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ITERATING THE BASIC CONSTRUCTION

by

Mihai PIMSNER and Sorin POPA

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Introduction

Let $N \subset M$ be a pair of finite factors. Jones defined in [1] the index $[M : N]$ of N in M to be the coupling constant of N in his representation on $L^2(M)$. If this index is finite, then the trace preserving conditional expectation of M onto N, regarded as an operator on $L^2(M)$, generates together with M a finite factor M_1. This factor is called in Jones' terminology the extension of M by N and the construction of M_1 from M and N, the basic construction. The pair $M \subset M_1$ has the remarkable property that $[M_1 : M] = [M : N]$, so this procedure may be iterated to get an increasing sequence of finite factors $N \subset M \subset M_1 \subset M_2$ ... and together with it a sequence of projections $e_i \in M_{i+1}$, $i \geq 0$, implementing the conditional expectations at consecutive steps.

We prove in this paper that in this sequence of factors the basic construction arises periodically from n to n steps, for any n. In fact we give a formula for a projection f_n in M_{2n+1} that implements the conditional expectation of M_n onto N: f_n is a scalar multiple of the word of maximal length in $\{e_i\}_{0 \leq i \leq 2n}$, namely

$$f_n = [M:N]^{\frac{n(n+1)}{2}} (e_n e_{n-1} \ldots e_0)(e_{n+1} e_n \ldots e_1) \ldots (e_{2n} \ldots e_n) .$$

We mention that this result was independently obtained by A. Ocneanu [2]. We apply this theorem to show that if the logarithm of the index $[M:N]$ equals the relative entropy $H(M|N)$ considered in [3], then one also has $H(M_n|N) = \ln[M_n:N]$ for every n. Since this equality characterises an extremal case for an inclusion of factors, from the analysis of such situation in [3] we deduce several properties of the inclusion $N \subset M_n$ and of the relative commutant $N' \cap M_n$.

§1 Preliminaries

Throughout this paper M will be a finite factor with normalized trace τ, $\tau(1) = 1$. We denote by $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$, the Hilbert norm given by τ and by $L^2(M,\tau)$ the Hilbert space completion of M in this norm. The canonical conjugation of $L^2(M,\tau)$ is denoted by J. It acts on M $L^2(M,\tau)$ by $Jx = x^*$ and satisfies $JMJ = M'$. In fact, if we regard M as acting by left multiplication on $L^2(M,\tau)$ then for $x \in M$, JxJ is the operator of right multiplication by x^*.

$N \subset M$ will denote a subfactor of M with $1_N = 1_M$ and E_N will be the unique normal trace preserving conditional expectation of M onto N. Note that E_N is just the restriction to $M \subset L^2(M,\tau)$ of the orthogonal projection e_N of $L^2(M,\tau)$ onto $L^2(N,\tau)$ (the closure of N in $L^2(M,\tau)$). The conditional expectation E_N, the projection e_N and the conjugation J are related by the properties

(i) if $x \in M$ then $x \in N$ iff $e_N x = x e_N$.

(ii) $e_N x e_N = E_N(x) e_N$, $x \in M$

(iii) J commutes with e_N .

If the index of N in M is finite then from the pair $N \subset M$ one can construct a new pair of finite factors $M \subset M_1$ with the same index $[M_1 : M] = [M:N]$. The construction of M_1 is called the basic construction and the factor M_1 is called the extension of M by N .

We recall from [1] the definition and main properties of M_1 :

1.1 <u>Proposition</u>. Define $M_1 = J N' J$. Then we have:

1° $M_1 = (M \cup \{e_N\})''$;

2° $[M_1 : M] = [M:N]$ and if τ denotes the unique normalized trace on M_1 and E_M the τ preserving conditional expectation of M_1 onto M , then $E_M(e_N) = [M:N]^{-1} 1_M$ or equivalently $\tau(e_N x) = [M:N]^{-1} \tau(x)$ for every $x \in M$.

Part 1° of this proposition can be made more precise: by [3] , if $n+1 \geq [M:N]$ then any element in M_1 is a sum of at most n^2 monomials of the form $x e_N y$, $x,y \in M$. Note that M_1 can also be described abstractly as the unique (up to isomorphism) finite factor M_1 which contains M and a projection e so that $[M_1 : M] = [M:N]$, $[e,y] = 0$ for $y \in N$, $exe = E_N(x)e$ for $x \in M$, and with the trace

τ satisfying $\tau(ex) = [M_1:M]^{-1}\tau(x)$, $x \in M$. In fact one of the conditions is redundant : the next proposition gives two equivalent ways of characterising M_1 .

<u>1.2 Proposition</u> Let $N \subset M$ be a pair of finite factors with finite index and M_1 the extension of M by N. Let $\tilde{M}$ be a finite factor that contains M and with normalized trace $\tilde{\tau}$, E_M the $\tilde{\tau}$-preserving conditional expectation of $\tilde{M}$ onto M and $e \in \tilde{M}$ an orthogonal projection. Then the following conditions are equivalent :

1° There exists an isomorphism ϕ of M_1 onto M such that $\phi(x) = x$ for $x \in M$ and $\phi(e_N) = e$.

2° (i) $[e,y] = 0$, $y \in N$;

(ii) $E_M(e) = [\tilde{M}:M]^{-1} 1_M = [M:N]^{-1} 1_M$;

3° (i) $exe = E_N(x)e$, $x \in M$ and $e \neq 0$;

(ii) e and M generate $\tilde{M}$ as a von Neumann algebra.

<u>Proof</u> : 1° implies 2° by the known properties of e_N .

Suppose 2° holds. Then by 1.8 of [3] we get that $\tilde{M}$ is the extension of M by P where $P = \{e\}' \cap M$. But (i) implies that $N \subset P$ and since $[M:P] = [\tilde{M}:M] = [M:N]$ we conclude that $N = P$. Thus e and M generate $\tilde{M}$ as a von Neumann algebra and again by 1.8 of [3] we get $E_N(x) = exe$, for every $x \in M$.

Assume that 3^o holds. Using the "othonormal basis" of [3] it is easy to see that the map $\phi : M_1 \longrightarrow \tilde{M}$ that sends $\sum x_i e_N y_i$ to $\sum x_i e y_i$ is a well defined $*$-homomorphism. Moreover ϕ satisfies $m\phi(x) = \phi(mx)$ for every $m \in M$ and $x \in M_1$. This shows that $\phi(1)$ is a projection that commutes with e and with every $m \in M$. By (ii) we conclude that $\phi(1)$ is central and since $e \neq o$ and M is a factor $\phi(1) = 1$. This implies now that $\phi(m) = \phi(m1) = m\,\phi(1) = m$ and since obviously $\phi(e_N) = e$ we get 1^o.

q.e.d.

The pair $M \subset M_1$ having finite index one can construct its extension $M_1 \subset M_2$ and in fact the whole procedure may be iterated to get an increasing sequence of finite factors $N \subset M \subset M_1 \subset M_2 \subset \ldots$, and orthogonal projections $e_i \in M_{i+1}$, $i \geq o$ $(N = M_{-1},\ M = M_o)$ in which M_{i+1} is the extension of M_i by M_{i-1} or in other words M_{i+1} and e_i are obtained by the basic construction from the pair $M_{i-1} \subset M_i$. Thus if τ denotes the unique normalized trace on $\bigcup M_i$ and $E_{M_{i-1}}$ the τ-preserving conditional expectation of M_i onto M_{i-1}, $i \geq o$, then :

(a) $[e_i, y] = o$ for $y \in M_{i-1}$;

(b) $e_i x e_i = E_{M_{i-1}}(x) e_i$, $x \in M_i$;

(c) $[M_{i+1} : M_i] = [M:N]$ and $E_{M_i}(e_i) = [M:N]^{-1}\, 1$.

In particular it follows that the sequence of projections e_i satisfies. $[e_i, e_j] = o$, $|i-j| \geq 2$, $e_i e_{i\pm 1} e_i =$

$= [M:N]^{-1} e_i$ and $\tau(e_i w) = [M:N]^{-1} \tau(w)$ for every word in $1, e_o, e_1, \ldots, e_{i-1}$.

§2. n-step extensions

In this section we prove the main result of the paper : we show that if $N \subset M \subset M_1 \subset \ldots$ is the sequence of finite factors obtained by iterating the basic construction as in §1. then, for each $n \geq o$, M_{2n+1} is the extension of M_n by N. In fact we give an explicit formula for a projection $f_n \in M_{2n+1}$ which implements the conditional expectation of M_n onto N and generates with M_n the factor M_{2n+1} : f_n will be a scalar multiple of the word of maximal length in $e_o, e_1, \ldots, e_{2n}$ where $e_i \in M_{i+1}$ are as in §1.

We define for each $n, k \geq o$ the element

$$g_n^k = (e_{n+k} e_{n+k-1} \cdots e_k)(e_{n+k+1} e_{n+k} \cdots e_{k+1}) \cdots (e_{2n+k} e_{2n+k-1} \cdots e_{n+k})$$

(there are $n+1$ products of paranthesis and in each paranthesis the product of $n+1$ consecutive projections e_i in decreasing order). We put $f_n^k = [M:N]^{\frac{n(n+1)}{2}} g_n^k \in M_{2n+k+1}$ and $f_n = f_n^o \in M_{2n+1}$.

To prove that the above defined f_n implements the basic construction in the extension of M_n by N, we only have to show that f_n is an orthogonal projection, that $f_n \in N' \cap M_{2n+1}$ and that $E_{M_n}(f_n) = [M_n:N]^{-1} = [M_{2n+1}:M_n]^{-1}$. (see proposition 1.2). Note that since $[M_{i+1}:M_i] = [M:N]$, by the multiplicative property of the index we do have $[M_n:N] = [M:N]^{n+1} = [M_{2n+1}:M_n]$. To prove the other properties, let us first recall some facts about the algebra generated by $\{e_i\}_{i \geq o}$ (cf. [1]).

A finite product of e_i's is called a word . It is called a reduced word if it is of minimal length for the gramatical rules $e_i e_{i\pm 1} e_i \longleftrightarrow e_i$, $e_i^2 \longleftrightarrow e_i$ and $e_i e_j \longleftrightarrow$ $\longleftrightarrow e_j e_i$ for $|i-j| \geq 2$. Note that any word is a scalar multiple of a reduced word. Jones pointed out (in [1] , 4.1.4) that reduced words can be uniquely written in the ordered form

$$(*) \quad w = (e_{j_1} e_{j_1 -1} \cdots e_{k_1})(e_{j_2} e_{j_2 -1} \cdots e_{k_2}) \cdots (e_{j_p} e_{j_p -1} \cdots e_{k_p})$$

where $j_i \geq k_i$, $j_{i+1} > j_i$, $k_{i+1} > k_i$.

From this description of reduced words it follows that if a reduced word w is written with the letters $e_r, e_{r+1}, \ldots, e_s$ $(s \geq r)$ then e_{r+i} and e_{s-i} appear at most $i+1$ times in w .

To prove the theorem we first show that g_n^o are self-adjoint elements. This will be an easy consequence of the next two lemmas.

<u>2.1 Lemma</u> g_n^o is the unique reduced word of maximal length in $e_o, e_1, \ldots, e_{2n}$.

<u>Proof:</u> Since by definition g_n^o is of the form $(*)$ it is a reduced word. As noted before if w is an arbitrary reduced word in $e_o, e_1, \ldots, e_{2n}$ then e_o, e_{2n} appear at most once in w , e_1, e_{2n-1} at most twice and more generally e_k, e_{2n-k} at most $k+1$ times. Thus the length of w is at most equal to $1+2+\ldots+n+(n+1)+n+\ldots+2+1$ and by inspec-

ting the conditions $j_i \geq k_i$, $j_{i+1} > j_i$, $k_{i+1} > k_i$ of (*) it follows that the only reduced word w with this length is obtained when $j_i = n+i$, $k_i = i$, i.e. $w = g_n^o$

q.e.d.

<u>2.2 Lemma</u> If w is a reduced word in $e_o, e_1, \ldots, e_{2n}$ then the reduced form of w^* has the same length as w .

<u>Proof:</u> Indeed, w^* has length at most equal to that of w and since $(w^*)^* = w$, the statement follows.

q.e.d.

To prove that g_n^o are scalar multiples of projections we have to compute $(g_n^o)^2$. To do this we use an induction argument based on the formula:

<u>2.3 Lemma</u> $g_n^o = (e_n e_{n+1} \ldots e_{2n}) g_{n-1}^o (e_{2n-1} \ldots e_n)$.

<u>Proof!</u> The equality follows by pushing e_{2n} to the left as much as possible.

q.e.d.

<u>2.4 Remark</u> some other two equalities that can be obtained in a similar fashion and seem to be of interest are

$$g_n^o = g_{n-1}^1 (e_{2n} \ldots e_{n+1})(e_o \ldots e_n) = (e_n e_{n-1} \ldots e_o) g_{n-1}^2 (e_1 e_2 \ldots e_n)$$

To show that g_n^o projects on a scalar in M_n we prove:

<u>2.5 Lemma</u> $E_{M_{2n}}(g_n^o) = [M:N]^{-(n+1)} g_{n-1}^1$. More generally

$E_{M_{2n+k}}(g_n^k) = [M:N]^{-(n+1)} g_{n-1}^{k+1}$.

<u>Proof:</u> It is enough to prove that $E_{M_{2n}}(g_n^o) = \lambda^{n+1} g_{n-1}^1$, where $\lambda = [M:N]^{-1}$, because the rest of the statement follows by starting the sequence of factors from $M_{k-1} \subset M_k$, instead of $N = M_{-1} \subset M_o = M$.

We first show that for $j \geqslant p \geqslant k+1$ we have :

$$(**) \quad (e_j e_{j-1} \cdot\cdot e_k)(e_p e_{p-1} \cdot\cdot e_{k+1}) = \lambda (e_{p-2} \cdots e_k)(e_j \cdots e_{k+1}).$$

Indeed we have $(e_j e_{j-1} \cdot\cdot e_p e_{p-1} \cdot\cdot e_k) e_p = \lambda (e_j e_{j-1} \cdot\cdot e_p)$ $(e_{p-2} e_{p-3} \cdot\cdot e_k) = \lambda (e_{p-2} \cdots e_k)(e_j e_{j-1} \cdot\cdot e_p)$, which easily implies (**). Applying recursively (**) we get :

$$E_{M_{2n}}(g_n^o) = (e_n e_{n-1} \cdot\cdot e_o) \cdot\cdot (e_{2n-1} \cdot\cdot e_{n-1}) E_{M_{2n}}(e_{2n})(e_{2n-1} \cdot\cdot e_n) =$$

$$= \lambda (e_n \cdot\cdot e_o) \cdots (e_{2n-1} \cdot\cdot e_{n-1})(e_{2n-1} \cdot\cdot e_n) =$$

$$= \lambda^2 (e_n \cdot\cdot e_o) \cdots (e_{2n-2} e_{2n-3} \cdot\cdot e_{n-2})(e_{2n-3} \cdot\cdot e_{n-1})(e_{2n-1} \cdot\cdot e_n) =$$

$$= \lambda^3 (e_n \cdot\cdot e_o) \cdot\cdot (e_{2n-5} \cdot\cdot e_{n-2})(e_{2n-2} \cdot\cdot e_{n-1})(e_{2n-1} \cdot\cdot e_n) = \cdots .$$

$$\cdots = \lambda^n (e_n \cdot\cdot e_o) e_1 (e_{n+1} \cdot\cdot e_2) \cdots (e_{2n-2} \cdot\cdot e_{n-1})(e_{2n-1} \cdot\cdot e_n) =$$

$$= \lambda^{n+1} (e_n \cdot\cdot e_1)(e_{n+1} \cdot\cdot e_2) \cdot\cdot (e_{2n-1} \cdot\cdot e_n) = \lambda^{n+1} g_{n-1}^1 .$$

q.e.d.

We can now prove the theorem.

2.6 Theorem Let $N \subset M$ be a pair of finite factors with $[M:N] < \infty$. Let $N \subset M \subset M_1 \subset \ldots$ be the sequence of finite factors obtained by iterating the basic construction and $e_i \in M_{i+1}$ the projection implementing the conditional expectation of M_i onto M_{i-1} at each step of the basic construction as in §1. for $i \geq 0$ $(M_{-1} = N, M_0 = M)$. Let $f_n = [M:N]^{\frac{n(n+1)}{2}} (e_n e_{n-1} \ldots e_0)(e_{n+1} e_n \ldots e_1) \ldots (e_{2n} e_{2n-1} \ldots e_n)$ $\in M_{2n+1}$. Then M_{2n+1} is the extension of M_n by N and $f_n \in M_{2n+1}$ is the projection that implements the conditional expectation of M_n onto N, i.e. $f_n \in N' \cap M_{2n+1}$, $f_n x f_n = E_N(x) f_n$, $x \in M_n$, $E_{M_n}(f_n) = [M_n:N]^{-1}$ and $M_{2n+1} = (M_n \cup \{f_n\})''$.

Proof: We will prove the theorem by induction over $n \geq 0$. If $n = 0$ then $f_0 = e_0$ and we have nothing to prove. Assume the statement is true up to $n-1$. Let $\lambda = [M:N]^{-1}$ and $c_n = \lambda^{-\frac{n(n+1)}{2}}$. Since $f_n = c_n g_n^0$ and g_n^0 is a word in $e_0, e_1, \ldots, e_{2n}$, which all commute with N, it follows that $f_n \in N' \cap M_{2n+1}$. Note also that since e_{2n} $M_{2n-1}' \cap M_{2n+1}$, e_{2n} commutes with $g_{n-1}^0 \in M_{2n-1}$. To see that g_n^0 is selfadjoint we use lemma 2.2 to obtain that $g_n^{0\,*}$ has the same length as g_n^0 and thus by lemma 2.1 $g_n^0 = (g_n^0)^*$. Further lemma 2.3 implies that $(g_n^0)^2 = g_n^{0\,*} g_n^0 =$

$= (e_n e_{n+1} \ldots e_{2n-1}) g_{n-1}^0 (e_{2n} e_{2n-1} \ldots e_{n+1} e_n e_{n+1} \ldots e_{2n-1} e_{2n}) g_{n-1}^0$ $(e_{2n-1} \ldots e_n) = \lambda^n (e_n e_{n+1} \ldots e_{2n-1}) g_{n-1}^0 e_{2n} g_{n-1}^0 (e_{2n-1} \ldots e_n) =$

$= \lambda^n (e_n e_{n+1} \ldots e_{2n})(g_{n-1}^0)^2 (e_{2n-1} \ldots e_n) =$

$$= \lambda^n c_{n-1}^{-1}(e_n e_{n+1} \cdot\cdot e_{2n}) g_{n-1}^o (e_{2n-1} \cdot\cdot e_n) = \lambda^n c_{n-1}^{-1} g_n^o = c_n^{-1} g_n^o .$$

Thus $f_n = c_n g_n^o$ is a selfadjoint projection in $N' \cap M_{2n+1}$. Next we apply recursively lemma 2.5 to get

$$E_{M_n}(f_n) = c_n E_{M_n}(g_n^o) = c_n E_{M_n} E_{M_{2n}}(g_n^o) = c_n \lambda^{n+1} E_{M_n}(g_{n-1}^1) =$$

$$= c_n \lambda^{n+1} E_{M_n} E_{M_{2n-1}}(g_{n-1}^1) = c_n \lambda^{(n+1)+n} E_{M_n}(g_{n-2}^2) = \dots =$$

$$= c_n \lambda^{(n+1)+n+..+2} E_{M_n}(g_o^n) = c_n \lambda^{(n+1)+n+..+2} E_{M_n}(e_n) =$$

$$= c_n \lambda^{\frac{(n+2)(n+1)}{2}} 1_{M_n} = \lambda^{n+1} 1_{M_n} .$$

(we used that $g_o^n = e_n$).

Moreover by [1] , $[M_{2n+1}:M_n] = \prod_{n \leq i \leq 2n} [M_{i+1}:M_i] =$ $= [M:N]^{n+1} = \prod_{o \leq i \leq n} [M_{i+1}:M_i] = [M_n:N]$.

By proposition 1.2 the rest of the properties of f_n follow automatically.

q.e.d.

<u>2.7 Remark</u> We could include the proof of $g_n^o = g_n^{o\,*}$ in the induction argument. Indeed by lemma 2.3 and using $g_{n-1}^o = (g_{n-1}^o)^*$ and $[e_{2n}, g_{n-1}^o] = o$ we get

$$(g_n^o)^* = e_n e_{n+1} \cdot\cdot e_{2n} (g_{n-1}^o)^* e_{2n-1} e_{2n-2} \cdot\cdot e_n =$$

$$= e_n e_{n+1} \cdot\cdot e_{2n} g_{n-1}^o e_{2n-1} \cdots e_n = g_n^o .$$

We prefered however the deductive argument of lemmas 2.1

and 2.2 as it points out some properties of f_n.

§3. Some aplications

In this section we derive some consequences on the inclusion $N \subset M_n$. We consider the case when the relative entropy $H(M|N)$ considered in [3] satisfies $H(M|N) = \ln[M:N]$. An important case when this equality occurs is when $N' \cap M = \mathbb{C}$ (cf. [3]). First we compute the relative entropy from n to n steps.

3.1 Theorem If $H(M|N) = \ln[M:N]$ then $H(M_{n+k}|M_{k-1}) = \ln[M_{n+k}:M_{k-1}]$, for every $n,k \geq 0$. In particular $H(M_n|N) = \ln[M_n:N]$ and $H(M_k|M_{k-1}) = \ln[M_k:M_{k-1}]$, for every $k \geq 0$.

Proof: We first prove that $H(M_n|N) = \ln[M_n:N]$. By 4.4 in [3] and theorem 2.6, it is enough to prove that $E_{M_n' \cap M_{2n+1}}(f_n) = \lambda^{n+1} 1_{M_{2n+1}}$. Since $M_n' \cap M_{2n+1} \subset M_{n-1}' \cap M_{2n+1} \subset \ldots \subset M' \cap M_{2n+1}$ we have $E_{M_n' \cap M_{2n+1}} = E_{M_n' \cap M_{2n+1}} \cdot E_{M_{n-1}' \cap M_{2n+1}} \ldots E_{M' \cap M_{2n+1}}$. Since e_0 appears only once in g_n^0 and $E_{M' \cap M_{2n+1}}(e_0) = \lambda 1$ and $e_i \in M_{i-1}'$, it follows that

$$E_{M'\cap M_{2n+1}}(g_n^o) = (e_n \dots e_1 \, E_{M'\cap M_{2n+1}}(e_o))(e_{n+1} \dots e_1) \dots$$

$\dots (e_{2n}e_{2n-1}\dots e_n)$. Using now the same computations as in the proof of 2.6 it follows that $E_{M'\cap M_{2n+1}}(g_n^o) = \lambda^{n+1} g_{n-1}^1$. By induction it follows that $E_{M_n'\cap M_{2n+1}}(g_n^o) = \lambda^{n+1} E_{M_n'\cap M_{2n+1}}(g_{n-1}^1)$ $= \lambda^{n+1} E_{M_n'\cap M_{2n+1}} E_{M_1'\cap M_{2n+1}}(g_{n-1}^1) = \lambda^{n+1}\lambda^n E_{M_n'\cap M_{2n+1}}(g_{n-2}^2) =$ $= \dots = \lambda^{(n+1)+n+\dots+1} I$ and thus $E_{M_n'\cap M_{2n+1}}(f_n) = \lambda^{n+1} I$.

From the equalities $H(M_n|N) = \ln[M_n:N] = \ln \prod_{o\leq i\leq n}[M_i:M_{i-1}]$ $= \sum_{o\leq i\leq n} \ln[M_i:M_{i-1}]$, by §3.1 and 4.1 in [3] we deduce

$$\sum_{o\leq i\leq n} \ln[M_i:M_{i-1}] = H(M_n|N) \leq \sum_{o\leq i\leq n} H(M_i|M_{i-1}) \leq$$

$$\leq \sum_{o\leq i\leq n} \ln[M_i:M_{i-1}]$$. Thus all these inequalities are equalities and from $H(M_i|M_{i-1}) \leq \ln[M_i:M_{i-1}]$ it follows that in fact $H(M_i|M_{i-1}) = \ln[M_i:M_{i-1}]$ for all $i \geq o$.

The general formula $H(M_{n+k}|M_{k-1}) = \ln[M_{n+k}:M_{k-1}]$ follows now easily by the first part of the proof, since we have $H(M_k|M_{k-1}) = \ln[M_k:M_{k-1}]$ and since we can start the sequence from the inclusion $M_{k-1} \subset M_k$, instead of $N \subset M$.

q.e.d.

<u>3.2 Corollary</u> Let $N \subset M$ be as in theorem 3.1 . Let J_n be the canonical conjugation on $L^2(M_n,\tau)$. Suppose

M_{2n+1} is represented on $L^2(M_n,\tau)$ so that to coincide with the basic construction of $N \subset M_n$. Then we have :

(i) For every projection $f \in N' \cap M_n$, $[(M_n)_f : N_f] = [M_n : N]\, \tau(f)^2$;

(ii) The antiisomorphism $N' \cap M_n \ni x \longmapsto J_n x J_n \in M_n' \cap M_{2n+1}$ is trace preserving ;

(iii) For every $k \geq 0$ there exists a trace preserving isomorphism $N' \cap M \ni x \longmapsto x' \in M_{k-1}' \cap M_k$ so that for every minimal projection $f \in N' \cap M$, $[M_f : N_f] = [(M_k)_{f'} : (M_{k-1})_{f'}]$.

Proof: By 4.5 in [3] the condition $H(M_n | N) = \ln [M_n : N]$ is equivalent to the above conditions (i) and (ii). Then (iii) follows by (i), (ii) and by the fact that given any trace preserving antiisomorphism between two finite dimensional algebras there exists a trace preserving isomorphism between them which acts on the centers in the same way the antiisomorphism does.

q.e.d.

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