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by

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O N E   S U B S P A C E

by Silviu TELEMAN

0. In [3] M.A.Rieffel and A. van Daele gave a bounded operator approach to the Tomita-Takesaki Theory, which has as a starting frame a closed <u>real</u> vector subspace $\mathcal{K}$ of the complex Hilbert space $\mathcal{H}$, satisfying the "non-degeneracy" conditions

$$\text{a)}\quad \mathcal{K} \cap (i\mathcal{K}) = \{0\},$$

and

$$\text{b)}\quad (\mathcal{K} + i\mathcal{K})^{\perp} = \{0\}.$$

Conditions a) and b) are slightly less restrictive than the "general position" condition for a pair of closed vector subspaces $\mathcal{K}, \mathcal{L} \subset \mathcal{H}$ of the real, or complex, Hilbert space $\mathcal{H}$, all over the same field of scalars, studied by P.R.Halmos in [1], for which a "graph representation" was obtained (see [1], Theorem 3 ; [3], Theorem 2.4).

In the present Note we shall give a similar "graph representation" Theorem for the case of the closed <u>real</u> vector subspace $\mathcal{K}$ of the <u>complex</u> Hilbert space $\mathcal{H}$, satisfying the "non-degeneracy" conditions a) and b). With its help, we hope to satisfy the desire expressed in ([p.200) for a geometric characterization of the modular group $\mathbb{R} \ni t \mapsto \Delta^{it}$, corresponding to $\mathcal{K}$.

Namely, we shall prove that any pair, consisting of a strongly continuous one-parameter unitary group

$$\mathbb{R} \ni t \longmapsto u_t \in \mathcal{L}(\mathcal{H}),$$

and a conjugation $J : \mathcal{H} \to \mathcal{H}$, such that

$$Ju_t = u_t J, \qquad t \in \mathbb{R},$$

derives from a closed real vector subspace $\mathcal{K} \subset \mathcal{H}$, satisfying the non-degeneracy conditions a) and b) above, by the construction given in (see, also, [2], p.371). Moreover, $\mathcal{K}$ is uniquely determined by the pai

$((u_t)\;, J)$

1.Let $\mathcal{H}$ be any complex Hilbert space.A conjugation in $\mathcal{H}$ is any anti-linear mapping $J:\mathcal{H}\to\mathcal{H}$,such that

$$(Jx|Jy) = (y|x), \qquad x,y\in\mathcal{H} ,$$

and $J^2 = 1$.It immediately follows that $J^* = J$ and,therefore,J is a symmetry of the real Hilbert space $\mathcal{H}_{\mathbb{R}}$,obtained from $\mathcal{H}$ by restricting the scalars to $\mathbb{R}$ and by endowing it with the real scalar product

$$\langle x|y\rangle = \mathrm{Re}\,(x|y) , \qquad x,y\in\mathcal{H};$$

i.e., J is an orthogonal self-adjoint continuous linear operator in $\mathcal{H}_{\mathbb{R}}$.

LEMMA 1.Any conjugation J of $\mathcal{H}$ decomposes uniquely as the difference

$$(1) \qquad J = J_+ - J_-$$

of two real projections $J_+ ,J_-\in\mathcal{L}(\mathcal{H}_{\mathbb{R}})$,such that

$$(2) \qquad 1 = J_+ + J_- .$$

Moreover,we have

$$(3) \qquad J_+ i = iJ_- , \quad J_- i = iJ_+ .$$

Proof.Let us define

$$(*) \qquad J_+ = \frac{1}{2}(1+J), \quad J_- = \frac{1}{2}(1-J).$$

It is obvious that conditions (1) and (2) are satisfied.Moreover,we have

$$J_+^* = J_+ \quad\text{and}\quad J_-^* = J_-,$$

(either in $\mathcal{H}_{\mathbb{R}}$ or in $\mathcal{H}$) and $J_+^2 = J_+$,$J_-^2 = J_-$.From the definition (*) it immediately follows that relations (3) hold too.

Conversely,conditions (1) and (2) uniquly determine the operators J_+ and J_-.

Let now $P_1\in\mathcal{L}(\mathcal{H})$ be a complex projection;then it is obvious that $P_0=JP_1J$ is also a complex projection.We shall assume that

$$P_0P_1 = 0,$$

and we shall denote $Q_0 = 1 - P_0 - P_1$.It is obvious that Q_0 is a complex projection,such that $JQ_0J = Q_0$.

It immediately follows that $Q_+=Q_oJ_+$ $(= J_+Q_o)$ and $Q_- = Q_oJ_-$ $(= J_-Q_o)$ are <u>real</u> projections, such that

$$Q_o = Q_+ + Q_- , \quad Q_+Q_- = o.$$

We shall denote $\mathcal{H}_o = P_o(\mathcal{H})$, $\mathcal{H}_1 = P_1(\mathcal{H})$, $\mathcal{H}_+ = Q_+(\mathcal{H})$ and $\mathcal{H}_- = Q_-(\mathcal{H})$. Then we have the real orthogonal decomposition $\mathcal{H} = \mathcal{H}_o \oplus \mathcal{H}_1 \oplus \mathcal{H}_+ \oplus \mathcal{H}_-$.

From $iQ_+ = Q_-i$ and $iQ_- = Q_+i$ we immediately infer that $i\mathcal{H}_+ = \mathcal{H}_-$, $i\mathcal{H}_- = \mathcal{H}_+$; and also

$$\mathcal{H}_+ = \{ x \in Q_o(\mathcal{H}) \ ; \ Jx = x \} ,$$
$$\mathcal{H}_- = \{ x \in Q_o(\mathcal{H}) \ ; \ Jx = -x \} .$$

It immediately follows that

$$x,y \in \mathcal{H}_+ \Rightarrow (x|y) = \langle x|y \rangle ,$$
$$x,y \in \mathcal{H}_- \Rightarrow (x|y) = \langle x|y \rangle ,$$
$$x \in \mathcal{H}_+ , y \in \mathcal{H}_- \Rightarrow \langle x|y \rangle = o.$$

We shall now consider the <u>real</u> Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_+$ and we shall define the <u>real</u> linear isometry

$$U : \mathcal{H} \to (\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_+)$$

by the formula

(1) $\quad Ux = (P_1x+\frac{1}{\sqrt{2}}Q_+x+\frac{i}{\sqrt{2}}Q_-x, JP_ox+\frac{1}{\sqrt{2}}Q_+x-\frac{i}{\sqrt{2}}Q_-x)$,

for $x \in \mathcal{H}$. It is easy to see that for any pair $(x_1'+x_+' , x_1''+x_+'') \in (\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_+)$, the corresponding $x \in \mathcal{H}$, such that $Ux = (x_1'+x_+' , x_1''+x_+'')$, is given by the formula

(2) $\quad x = x_1' + Jx_1'' + \frac{\sqrt{2}}{2}(x_+' + x_+'') - i\frac{\sqrt{2}}{2}(x_+' - x_+'')$,

which is a proof for the surjectivity of U.

In the space $(\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_+)$ one can introduce the structure of a complex vector space by defining the multiplication by i, by the formula

(3) $\quad (x_1'+x_+', x_1''+x_+'') \mapsto (ix_1' - x_+'', -ix_1''+x_+')$.

With this definition we have

$$U(ix) = iU(x) , \qquad x \in \mathcal{H};$$

hence, U becomes $\mathbb{C}$-linear.

In order to ensure that U preserve the scalar product, one has to define the complex scalar product in the complex vector space

$$(\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_+)$$

by the formula

$$\begin{aligned}(4)\quad &((x_1'+x_+',\ x_1''+x_+'') \mid (y_1'+y_+',\ y_1''+y_+'')) = \\ &= (x_1' \mid y_1') + (y_1'' \mid x_1'') + (x_+' \mid y_+') + (x_+'' \mid y_+'') - i(x_+' \mid y_+'') + i(x_+'' \mid y_+'),\end{aligned}$$

for any $x_1', x_1'', y_1', y_1'' \in \mathcal{H}_1$ and any $x_+', x_+'', y_+', y_+'' \in \mathcal{H}_+$.

The corresponding real scalar product is given by the formula

$$\langle (x_1'+x_+', x_1''+x_+'') \mid (y_1'+y_+', y_1''+y_+'') \rangle =$$
$$= \langle x_1' \mid y_1' \rangle + \langle x_1'' \mid y_1'' \rangle + \langle x_+' \mid y_+' \rangle + \langle x_+'' \mid y_+'' \rangle \quad ;$$

hence, the real Hilbert space structure of the space

$$(\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_+),$$

derived from its complex Hilbert space structure, coincides with that given by the direct Hilbert sum of its components, regarded as real Hilbert spaces.

2. Let now $\mathcal{K} \subset \mathcal{H}$ be any closed real vector subspace of $\mathcal{H}$, satisfying conditions a) and b), in section 0.

As in [3], we shall denote by P, respectively Q, the real projection onto $\mathcal{K}$, respectively onto $i\mathcal{K}$, and we shall define

$$R = P + Q,$$

whereas $JT = P - Q$ will stand for the polar decomposition of the diffe rence P - Q. Then J is a conjugation of $\mathcal{H}$, R and T are (complex) line operators in $\mathcal{L}(\mathcal{H})$, related by the equality

$$T = (2 - R)^{1/2} R^{1/2}$$

(We recall that $0 \leq R \leq 2$, whereas R and 2-R are injective). Moreover, we have that

$$JP = (1-Q)J, \quad JQ = (1-P)J;$$

hence, $JR = (2-R)J$; and $JT = TJ$, $Pi = iQ$, $Qi = iP$. We refer to [3] for the proofs of these assertions.

The "modular operator" Δ is introduced by the formula

$$\Delta = (2 - R) R^{-1},$$

and it is defined on $\mathcal{D}(\Delta) = \mathcal{R}(P)$. It is obvious that we also have $\Delta = R^{-1}(2-R) = 2R^{-1}-1$, and Δ is a positive injective self-adjoint operator, which can also be defined by the functional calculus

$$\Delta = f(R), \qquad f : (0,+\infty) \ni t \mapsto t^{-1}(2-t).$$

In general, Δ is an unbounded self-adjoint operator, for which we sha denote by $(E_\Delta(\lambda))_{\lambda\in\mathbb{R}}$ the spectral scale. We recall that we have

$$(\Delta x|y) = \int_{\mathbb{R}} \lambda \, d(E_\Delta(\lambda)x|y), \qquad x\in\mathcal{D}(\Delta),\ y\in\mathcal{H},$$

and

$$x\in\mathcal{D}(\Delta) \Longleftrightarrow \int_{\mathbb{R}} \lambda^2 d\|E_\Delta(\lambda)x\|^2 < +\infty .$$

It is easy to prove that we have

$$J\Delta J = \Delta^{-1}$$

(see [2], Lemma 8.13.4).

3. In order to obtain our graph representation Theorem for the subspace $\mathcal{K}$, we shall slightly modify the proof of Theorem 2.4 from [3] The changes are necessary, due to the fact that the non-degeneracy conditions for $\mathcal{K}$ do not imply, in general, that $\mathcal{K}$ and $i\mathcal{K}$ are in "genera position".

Let us denote $P_1 = \chi_{(1,2]}(R)$ and $P_0 = \chi_{[0,1)}(R)$. It is obvious that we have

$$(1) \qquad P_1 = \chi_{(0,1)}(\Delta),$$

and, also, that we have $P_0P_1 = 0$; as above, we define

$$Q_0 = 1 - P_0 - P_1 .$$

Let us denote by $(E_R(\lambda))_{\lambda\in\mathbb{R}}$ the spectral scale of R; then, from the equality $JRJ = 2-R$, we infer that

$$JE_R(\lambda)J = E_{JRJ}(\lambda) = E_{2-R}(\lambda) = 1 - E_R(2-\lambda-0), \qquad \lambda\in\mathbb{R}.$$

If we denote by $\mathcal{H}_i$ the range of the projection P_i, $i = 0,1$, then, from the equalities

$$\mathcal{H}_1 = \{x\in\mathcal{H};\ E_R(\lambda)x = 0,\ \forall\lambda\le 1\}$$

and

$$\mathcal{H}_o = \{ x \in \mathcal{H} ;\ E_R(1-o)x = x \} ,$$

we immediately infer that

$$J(\mathcal{H}_1) = \mathcal{H}_o \quad \text{and} \quad J(\mathcal{H}_o) = \mathcal{H}_1 .$$

On the other hand, from the equality

$$Q_o = \chi_{\{1\}}(R) ,$$

we immediately infer that

$$Q_o(\mathcal{H}) = \{ x \in \mathcal{H} ;\ Rx = x \}$$

and also that

$$Q_o(\mathcal{H}) = \{ x \in \mathcal{D}(\Delta) \ ;\ \Delta x = x \} .$$

It is obvious that $J(Q_o(\mathcal{H})) = Q_o(\mathcal{H})$.

As in section 1, we have the canonical decomposition $J = J_+ - J_-$, and we shall denote

$$Q_+ = Q_oJ_+ = J_+Q_o ,$$

$$Q_- = Q_oJ_- = J_-Q_o ,$$

$$\mathcal{H}_+ = Q_+(\mathcal{H}) = \{ x \in Q_o(\mathcal{H}) \ ;\ Jx = x \},$$

$$\mathcal{H}_- = Q_-(\mathcal{H}) = \{ x \in Q_o(\mathcal{H}) \ ;\ Jx = -x \} .$$

Of course, the results obtained in section 1 apply in this situation.

LEMMA 2. $\mathcal{H}_+ = \mathcal{K} \cap (i\mathcal{K})^{\perp}$ and $\mathcal{H}_- = (i\mathcal{K}) \cap \mathcal{K}^{\perp}$.

(here the orthogonal complement is taken with respect to the real scalar product).

Proof. Indeed, for $x \in \mathcal{H}_+$ we have that $Rx = x$ and $Q_+x = x$. It follows that $Px + Qx = x$; hence, $Tx = x$, and $J_+x = x$. This implies that $Px - Qx = JTx = Jx = x$ and, therefore, $Px = x$ and $Qx = o$. It follows that $x \in \mathcal{K} \cap (i\mathcal{K})^{\perp}$. Conversely, $Px = x$ and $Qx = o$ obviously imply that $x \in \mathcal{H}_+$. The second equality has a similar proof.

Let C_1 be the restriction of $\Delta^{1/2}$ to $\mathcal{H}_1$; in fact, it is easy to see that $\mathcal{H}_1 \subset \mathcal{D}(\Delta) \subset \mathcal{D}(\Delta^{1/2})$ and, of course, we have that $C_1(\mathcal{H}_1) \subset \mathcal{H}_1$. It follows that C_1 is a bounded operator in $\mathcal{L}(\mathcal{H}_1)$, such that $o \leq C_1 \leq 1_{\mathcal{H}_1}$ and both C_1 and $1_{\mathcal{H}_1} - C_1$ are injective.

LEMMA 3. $P(\mathcal{H}_1) = (1 + JC_1)(\mathcal{H}_1)$.

Proof. For $x \in \mathcal{H}_1$ we have $Rx \in \mathcal{H}_1$ and

$$2Px = (R + JT)x = (R + JR^{1/2}(2 - R)^{1/2})x =$$
$$= (1 + JC_1)Rx \in (1 + JC_1)(\mathcal{H}_1).$$

We infer that

$$P(\mathcal{H}_1) \subset (1 + JC_1)(\mathcal{H}_1).$$

Let now $x \in (1 + JC_1)(\mathcal{H}_1)$; then $x = (1 + JC_1)x_1$, where $x_1 \in \mathcal{H}_1$ and, therefore, since $R(\mathcal{H}_1) = \mathcal{H}_1$, there exists an $x_0 \in \mathcal{H}_1$, such that $x_1 = Rx_0$. It follows that

$$x = (1 + JC_1)Rx_0 = 2Px_0 \in P(\mathcal{H}_1).$$

We infer that

$$(1 + JC_1)(\mathcal{H}_1) \subset P(\mathcal{H}_1),$$

and the Lemma is proved.

LEMMA 4. $P(\mathcal{H}_0) \subset (1 + JC_1)(\mathcal{H}_1)$.

Proof. For $x_0 \in \mathcal{H}_0$ we have $x_0 = Jx_1$, where $x_1 \in \mathcal{H}_1$; therefore, $Tx_1 \in \mathcal{H}_1$. On the other hand, we have

$$2Px_0 = (R + JT)x_0 = (R + JT)Jx_1 = (RJ + T)x_1 =$$
$$= (J(2 - R) + T)x_1 = (J(2 - R) + (2 - R)^{1/2}R^{1/2})x_1 =$$
$$= (J(2 - R)^{1/2}R^{-1/2} + 1)(2 - R)^{1/2}R^{1/2}x_1 =$$
$$= (1 + JC_1)Tx_1 \in (1 + JC_1)(\mathcal{H}_1).$$

The Lemma is proved.

From $P(\mathcal{H}_+) = \mathcal{H}_+$, $P(\mathcal{H}_-) = \{0\}$ and from the (real orthogonal) decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_0$$

we infer that

$$\begin{aligned} \mathcal{K} = P(\mathcal{H}) &= P(\mathcal{H}_1) + P(\mathcal{H}_+) + P(\mathcal{H}_-) + P(\mathcal{H}_0) \supset \\ &\supset (1 + JC_1)(\mathcal{H}_1) + \mathcal{H}_+ , \end{aligned} \tag{1}$$

where we have taken into consideration Lemma 3; and also we have that

$$\mathcal{K} \subset (1 + JC_1)(\mathcal{H}_1) + \mathcal{H}_+ , \tag{2}$$

by taking into account Lemma 4.

LEMMA 5. *We have the orthogonal decomposition*

$$\mathcal{K} = (1 + JC_1)(\mathcal{H}_1) \oplus \mathcal{H}_+ \quad ,$$

with respect to the complex scalar product.

Proof. From inclusions (1) and (2) above we infer that we have

$$\mathcal{K} = (1 + JC_1)(\mathcal{H}_1) + \mathcal{H}_+ .$$

On the other hand, for $x_+ \in \mathcal{H}_+$ and $x_1 \in \mathcal{H}_1$, we have

$$(1) \qquad ((1 + JC_1)x_1 | x_+) = ((1 + JC_1)x_1 | Rx_+) = ((R + RJC_1)x_1 | x_+) = \\ = ((R + J(2 - R)C_1 x_1 | x_+).$$

Since $(2 - R)C_1 x_1 \in \mathcal{H}_1$, we have $J(2 - R)C_1 x_1 \in \mathcal{H}_0$ and, therefore, we have that

$$(R + J(2 - R)C_1)x_1 \in \mathcal{H}_0 \oplus \mathcal{H}_1 .$$

From

$$\mathcal{H}_+ \subset Q_0(\mathcal{H}) = (\mathcal{H}_0 \oplus \mathcal{H}_1)^{\perp}$$

and from (1) we infer that

$$((1 + JC_1)x_1 | x_+) = 0 , \quad \forall\ x_1 \in \mathcal{H}_1 ,\ x_+ \in \mathcal{H}_+ ,$$

and the Lemma is proved.

Remark. Since we have that $P_1 = \chi_{(0,1)}(\Delta)$, $Q_0(\mathcal{H}) = \{ x \in \mathcal{D}(\Delta);\ \Delta x = x \}$ and $Q_+ = Q_0 J_+$, from the preceding Lemma and from the definition of C_1 we immediately infer that the conjugation J and the modular operator Δ uniquely determines the (real) Hilbert subspace $\mathcal{K} \subset \mathcal{H}$

Below we shall prove that for *any* pair (Δ, J), consisting of an injective positive self-adjoint operator $\Delta : \mathcal{D}(\Delta) \to \mathcal{H}$ and a conjugation $J : \mathcal{H} \to \mathcal{H}$, such that $J \Delta J = \Delta^{-1}$, there exists a (real) Hilbert subspace $\mathcal{K} \subset \mathcal{H}$, satisfying the non-degeneracy conditions a) and b), such that Δ and J be the modular operator, and the conjugation, corresponding to $\mathcal{K}$ by the Rieffel-van Daele construction. The preceding Lemma then shows that such a subspace $\mathcal{K}$ is uniquely determined by the pair (Δ, J).

4. We shall now use the constructions given in section 1 for the projections $P_1 = \chi_{(1,2]}(R)$, $P_0 = \chi_{[0,1)}(R)$, $Q_+ = (1 - P_0 - P_1)J_+$, $Q_- = (1 -$

$- P_o - P_1)J_-$ .We shall consider the real Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_+$ and the complex linear isometry

$$U : \mathcal{H} \to (\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_+),$$

defined,as in section 1,by the formula

$$(1) \quad Ux = (P_1 x + \frac{1}{\sqrt{2}} Q_+ x + \frac{i}{\sqrt{2}} Q_- x,\ JP_o x + \frac{1}{\sqrt{2}} Q_+\ x - \frac{i}{\sqrt{2}} Q_- x),$$

for any $x \in \mathcal{H}$.Of course,the results obtained in section 1 apply to this particular case.

Easy computations give the following formulae,by which the operators on $\mathcal{H}$,introduced above,are transferred to $(\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus (\mathcal{H}_1 \oplus \mathcal{H}$

$$(1) \quad \begin{aligned} &(UPU^{-1})(x_1' + x_+', x_1'' + x_+'') = \\ &= (\tfrac{1}{2}(Rx_1' + Tx_1'') + \tfrac{1}{2}(x_1' + x_1''),\ \tfrac{1}{2}(Tx_1' + (2-R)x_1'' + \tfrac{1}{2}(x_+' + x_+ \end{aligned}$$

$$(2) \quad \begin{aligned} &(UQU^{-1})(x_1' + x_+', x_1'' + x_+'') = \\ &= (\tfrac{1}{2}(Rx_1' - Tx_1'') + \tfrac{1}{2}(x_+' - x_+''),\ \tfrac{1}{2}((2-R)x_1'' - Tx_1') + \tfrac{1}{2}(x_+'' - x_+') \end{aligned}$$

$$(3) \quad (URU^{-1})(x_1' + x_+', x_1'' + x_+'') = (Rx_1' + x_+', (2-R)x_1'' + x_+''),$$

$$(4) \quad (U\Delta U^{-1})(x_1' + x_+', x_1'' + x_+'') = (\Delta x_1' + x_+',\ \Delta^{-1} x_1'' + x_+''),$$

$$(5) \quad (U\Delta^{it} U^{-1})(x_1' + x_+', x_1'' + x_+'') = (\Delta^{it} x_1' + x_1',\ \Delta^{it} x_1'' + x_+''), \quad t \in \mathbb{R} .$$

The conjugation J in $\mathcal{H}$ is transferred to the space $(\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus$ $\oplus (\mathcal{H}_1 \oplus \mathcal{H}_+)$ by the formula

$$(6) \quad (UJU^{-1})(x_1' + x_+', x_1'' + x_+'') = (x_1'' + x_+'', x_1' + x_+') ;$$

i.e., $S = UJU^{-1}$ is the <u>natural symmetry</u> in the space $(\mathcal{H}_1 \oplus \mathcal{H}_+) \oplus$ $\oplus (\mathcal{H}_1 \oplus \mathcal{H}_+)$.

We shall now define the operator $\tilde{C}_1 \in \mathcal{L}_{\mathbb{R}}(\mathcal{H}_1 \oplus \mathcal{H}_+)$ by the formula

$$\tilde{C}_1(x_1 + x_+) = C_1 x_1 + x_+ \ , \ x_1 \in \mathcal{H}_1 \ , \ x_+ \in \mathcal{H}_+ .$$

It is obvious that we have $0 \le \tilde{C}_1 \le 1$.

<u>Remark</u>.Since $\Delta x = x$,for any $x \in \mathcal{H}_+$,we can also define $\tilde{C}_1$ to be t restriction and corestriction of $\Delta^{1/2}$ to $\mathcal{H}_1 \oplus \mathcal{H}_+$.

The following Theorem exhibits the subspaces $\mathcal{K}$ and $i\mathcal{K}$ as graph of bounded operators and is,therefore,an extension of Theorem 3 from [1] (see,also, [3] ,Theorem 2.4).

THEOREM 1. a) $U(\mathcal{K}) = \Gamma(\tilde{C}_1)$; b) $U(i\mathcal{K}) = \Gamma(-\tilde{C}_1)$.

Proof. a)By Lemma 5,any $x\in\mathcal{K}$ is of the form

$$x = (1 + JC_1)x_1 + x_+ ,$$

where $x_1\in\mathcal{H}_1$ and $x_+\in\mathcal{K}_+$ .We then have

$$Ux = (P(1+JC_1)x_1+P_1x_+ + \frac{1}{\sqrt{2}}Q_+(1+JC_1)x_1+\frac{1}{\sqrt{2}}Q_+x_+ + \frac{i}{\sqrt{2}}Q_-(1+JC_1)x_1+ + \frac{i}{\sqrt{2}}Q_-x_+ ,JP_o(1+JC_1)x_1+JP_ox_+ + \frac{1}{\sqrt{2}}Q_+(1+JC_1)x_1+\frac{1}{\sqrt{2}}Q_+x_+ - \frac{i}{\sqrt{2}}Q_-(1+JC_1)x_1 - \frac{i}{\sqrt{2}}Q_-x_+) = (x_1+\frac{1}{\sqrt{2}}x_+ , C_1x_1+\frac{1}{\sqrt{2}}x_+) = (x_1+\frac{1}{\sqrt{2}}x_+ ,\tilde{C}_1(x_1+\frac{1}{\sqrt{2}}x_+))\in$$
$$\in\Gamma(\tilde{C}_1) , \qquad \forall\, x\in\mathcal{K} .$$

Conversely,it is easy to see that any $\tilde{x}\in\Gamma(\tilde{C}_1)$ is the image of an $x\in\mathcal{K}$;hence, $U(\mathcal{K}) = \Gamma(\tilde{C}_1)$.

b)For any $x\in\mathcal{K}$,by taking into account the complex vector space structure of $(\mathcal{H}_1\oplus\mathcal{H}_+)\oplus(\mathcal{H}_1\oplus\mathcal{H}_+)$,introduced in section 1,we get that

$$U(ix) = iUx = i(x_1+\frac{1}{\sqrt{2}}x_+ ,C_1x_1 + \frac{1}{\sqrt{2}}x_+) =$$
$$= (ix_1-\frac{1}{\sqrt{2}}x_+ , -iC_1x_1+\frac{1}{\sqrt{2}}x_+) =$$
$$= (x_1'+x_+' , -C_1x_1'-x_+')\in\Gamma(-\tilde{C}_1), \qquad \forall\, x\in\mathcal{K} .$$

As above,we infer that $U(i\mathcal{K}) = \Gamma(-\tilde{C}_1)$.

5.As in the paper of Rieffel and van Daele,one can consider the operators $S:\mathcal{D}(S)\to\mathcal{H}$ and $F:\mathcal{D}(F)\to\mathcal{H}$,given by

(1) $\mathcal{D}(S) = \mathcal{K}+i\mathcal{K}$, $S(x+iy) = x-iy$; $x,y\in\mathcal{K}$,

and

(2) $\mathcal{D}(F) = \mathcal{K}^{\perp}+i\mathcal{K}^{\perp}$, $F(x+iy) = -x+iy$; $x,x\in\mathcal{K}^{\perp}$.

(it is clear that $(i\mathcal{K})^{\perp} = i\mathcal{K}^{\perp}$,and that the pair $(i\mathcal{K}^{\perp},\mathcal{K}^{\perp})$ also satisfies the non-degeneracy conditions a) and b)).As remarked in [3] it is obvious that S and F are closed,densely defined operators;of course,they are also antilinear due to the fact that here we work wit the particular case of a complex Hilbert space.

It is easy to see that the real projections,corresponding to the pair $(i\mathcal{K}^{\perp},\mathcal{K}^{\perp})$,are given by

$$P' = 1 - Q , \quad Q' = 1 - P.$$

(P' is the real projection operator onto $i\mathcal{K}^{\perp}$, whereas Q' is the real projection operator onto $\mathcal{K}^{\perp}$). The corresponding operators R,T,J are then given by the formulae

$$R' = 2 - R,$$

$$J' = J\ ,\ T' = T\ ,$$

whereas $\Delta' = \Delta^{-1}$. It is easy to see that the corresponding closed complex vector subspaces of $\mathcal{H}$ are $\mathcal{H}_1' = \mathcal{H}_o$, $\mathcal{H}_o' = \mathcal{H}_1$, $\mathcal{H}_+' = \mathcal{H}_+$, $\mathcal{H}_-' = \mathcal{H}_-$, and, therefore, we have

$$P_1' = P_o\ ,\ P_o' = P_1\ ,\ Q_+' = Q_+\ ,\ Q_-' = Q_-\ ,$$

with an obvious notation.

6. Let us now remark that we have the inclusion $\mathcal{H}_1 \subset \mathcal{D}(\Delta)$ and that $\Delta_1 = \Delta|\mathcal{H}_1$ is an injective positive operator in $\mathcal{L}(\mathcal{H}_1)$, such that $o \leq \Delta_1 \leq 1$. This remark, together with the preceding calculations, will enable us to give a better presentation for the K.M.S. phenomenon.

Remark. It is obvious that $1-\Delta_1$ is also injective in $\mathcal{H}_1$.

Indeed, since Δ is injective and positive, we can define the self-adjoint densely defined operator $h = \ln \Delta$. Of course, we have that $\Delta = e^h$, and we can also consider the normal densely defined operator

$$\Delta^{iz} = e^{izh}\ , \qquad z \in \mathbb{C}\ .$$

By taking into account formulae (4) and (5) from section 5, we infer that, by defining $W_z = U\Delta^{iz}U^{-1}$, for $z = u+iv \in \mathbb{C}$, we have

$$W_z(x_1'+x_+', x_1''+x_+'') = (\Delta_1^{iu-v}x_1'+x_+', \Delta_1^{iu+v}x_1''+x_+'')\ ,$$

for any $x_1', x_1'' \in \mathcal{H}_1$, $x_+', x_+'' \in \mathcal{H}_+$, if $x_1' \in \mathcal{D}(\Delta_1^{-v})$, $x_1'' \in \mathcal{D}(\Delta_1^{v})$. With formula (2) from section 4, we infer that we have

$$(W_z(x_1'+x_+', x_1''+x_+'') \mid (y_1'+y_+', y_1''+y_+'')) =$$

$$= (\Delta_1^{iu-v}x_1' \mid y_1') + (y_1'' \mid \Delta_1^{iu+v}x_1'') + (x_+' \mid y_+') + (x_+'' \mid y_+'') - i(x_+' \mid y_+'') + i(x_+'' \mid y_+'),$$

for any $x_1' \in \mathcal{D}(\Delta_1^{-v})$, $x_1'' \in \mathcal{D}(\Delta_1^{v})$, $y_1', y_1'' \in \mathcal{H}_1$, $x_+', x_+'', y_+', y_+'' \in \mathcal{H}_+$.

If we now assume that $(x_1'+x_+', x_1''+x_+''), (y_1'+y_+', y_1''+y_+'') \in \Gamma(\tilde{C}_1)$, and $-(1/2) \leq v \leq o$, then we get that

$$(W_z(x_1'+x_+', x_1''+x_+'') \mid (y_1'+y_+', y_1''+y_+'')) =$$
$$= (\Delta_1^{iz}x_1' \mid y_1') + (\Delta_1^{-iz+1}y_1' \mid x_1') + 2(x_+' \mid y_+'),$$

and this formula immediately yields another proof of Proposition 3.7 from [3].

7. In this section we shall prove that for any one-parameter strongly continuous unitary group $\mathbb{R}\ni t\mapsto u_t\in\mathcal{L}(\mathcal{H})$, on the complex Hilbert space $\mathcal{H}$, and any conjugation $J_o:\mathcal{H}\to\mathcal{H}$, such that

(1) $$u_tJ_o = J_ou_t \quad , \quad t\in\mathbb{R} \; ,$$

there exists a closed real Hilbert subspace $\mathcal{K}_o$, such that

a) $\mathcal{K}_o \cap (i\mathcal{K}_o) = \{o\}$ and b) $(\mathcal{K}_o + i\mathcal{K}_o)^{\perp} = \{o\}$,

and such that $(u_t)_{t\in\mathbb{R}}$ be the modular group $(\Delta^{it})_{t\in\mathbb{R}}$, corresponding to $\mathcal{K}_o$, whereas J_o be the corresponding conjugation. By the Remark following Lemma 5, the subspace $\mathcal{K}_o$ is uniquely determined by the pair $((u_t)_{t\in\mathbb{R}}, J_o)$.

Indeed, by the Stone Representation Theorem, there exists a self-adjoint densely defined operator $h:\mathcal{D}(h)\to\mathcal{H}$, such that

$$u_t = e^{ith} \; , \quad t\in\mathbb{R}, (\mathcal{D}(h)\subset\mathcal{H}).$$

From the commutation condition (1) we immediately infer that

(2) $$J_ohJ_o = -h;$$

if we denote by E_h the spectral scale of h, from (2) we obtain that

$$J_oE_h(\lambda)J_o = 1 - E_h(-\lambda - o) \; , \qquad \lambda\in\mathbb{R} \; .$$

Let $P_1 = E_h(-o)$, $P_o = J_oP_1J_o = 1 - E_h(o)$, and $Q_o = 1 - P_o - P_1$. We can now apply the constructions described in section 1. We, moreover, remark that we have

$$\mathcal{H}_+ \oplus \mathcal{H}_- \subset \mathcal{D}(h)$$

and

$$(x\in\mathcal{D}(h) \text{ and } h(x) = o) \Longleftrightarrow x\in\mathcal{H}_+\oplus\mathcal{H}_- \; .$$

Since the mapping $U_o:\mathcal{H}\to(\mathcal{H}_1\oplus\mathcal{H}_+)\oplus(\mathcal{H}_1\oplus\mathcal{H}_+)$ is given by

(3) $$U_ox = (P_1x + \frac{1}{\sqrt{2}}Q_+x + \frac{i}{\sqrt{2}}Q_-x, J_oP_ox + \frac{1}{\sqrt{2}}Q_+x - \frac{i}{\sqrt{2}}Q_-x),$$

the inverse mapping U_o^{-1} is given by the formula

$$U_o^{-1}(x_1'+x_+',x_1''+x_+'') = x_1'+J_o x_1''+\frac{1}{\sqrt{2}}(x_+'+x_+'') - \frac{i}{\sqrt{2}}(x_+'-x_+'').$$

Of course, $S_o = U_o J_o U_o^{-1}$ is the natural symmetry, given by

$$(4) \qquad S_o(x_1'+x_+',x_1''+x_+'') = (x_1''+x_+'',x_1'+x_+'),$$

for any $x_1',x_1''\in\mathcal{H}_1$, $x_+',x_+''\in\mathcal{H}_+$.

Let us now remark that we have

$$o\leq P_1 e^h\leq P_1 ;$$

whereas $P_1 e^h$ and $P_1-P_1 e^h$ are injective in $\mathcal{H}_1$.

We shall define a bounded $\mathbb{R}$-linear operator

$$\tilde{D}_1 \quad \mathcal{H}_1\oplus\mathcal{H}_+\longrightarrow\mathcal{H}_1\oplus\mathcal{H}_+$$

by the formula

$$\tilde{D}_1(x_1+x_+) = e^{(1/2)h}x_1+x_+ , \qquad x_1\in\mathcal{H}_1 , \ x_+\in\mathcal{H}_+.$$

Of course, we have that $o\leq\tilde{D}_1\leq 1$ in $\mathcal{H}_1\oplus\mathcal{H}_+$, and the graph $\Gamma(\tilde{D}_1)$ of $\tilde{D}_1$ is a closed real subspace of $(\mathcal{H}_1\oplus\mathcal{H}_+)\oplus(\mathcal{H}_1\oplus\mathcal{H}_+)$, which we shall denote by $\mathcal{K}_o$. By taking into account the definition of the complex vector space structure of $(\mathcal{H}_1\oplus\mathcal{H}_+)\oplus(\mathcal{H}_1\oplus\mathcal{H}_+)$; we can easily prove that $i\mathcal{K}_o = \Gamma(-\tilde{D}_1)$.

We shall prove that the pair $(\mathcal{K}_o, i\mathcal{K}_o)$ satisfies the non-degeneracy conditions

a') $$\mathcal{K}_o\cap(i\mathcal{K}_o) = \{o\};$$

and

b') $$(\mathcal{K}_o+i\mathcal{K}_o)^{\perp} = \{o\}.$$

Indeed, for $(x_1+x_+,e^{(1/2)h}x_1+x_+)\in i\mathcal{K}_o$, we infer that there exists an $x_1'\in\mathcal{H}_1$, and an $x_+'\in\mathcal{H}_+$, such that

$$(x_1+x_+,e^{(1/2)h}x_1+x_+) = (y_1+y_+,-e^{(1/2)h}y_1-y_+),$$

whence we infer that $x_+ = o$ and $e^{(1/2)h}x_1 = o$. It follows that $x_1 = o$, and so relation a') is proved.

In order to prove relation b'), we shall prove that $\mathcal{K}_o+i\mathcal{K}_o$ is dense in $(\mathcal{H}_1\oplus\mathcal{H}_+)\oplus(\mathcal{H}_1\oplus\mathcal{H}_+)$, by showing that the <u>complex</u> orthogonal complement of $\mathcal{K}_o+i\mathcal{K}_o$ in $(\mathcal{H}_1\oplus\mathcal{H}_+)\oplus(\mathcal{H}_1\oplus\mathcal{H}_+)$ is equal to $\{o\}$. Indeed, this is an immediate consequence of the equalities

$$\mathcal{K}_o = \Gamma(\tilde{D}_1) \quad , \quad i\mathcal{K}_o = \Gamma(-\tilde{D}_1),$$

and of the fact that $e^{(1/2)h}$ is injective in $\mathcal{H}_1$.

If we denote $\mathcal{K} = U_o^{-1}(\mathcal{K}_o)$, then $\mathcal{K}$ is a closed real vector subspac of $\mathcal{H}$, satisfying the non-degeneracy conditions a) and b) in $\mathcal{H}$.

We shall prove that $J = J_o$ and $u_t = \Delta^{it}$, $t \in \mathbb{R}$, where J and Δ are, respectively, the conjugation and the modular operator associated the pair $(\mathcal{K}, i\mathcal{K})$, as in [3].

Indeed, let P_o' be the real projection onto $\mathcal{K}_o = \Gamma(\tilde{D}_o)$. From

$$P_o'(x_1'+x_+' , x_1''+x_+'') = (y_1'+y_+', e^{(1/2)h}y_1'+y_+')$$

and

$$\langle (x_1'-y_1'+x_+'-y_+', x_1''-e^{(1/2)h}y_1'+x_+''-y_+') \mid (z_1'+z_+', e^{(1/2)h}z_1'+z_+') \rangle = 0,$$

$$\forall\, z_1' \in \mathcal{H}_1 \ , \ z_+' \in \mathcal{H}_+ ,$$

it is easy to infer that

$$y_1' = (1+e^h)^{-1}(x_1'+e^{(1/2)h}x_1''),$$

and

$$y_+' = (1/2)(x_+'+x_+'').$$

If we denote by Q_o' the <u>real</u> projection onto $\Gamma(-\tilde{D}_1)$, from

$$Q_o'(x_1'+x_+', x_1''+x_+'') = (y_1''+y_+'', -e^{(1/2)h}y_1''-y_+'')$$

and

$$\langle (x_1'-y_1''+x_+'-y_+'', x_1''+e^{(1/2)h}y_1''+x_+''+y_+'') \mid (z_1'+z_+', -e^{(1/2)h}z_1'-z_+') \rangle = 0,$$

$$\forall\, z_1' \in \mathcal{H}_1 \ , \ z_+' \in \mathcal{H}_+ ,$$

we easily infer that

$$y_1'' = (1+e^h)^{-1}(x_1'-e^{(1/2)h}x_1''),$$

and

$$y_+'' = (1/2)(x_+'-x_+'').$$

If we denote $R_o' = P_o' + Q_o'$, we obtain that

(4) $R_o'(x_1'+x_+', x_1''+x_+'') = (2(1+e^h)^{-1}x_1'+x_+', 2e^h(1+e^h)^{-1}x_1''+x_+''),$

for any $x_1', x_1'' \in \mathcal{H}_1$, $x_+', x_+'' \in \mathcal{H}_+$; similarly, we obtain that

(5) $(P_o'-Q_o')(x_1'+x_+', x_1''+x_+'') =$
$= (2(1+e^h)^{-1}e^{(1/2)h}x_1''+x_+'', 2(1+e^h)^{-1}e^{(1/2)h}x_1'+x_+').$

For $T_o' = R_o'^{1/2}(2-R_o')^{1/2}$ we get that

(6) $T_o'(x_1'+x_+', x_1''+x_+'') =$
$= (2(1+e^h)^{-1}e^{(1/2)h}x_1'+x_+', 2(1+e^h)^{-1}e^{(1/2)h}x_1''+x_+'').$

From formulae (3),(5) and (6) we immediately infer that

(7) $$P_o' - Q_o' = S_o'T_o'$$

is the polar decomposition of $P_o'-Q_o'$. If we denote $\Delta_o' = (2-R_o')(R_o')^{-1}$, then from formula (4) we get that

$$(R_o')^{-1}(x_1'+x_+', x_1''+x_+'') = (\tfrac{1}{2}(1+e^h)x_1'+x_+', \tfrac{1}{2}e^{-h}(1+e^h)x_1''+x_+''),$$

and this immediately implies that

(8) $\Delta_o'(x_1'+x_+', x_1''+x_+'') = (e^h x_1'+x_+', e^{-h}x_1''+x_+'')$;

of course, $\mathcal{D}(\Delta_o') = \{(x_1'+x_+', x_1''+x_+'')\ ;\ x_1'' \in \mathcal{R}(e^h) \cap \mathcal{H}_1\}$.

From relations (7) and (8) it is now easy to infer that $J_o = J$, where $J = U_o^{-1}S_oU_o$ is the conjugation in $\mathcal{H}$, corresponding to $\mathcal{K}$, whereas $\Delta = U_o^{-1}\Delta_o'U_o$ is the modular operator corresponding to the same subspace. From (8) we now immediately infer that

$$u_t = \Delta^{it}, \quad t \in \mathbb{R};$$

i.e., $(u_t)_{t\in\mathbb{R}}$ is the modular group which corresponds to the pair $(\mathcal{K}, i\mathcal{K})$, according to the construction given in [3].

In this manner, we have proved the following

THEOREM 2. *Let $\mathcal{H}$ be any complex Hilbert space and let $((u_t)_{t\in\mathbb{R}}, J)$ be any pair consisting of a strongly continuous one-parameter unitary group, and a conjugation J, in $\mathcal{H}$, such that*

$$u_tJ = Ju_t, \qquad t \in \mathbb{R}.$$

Then there exists a unique closed real Hilbert subspace $\mathcal{K} \subset \mathcal{H}$, satisfying the non-degeneracy conditions a) and b), such that $(u_t)_{t\in\mathbb{R}}$ be the modular group, and J be the conjugation in $\mathcal{H}$, corresponding to $\mathcal{K}$ by the Rieffel-van Daele construction.

Remark. Let $M \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra, having a cyclic separating vector $x_o \in \mathcal{H}$, $\|x_o\| = 1$. Let $M_{sa} \subset M$ be the real vector subspace of the self-adjoint elements in M and define $\mathcal{K} = \overline{M_{sa}x_o}$ .Then $\mathcal{K} \subset \mathcal{H}$ satisfies the non-degeneracy conditions a) and b) (see [3] §4), but from $\Delta x_o = x_o$ and $Jx_o = x_o$, we infer that $\mathcal{H}_+ \neq \{0\}$ .This fact shows that, in this case, Halmos' Theorem can never be applied; i.e. the pair $(\mathcal{K}, i\mathcal{K})$ is not in "general position".

8. Let now $J : \mathcal{H} \to \mathcal{H}$ be a conjugation and $\Delta : \mathcal{D}(\Delta) \to \mathcal{H}$ be a positive injective self-adjoint densely defined operator in $\mathcal{H}$, such that

$$J\Delta J = \Delta^{-1} .$$

For the unitary one-parameter group

$$u_t = \Delta^{it} = e^{it\ln\Delta} , \qquad t \in \mathbb{R},$$

we then have that $Ju_t = u_tJ$, $t \in \mathbb{R}$, and, therefore, according to Theorem 2, there exists a uniquely determined closed real vector subspace $\mathcal{K} \subset \mathcal{H}$ satisfying the non-degeneracy conditions a) and b), such that J be the conjugation, and $(u_t)_{t\in\mathbb{R}}$ be the unitary one-parameter group, corresponding to $\mathcal{K}$ by the Rieffel-van Daele construction. It is then obvious that Δ is the modular operator corresponding to $\mathcal{K}$ by the same construction. Hence, we have the following

<u>THEOREM 3</u>. <u>For any conjugation</u> $J : \mathcal{H} \to \mathcal{H}$, <u>and any positive, injective self-adjoint, densely defined linear operator</u> $\Delta : \mathcal{D}(\Delta) \to \mathcal{H}$, <u>such that</u>

$$J\Delta J = \Delta^{-1} ,$$

<u>there exists a uniquely determined closed real vector subspace</u> $\mathcal{K} \subset \mathcal{H}$ <u>satisfying the non-degeneracy conditions</u> a) <u>and</u> b), <u>such that</u> J <u>be the conjugation and</u> Δ <u>be the modular operator corresponding to</u> $\mathcal{K}$ <u>by the Rieffel-van Daele construction</u>.

<u>Remark</u>. One could consider the problem to characterize the closed real vector subspaces $\mathcal{K} \subset \mathcal{H}$, which arise from a standard von Neumann algebra $M \subset \mathcal{L}(\mathcal{H})$, by the formula $\mathcal{K} = \overline{M_{sa}x_o}$, where $x_o \in \mathcal{H}$ is a separating cyclic vector for M.

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