then char A=o, char A/M= =p ≠ o and A contains a Cohen p-ring (K,pK,A/M). In both cases, there are the indeterminates $X=(X_1,\ldots,X_n)$ and a prime ideal I of R=K[[X]] with A=R/I. Let S=R[T,Y] and J, resp.Q', the trace of J', resp.of the maximal ideal of D, in S. Then Q' includes J and XS and $D=(S/J)_{Q'}$. If J=0, then D is regular. Let $J\neq 0$. If char A=0, then also $p\notin J$. The field D_J is regular, hence by [M] th 15 and o.4 there are $f_1,\ldots,f_m\in J$ with $JS_J=(f_1,\ldots,f_m)S_J$ and $D_1,\ldots,D_m\in Der(S,S)$ s.t. $\det(D_if_j)\notin J$. The ring $E=(S_{Q'})^{\wedge}$ is local, complete and regular and $\hat{D}=E/JE$. By fractions amd Q'-adic completion, D_i extends to a derivation $D_i':E \longrightarrow E$ with $i=1,\ldots,m$. Let $N \in \operatorname{Spec} E$ with $N \cap S=J$. Then $JE_N=(f_1,\ldots,f_m)E_N$ and $\det(D_i'f_j) \notin N$. Hence $(E/JE)_N$ is regular, by [N_1]. Hence $L\otimes_D\hat{D}$ is regular.

<u>Remark</u> . Another proof of 1.2 and 1.3, also different from EGA, is contained in [R] .But all these three proofs are based on the same lemma on differentials (see e.g. [M] , 4o.C,th-9, p.285 or [BR 3], lemma 2 ii).) A homological proof of 1.2 and 1.3 is given in [BR 2] .

1.4 PROPOSITION (cf. [N2], and EGA,IV,7.4.4) Let $\mathbb{P}$ be a property of local noetherian rings, satisfying the conditions: i) every regular local ring has $\mathbb{P}$; ii) $\mathbb{P}$ descends by local flat morphisms; iii) the $\mathbb{P}$-morphisms are stable by base change of finite type.

<u>If A is a noetherian $\mathbb{P}$-ring, then any A-algebra B of finite type is a $\mathbb{P}$-ring too.</u>

<u>Proof.</u>It suffices to take $B=A[Y]$, with Y an indterminate. Let $N\in \operatorname{Max} B$ and $M=N\cap A$. We have to prove that the completion morphism $u:B_N \longrightarrow (B_N)^\wedge$ is regular. B_N is a localization of $A_M[Y]$; hence, replacing A by A_M, we can suppose that A,M is local. The morphism g: $:B\longrightarrow B'=B\otimes_A\hat{A}=\hat{A}[Y]$, induced by $A\longrightarrow\hat{A}$ is still a $\mathbb{P}$-morphism, by iii), and faitfully flat. Let $N'\in \operatorname{Max} B'$ with $N'\cap B=N$. The composition of canonical morphisms $g_{N'}: B_N\longrightarrow B'_{N'}$ and $u':B'_{N'}\longrightarrow (B'_{N'})^\wedge$ is equal to the composition of $u:B_N\longrightarrow (B_N)^\wedge$ and $(g_{N'})^\wedge$. The morphism u' is regular, by 1.3. Hence $u'g_{N'}$ is a $\mathbb{P}$-morphism, by i). Since $(g_{N'})^\wedge$ is faithfully flat ,it follows that u is also a $\mathbb{P}$-morphism, by ii).

1.5 Take $\mathbb{P}$= regular in 1.4; then the conditions i),ii) iii) are fulfilled (for iii) see e.g. [M], L4,p.253). Then 1.4 becomes theorem 7.4.4 from EGA,IV.

1.6 Some other properties $\mathbb{P}$ which satisfy i),ii),iii) from 1.4 are : reduced,normal,(SR_n), complete intersection, Cohen-Macaulay, Gorenstein,etc.

1.7 Remember that a noetherian ring A is called a Nagata ring (universally japanese in EGA) if A/P is japanese for any P∈ Spec A. The ring A is called Nor-2 if for every A-algebra B of finite type the normal locus Nor B is open.(By EGA, IV,6.13.7, it suffices to verify the openess of Nor B for every finite A-algebra B, which is a domain). The following local-global principle for Nagata rings and its proof are implicitely included in [N2] .

PROPOSITION. A noetherian ring A is Nagata iff A is Nor-2 and A_M is Nagata for every maximal ideal M of A.
(Use e.g. the lemmata 3 and 4 from [M], p.238).

1.8 Another corollary of 2.3 is the following proof of the theorem ([N2]). If A is a noetherian Nagata ring, then every A-algebra B of finite type is a Nagata ring too.

It suffices to take B=A[X] , with Y an indeterminate .Apply 1.7. Every B-algebra C of finite type is also an A-algebra of finite type; hence Nor C is open (1.7). By Zariski-Nagata theorem ([N2],see e.g. EGA, IV,7.6.4 or [M], th.70,71), A is a red-ring. Hence B remains a red-ring, by 1.6. Again by Zariski-Nagata theorem and 1.7 it follows that B is a Nagata ring.

1.9 Let $k \longrightarrow S$ be as in [BR 3],th.1. Then the set Flis(k,S) = $=\{P \in \text{Spec } S;\ k \longrightarrow S_P$ is formally smooth$\}$ is open in Spec S.This results from "a) equivalent b)" in [BR 3], th.1.

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