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INTRODUCTION. The aim of this paper is to construct an iterative multigrid process for variational nonsymmetric problems defined by continuous, elliptic type, nonsymmetric bilinear functionals, such that the bounds obtained for the rate of convergence be independent of the approximation subspaces- i.e. of discretization parameters - and depending only of the continuity and ellipticity constants. Using an natural framework for the discrete problems such that the Nicolaides variational relations (1.5 and 1.17) are fullfiled and using a Hackbusch type decomposition for the two-grid iteration operator, constructed with an adequate relaxation process, we obtain bounds of same type as in multigrid literature.

§1. Approximation on subspaces. Let $\mathcal{H}$ be a real separable Hilbert space with $\langle\cdot,\cdot\rangle$ inner product and $\|\cdot\|$ the coresponding norm. We consider on $\mathcal{H}$ the following problem:

$(\mathcal{P})$ Given f a bounded linear functional on $\mathcal{H}$, find $u\in\mathcal{H}$ such that $a(u,v)=f(v)$ for every $v\in\mathcal{H}$,

where a is a nonsymmetric, bilinear functional on $\mathcal{H}$, that satisfies the following properties: there exist the constants α and β such that:

(1.1) $|a(w,v)| \leq \beta \|w\| \cdot \|v\|$

(1.2) $|a(w,w)| \geq \alpha \|w\|^2$

for every $w, v \in \mathcal{H}$.

By the well knowed Lax-Milgram lemma, this problem has a unique solution $u \in \mathcal{H}$. Now, let $\mathcal{A}$ be the linear operator on $\mathcal{H}$, bounded and nonsingular, defined by Riesz representation, unique determined by:

(1.3) $a(w,v) = \langle \mathcal{A}w, v \rangle$

for every $w, v \in \mathcal{H}$.

If $\mathcal{H}_k$ is an finite dimension subspace of $\mathcal{H}$, then by $\mathcal{P}_k$ we denote the orthogonal projection operator corresponding it. The restriction of the bilinear functional a at $\mathcal{H}_k$ defines on $\mathcal{H}_k$ an linear operator $\mathcal{A}_k$ by

$$a(w_k, v_k) = \langle \mathcal{A} w_k, v_k \rangle = \langle \mathcal{A}\mathcal{P}_k w_k, \mathcal{P}_k v_k \rangle = \langle (\mathcal{P}_k \mathcal{A} \mathcal{P}_k) w_k, v_k \rangle$$

for every $w_k, v_k \in \mathcal{H}_k$, hence

(1.4) $\mathcal{A}_k = \mathcal{P}_k \mathcal{A} \mathcal{P}_k$

As linear operator on $\mathcal{H}_k$, $\mathcal{A}_k$ is bounded and nonsingular because (1.1) and (1.2) hold on $\mathcal{H}_k$.

Let $\mathbb{R}_k$ be the finite dimension real space, $\dim \mathbb{R}_k = \dim \mathcal{H}_k = n_k$, equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle_k$ the corresponding norm $\| \cdot \|_k$. If $J_k : \mathcal{H}_k \to R_k$ is an bijection operator, then we define $J^k : R_k \to \mathcal{H}_k$ by

(1.5) $J^k = J_k^*$, i.e,

(1.5') $\langle J_k w_k, \bar{v}_k \rangle_k = \langle w_k, J^k \bar{v}_k \rangle$, for every $w_k \in \mathcal{H}_k$, $\bar{v}_k \in R_k$.

On R_k we define now, the linear operator corresponding to $\mathcal{A}_k$ by

(1.6) $A_k = J_k \mathcal{A}_k J^k$,

notting that for every $\bar{w}_k, \bar{v}_k \in R_k$:

(1.7) $\langle A_k \bar{w}_k, \bar{v}_k \rangle_k = \langle \mathcal{A}_k (J^k \bar{w}_k), J^k \bar{v}_k \rangle$.

With this, we can construct the discrete problems corresponding to $\mathcal{H}_k$ and $\mathcal{R}_k$ subspaces, of the problem $(\mathcal{P})$.

Let $f^* \in \mathcal{H}$ be the Riesz representation of the bounded linear functional f:

(1.8) $f(v) = \langle f^*, v \rangle$, for every $v \in \mathcal{H}$,

and let f_k^* be the projection of f^*: $f_k^* = P_k f^*$, on $\mathcal{H}_k$.

The discrete problem on $\mathcal{H}_k$, corresponding of the problem $(\mathcal{P})$ is:

$(\mathcal{P}_k)$ To find $u_k \in \mathcal{H}_k$ such that
$a(u_k, v_k) = f(v_k)$ for every $v_k \in \mathcal{H}_k$,

and the two following problems are equivalent with $(\mathcal{P}_k)$:

$(\mathcal{P}_k')$ To find $u_k \in \mathcal{H}_k$ such that
$\langle \mathcal{A}_k u_k, v_k \rangle = \langle f^*, v_k \rangle$ for every $v_k \in \mathcal{H}_k$,

(1.9) $\langle \mathcal{A}_k u_k, v_k \rangle = \langle f_k^*, v_k \rangle$ for every $v_k \in \mathcal{H}_k$.

On R_k we define the discrete problem corresponding at $\mathcal{P}_k$:

(1.10) Given $\bar{f}_k^* \in R_k$, to find $\bar{u}_k \in R_k$ such that
$A_k \bar{u}_k = \bar{f}_k^*$, where $\bar{f}_k^* = J_k f_k^*$

1.1. Remark: If $\bar{u}_k \in R_k$ is solution of (1.10) then $J^k \bar{u}_k$ is solution of $(\mathcal{P}_k)$.

<u>Proof</u>. For every $\bar{v}_k \in R_k$, we have the following equivalent relations:

$$\langle A_k \bar{u}_k - \bar{f}_k^*, \bar{v}_k \rangle_k = 0$$
$$\langle J_k \mathcal{A}_k J^k \bar{u}_k, \bar{v}_k \rangle_k = \langle \bar{f}_k^*, \bar{v}_k \rangle_k \text{ , in } R_k \text{, and}$$
$$\langle \mathcal{A}_k (J^k \bar{u}_k), J^k \bar{v}_k \rangle = \langle f_k^*, J^k v_k \rangle \text{ , in } \mathcal{H}_k.$$

Because J^k is an bijection operator, the last relation is equivalent with $(\mathcal{P}_k^{\prime})$.

Let the following increasing sequence of finite dimension approximation subspaces of $\mathcal{H}$:

$$\mathcal{H}_0 \subset \mathcal{H}_1 \subset \dots \subset \mathcal{H}_{k-1} \subset \mathcal{H}_k \subset \dots \subset \mathcal{H}$$

By the level or grid k, we understand one or both of the subspaces $\mathcal{H}_k$ and R_k and we point out this dependence by the dependence of the k. For the ulterior estimations we need the following equivalence relations of the $\mathcal{H}_k$ and R_k - norms:

(1.11) $$C_1 \| J^k \bar{v}_k \| \leq \gamma_k \| \bar{v}_k \|_k \leq C_2 \| J^k \bar{v}_k \|,$$

for every $\bar{v}_k \in R_k$, where C_1, C_2 not depend of k.

1.2. <u>Remark</u>. Choosing convenable the linear operator J^k we obtain a such equivalence relation as (1.11). For example, let $\{e_k^j \mid j=1,n_k\}$ be the natural basis of R_k and $\mathcal{H}_k = \text{span}\{\phi_k^j \; j=1,n_k\}$, where the above family is orthonormal in $\mathcal{H}$. Then, defining J^k by: $J^k e_k^j = \phi_k^j$, $j=1,n_k$, we obtain (1.11), by $\|\bar{v}_k\|_k = \|J^k \bar{v}_k\|$, for every $\bar{v}_k \in R_k$. Hence, always holds an relation of the type (1.11), proved by this particular form.

1.1. <u>Lemma</u>. If (1.11) holds, then

(1.12) $\|A_k\|_s \leq \dfrac{\beta}{C_1^2}\gamma_k^2$

(1.13) $\|A_k^{-1}\|_s \leq \dfrac{C_2^2}{\alpha}\gamma_k^{-2}$; where $\|\cdot\|_s$-denotes the spectral norm.

Proof. Because for every $\bar{w}_k, \bar{v}_k \in R_k$ we have (1.4), (1.5), (1.6),

$$a(J^k\bar{v}_k, J^k\bar{w}_k) = \langle A_k\bar{v}_k, \bar{w}_k\rangle_k,$$

we obtain for the first estimation:

$$|\langle A_k\bar{v}_k, \bar{w}_k\rangle_k| = |a(J^k v_k, J^k w_k)| \leq \beta \|J^k\bar{v}_k\| \cdot \|J^k\bar{w}_k\| \leq \beta \frac{\gamma_k^2}{C_1^2} \|\bar{v}_k\|_k \cdot \|\bar{w}_k\|_k$$

For the second estimation we observe that:

$$\frac{\alpha\gamma_k^2}{C_2^2}\|A_k^{-1}\bar{v}_k\|_k^2 \leq \alpha\|J^k A_k^{-1}\bar{v}_k\|^2 \leq |a(J^k A_k^{-1}\bar{v}_k, J^k A_k^{-1}\bar{v}_k)|$$
$$\leq |\langle \bar{v}_k, A_k^{-1}\bar{v}_k\rangle_k| \leq \|\bar{v}_k\|_k \cdot \|A_k^{-1}\bar{v}_k\|_k$$

This estimations proves (1.12) and (1.13).

Now, for every $k \geq 1$ we define the diagram $(\mathcal{D}_k)$:

(1.14)
$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{P_k} & \mathcal{H}_k & \xrightarrow{J_k} & R_k \\ id \updownarrow & & & & \downarrow I_k^{k-1} \\ \mathcal{H} & \xrightarrow[P_{k-1}]{} & \mathcal{H}_{k-1} & \xrightarrow[J_{k-1}]{} & R_{k-1} \end{array}$$

where: P_k, P_{k-1} are the projections operators on $\mathcal{H}_k$ and $\mathcal{H}_{k-1}$; J_k, J_{k-1} are bijection operators and I_k^{k-1} is the transpose of an injection operator $I_{k-1}^{k}: R_{k-1} \to R_k$, i.e.:

(1.15) $\langle I_k^{k-1}\bar{v}_k, \bar{w}_{k-1}\rangle_{k-1} = \langle \bar{v}_k, I_{k-1}^{k}\bar{w}_{k-1}\rangle_k$,

for every $\bar{v}_k \in R_k$, $\bar{w}_{k-1} \in R_{k-1}$. By "id" we denote the identity on $\mathcal{H}$.

1.1. Theorem . If the diagram $(\mathcal{D}_k)$ is commutative, and (1.11) holds, then:

(1.16) $J_{k-1}P_{k-1} = I_k^{k-1}J_kP_k$

(1.17) $A_{k-1} = I_k^{k-1}A_kI_{k-1}^{k}$

(1.18) $\| I_{k-1}^{k}D_{k-1}I_k^{k-1} \|_s \leq C_3 (\frac{\delta_{k-1}}{\delta_k})^2 \cdot \| D_{k-1} \|_s$; $C_3 = C_2^2/C_1^2$

for every linear operator D_{k-1} on R_{k-1}.

Proof: The first relation is equivalent with the comutativity of the diagram. Using (1.4), (1.6), (1.16), we obtain:

$$A_{k-1} = J_{k-1}\mathcal{A}_{k-1}J^{k-1} = J_{k-1}P_{k-1}\mathcal{A}P_{k-1}J^{k-1} = I_k^{k-1}J_kP_k\mathcal{A}P_kJ^kI_{k-1}^{k} =$$
$$= I_k^{k-1}A_kI_{k-1}^{k}.$$

Now, by (1.11), we have:

$$\| I_{k-1}^{k}D_{k-1}I_k^{k-1} \|_s \leq (\frac{C_2}{\delta_k})^2 \| J^kI_{k-1}^{k}D_{k-1}I_k^{k-1}J_k \| = (\frac{C_2}{\delta_k})^2 \| P_kJ^kI_{k-1}^{k}D_{k-1}I_k^{k-1}J_kP_k \|$$
$$= (\frac{C_2}{\delta_k})^2 \| P_{k-1}J^{k-1}D_{k-1}J_{k-1}P_{k-1} \| = (\frac{C_2}{\delta_k})^2 \| J^{k-1}D_{k-1}J_{k-1} \|$$
$$\frac{C_2^2}{C_1^2} (\frac{\delta_{k-1}}{\delta_k})^2 \| D_{k-1} \|_s .$$

Particularly, we obtain by the theorme 1.1, that

$$\| I^{k}_{k-1} I^{k-1}_{k} \|_{s} \leq C_3 \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 .$$

1.3. Remark: If we restrict the diagram $(\mathcal{D}_k)$ at $\mathcal{H}_k$, then the commutativity is equivalent with:

$$(1.16') \qquad J_{k-1}P_{k-1}=I^{k-1}_{k}J_{k}$$

and this relation holds iff holds

$$(1.16'') \qquad P_{k-1}J^{k-1}=J^{k}I^{k}_{k-1}$$

For to prove the equivalence relation (1.16') with (1.16"), let $\bar{v}_{k-1}\in R_{k-1}$, firstly; then for every $u_k\in \mathcal{H}_k$, by (1.5'), (1.15), (1.16'):

$$\langle P_{k-1}J^{k-1}\bar{v}_{k-1},u_k\rangle=\langle \bar{v}_{k-1},J_{k-1}P_{k-1}u_k\rangle_{k-1}=\langle \bar{v}_{k-1},I^{k-1}_{k}J_k u_k\rangle_{k-1}$$
$$=\langle J^{k}I^{k}_{k-1}\bar{v}_{k-1},u_k\rangle$$

what proves that (1.16') implies (1.16"). Conversselly, let $u_k\in\mathcal{H}_k$; then for every $\bar{v}_{k-1}\in R_{k-1}$, we have:

$$\langle J_{k-1}P_{k-1}u_k,\bar{v}_{k-1}\rangle_{k-1}=\langle u_k,P_{k-1}J^{k-1}\bar{v}_{k-1}\rangle=\langle u_k,J^{k}I^{k}_{k-1}\bar{v}_{k-1}\rangle$$
$$=\langle I^{k-1}_{k}J_k u_k,\bar{v}_{k-1}\rangle_{k-1}$$

1.4. Remark. The Galerkin method conducts to a commutative diagram.

Proof. Let $\mathcal{H}_\ell=\mathrm{span}\{\phi^j_\ell\ j=1,n_\ell\}$, $\ell=k-1;k$, where both families are linear independent in $\mathcal{H}$. Because $\mathcal{H}_{k-1}\subset\mathcal{H}_k$, there exists the matrix $T=[t_{ij}]$, $i=1, n_{k-1}$; $j=1,n_k$, unique determined by

$\phi^{i}_{k-1}=\sum_{j=1}^{n_k} t_{ij}\phi^{j}_{k}$. Now, the "stifness" Galerkin matrix $\bar{A}_{k-1}$, $\bar{A}_k$ are defined by the entries $a^{\ell}_{ij}=\langle A\phi^{j}_{\ell},\phi^{i}_{\ell}\rangle$, $i,j=1,n_\ell$, $\ell=k-1,k$. We define

$$I^{k}_{k-1}:R_{k-1}\to R_k, \text{ by } I^{k}_{k-1}e^{i}_{k-1}=\sum_{j=1}^{n_k} t_{ij}e^{j}_{k};$$

$$J^{\ell}:R_\ell \to \mathcal{H}_\ell, \text{ by } J^{\ell}e^{i}_{\ell}=\phi^{i}_{\ell}, \ i=1,n_\ell;\ \ell=k-1,k.$$

Then, is a simple verification that for every $\bar{v}_{k-1}\in R_{k-1}$, we have $J^{k-1}\bar{v}_{k-1}=J^{k}I^{k}_{k-1}\bar{v}_{k-1}$, what proves that (1.16") is verified, and, by 1.3. Remark, the diagram $(\mathcal{D}_k)$ is commutative. Moreover, between the Galerkin matrix there exist the following relation $\bar{A}_{k-1}=T\bar{A}_kT^{*}$.

§2. Two-grid convergence. For to solve numerically the equation (1.10) by multigrid method, such that the bound of the rate of convergence be independent of approximation subspaces, we suppose that are satisfied the following properties, for $k\geq 1$:

I) the diagram (1.14) is commutative

II) the norms equivalence relation (1.11) holds, where C_1 and C_2 are independent of k.

Let k be fixed. By stationary iterative process for the equation (1.10) we understand the iterations sequence:

(2.1) $\bar{u}^{j+1}_{k}=G_k\,\bar{u}^{j}_{k}+D_k\bar{f}^{*}_{k}$; $j=1,2,\ldots$, where

(2.2) $G_k=I_k-D_kA_k$;

I_k is the indentity on R_k, and D_k is an linear operator on R_k. The spectral radius of the iteration operator G_k, $\rho(G_k)$ gives the rate of convergence if $\rho(G_k)<1$. If D_k is constructed only by help of the k level, then we name a such process as relaxation process on k level.

The two-grid iterative process is defined as follows:

in the (j+1) step we determine the (j+1) iteration $\bar{u}_k^{j+1}$ by $\bar{u}_k^j$ by:

i) ν-sweeps of relaxation on k level are efectuated:

$$\bar{u}_k^{j,i+1}=G_k\bar{u}_k^{j,i} + D_k\bar{f}_k^* , \; k=0,\dots,\nu-1; \quad \bar{u}_k^{j,0}=\bar{u}_k^j;$$

Let $\bar{u}_k^{j,\nu}$ be after relaxation;

ii) The k-level defect, $\bar{d}_k=\bar{f}_k^*-A_k\bar{u}_k^{j,\nu}$, is representated on coarse level k-1: $\bar{d}_{k-1}=I_k^{k-1}\bar{d}_k$

iii) Exact solution on k-1 level is efectuated: $A_{k-1}\bar{v}_{k-1}=\bar{d}_{k-1}$

iv) the coarse correction $\bar{v}_k=I_{k-1}^k\bar{v}_{k-1}$ is added at j-iteration fór to obtain the (j+1)-iteration:

$$\bar{u}_k^{j+1}=\bar{u}_k^j + \bar{v}_k.$$

The two-grid iterative process defines an iterative stationary process on R_k with the iteration operator M_k:

(2.3) $\quad M_k=(I_k-B_kA_k)G_k^\nu$, where

(2.4) $\quad B_k=I_{k-1}^k A_{k-1}^{-1} I_k^{k-1}$

and we use the following decomposition ([3]) of it:

(2.5) $\quad M_k=(A_k^{-1}-B_k)(A_kG_k^\nu).$

2.1. <u>Lemma</u>. In the hypothesis I)- II), we have the following estimations for every $k\geq 1$:

(2.6) $\quad \|A_k^{-1} - B_k\|_s \leq C_5\gamma_k^{-2}, \quad C_5 = \frac{C_2^2}{\alpha}(1+C_3)$

(2.7) $\quad \|A_kG_k^\nu\|_s \leq \frac{1}{\omega_k}g(\nu)$

where the relaxation operator G_k has the form:

$$(2.8) \quad G_k = I_k - \omega_k^2 A_k^* A_k, \quad \omega_k = 1/\|A_k\|_s$$

and g is a monotone decreasing function of ν:

$$(2.9) \quad g(\nu) = \frac{1}{2}\sqrt{3/(4\nu+1)}.$$

Proof: From (1.13), (1.18), we obtain:

$$\|A_k^{-1} - B_k\|_s \leq \|A_k^{-1}\|_s + \|I_{k-1}^k A_{k-1}^{-1} I_k^{k+1}\|_s \leq \|A_k^{-1}\|_s + C_3 \|A_k^{-1}\|_s \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2$$
$$\leq (C_3+1)\frac{c_2^2}{\alpha}\gamma_k^{-2}$$

Now, for the second estimation, we observe that $A_k^* A_k$ being an symmetric, positive linear operator, his eigenvalues $\{\lambda_j; j=1,m\}$ are positives. Then,

$$\|A_k G_k^\nu\|_s^2 = \|A_k G_k^{2\nu} A_k^*\|_s = \rho(A_k G_k^{2\nu} A_k^*) = \rho(A_k^* A_k G_k^{2\nu}) =$$
$$= \rho(A_k^* A_k (I_k - \omega_k^2 A_k^* A_k)^{2\nu}) = \max_{1 \leq j \leq n_k} [\lambda_j (1-\omega_k^2 \lambda_j)^{2\nu}]$$
$$\leq \frac{1}{\omega_k^2} \sup_{0 \leq X \leq 1} [X(1-X)^{2\nu}]$$

because $\lambda_j \omega_k^2 \leq \lambda_{max}/\|A_k\|_s^2 \leq 1$. In [3] is given the following estimation: $\sup_{0 \leq X \leq 1} [X(1-X)^s] \leq 3/[8(s+1/2)]$. Hence, with g of the form (2.1?) we obtain (2.7).

Hackbusch ([3]) names the relations (2.6) and (2.7) the approximation and smoothing properties.

2.2. Theorem of convergence. If the approximation and smoothing properties (2.6) and (2.7) holds, then in the hypothesis I), II), for every $\xi \in (0,1)$ there exists ν_0 depending only ξ such that

$$(2.10) \qquad \|M_k\|_s \le C_6 g(\nu) \le \xi < 1, \qquad C_6 = C_5 \frac{\rho}{C_1^2} .$$

Proof: From (2.6) and (2.7) we have:

$$\|M_k\|_s \le \|A_k^{-1} - B_k\|_s \cdot \|A_k G_k\|_s \le C_5 \gamma_k^{-2} \cdot \frac{1}{\omega_k} g(\nu)$$

Because by (1.12) $1/\omega_k \le \frac{\rho}{C_1^2}\gamma_k^2$, we obtain the first inequality in (2.10). Now, let ν_0 the smalest ν such that $C_6 g(\nu) \le \xi$, where ξ is fixed in (0.1) interval. This ν_0 there exists by the decreasing monotony of g. Hence

$$\rho(M_k) \le \|M_k\|_s \le \xi < 1$$

for every $\nu \ge \nu_0$, i.e. the two-grid iterative process converges, and the bound for the rate of convergence is independent of the level k.

§3. Multigrid convergence. If in the step iii) of the two-grid algorithm instead of exact solver we use an two-grid algorithm (k-1, k-2), we obtain an three-grid algorithm, and so on. Let $l \ge 2$ be fixed. The multigrid iterative process corresponding at the levels $\mathcal{H}_0 \subset \mathcal{H}_1 \dots \subset \mathcal{H}_l$ is caracterized by the following recursion ([10]):

$$(3.1) \qquad \begin{aligned} M_1 &= (I_1 - I_0^1 A_0^{-1} I_1^0 A_1) G_1^{\nu} \\ M_{k+1} &= (I_{k+1} - I^{k+1}(I_k - M_k^{\mu}) A_k^{-1} I^k A_{k+1}) G_{k+1}^{\nu}, \quad 1 \le k \le l-1 \end{aligned}$$

where $\gamma \geq 1$ is the number of the multigrid iterations on intermediary grid k, same for every $k \geq 1$. Let M_{k+1}^{k} be the two-grid iteration operator corresponding at levels (k+1,k) as in second paragraph. Then,

$$M_{k+1}=M_{k+1}^{k}+I_{k}^{k+1}M_{k}^{\mu}A_{k}^{-1}I_{k+1}^{k}A_{k+1}G_{k+1}^{\nu}, \quad k\geq 1 \tag{3.2}$$

3.1. <u>Remark</u>. In the I), II) hypothesis, we obtain the following estimation for the norm of multigrid iteration operator:

(3.3) $\|M_{k+1}\|_{s} \leq \sigma_{\nu}+\delta_{\nu}\|M_{k}\|_{s}^{\mu}$, where

$\sigma_{\nu}=C_{6}g(\nu)\geq\|M_{k+1}^{k}\|_{s}$, by (2.10)

(3.4) $\delta_{\nu}=C_{7}g(\nu)$, with $C_{7}=C_{3}C_{1}^{2}C_{2}^{2}/\alpha\beta$.

Indeed, $\|I_{k}^{k+1}D_{k}I_{k+1}^{k}\|_{s} \leq C_{3}(\frac{\gamma_{k}}{\gamma_{k+1}})^{2}\|D_{k}\|_{s}$; $D_{k}=M_{k}^{\mu}A_{k}^{-1}$ by commutativity of the diagram $\mathcal{D}_{k+1}$, and by (2.6) and (2.7),
$\|I_{k}^{k+1}D_{k}I_{k+1}^{k}A_{k+1}G_{k+1}^{\nu}\|_{s} \leq C_{7}g(\nu)\,\|M_{k}\|_{s}^{\mu}$.
Hence hold the estimations (3.3), (3.4).

3.1. <u>Theorem of convergence.</u> In the hypothesis I)-II), if $\sigma_{\nu}+\delta_{\nu}<1$, then for every $\mu \geq 1$ there exists ν_{0} depending only μ such that for every $\nu \geq \nu_{0}$

$$\|M_{1}\|_{s} \leq \eta < 1, \tag{3.5}$$

i.e. the multigrid algorithm converges, where η is the solution of the equation

$$f(\eta)\equiv\delta_{\nu}\eta^{\mu}-\eta+\sigma_{\nu}=0, \tag{3.6}$$

solution which lies in (0,1) interval.

<u>Proof</u>. Let $\{\eta_k\}$ be the sequence defined by ([3]):

$$(3.7) \quad \eta_1=\sigma_\nu;\ \eta_{k+1}=\sigma_\nu+\delta_\nu\eta_k^{\mu}, \quad k\geq 1$$

Because $\sigma_\nu+\delta_\nu<1$, the sequence $\{\eta_k\}$ is increasing monotone and bounded by the solution of (3.6). Moreover, $f(0)\cdot f(1)<0$, hence the solution lies in (0,1) interval. From (3.4), there exists such that $\sigma_\nu+\delta_\nu<1$ and it is the smalest with this property.

3.2. <u>Remark</u>. For $\mu=1$ (V-cycle case) and $\mu=2$ (W-cycle case) we obtain the following estimations:

$$(3.8) \quad \eta^{(\mu=1)}=\sigma/(1-\delta) \text{ and } \eta^{(\mu=2)}=(1-\sqrt{1-4\sigma\delta})/2\delta,$$

where $\sigma=\sigma_\nu$, $\delta=\delta_\nu$, of same type as in multigrid literature ([10]).

<u>Comments</u>. By the remarks 1.2 and 1.4, for any variational problem defined by a continuous, elliptic type bilinear functional on a real separable Hilbert space, we can construct an multigrid iterative process such that the bound of the rate of convergence be independent of approximation subspaces, depending only of the continuity and ellipticity constants and of the number of relaxation. In the symmetric case the relaxation process is changed by $G_k=I_k-\omega_kA_k$, i.e. by an Richardson process, obtaining same type estimations. We note that we no use the discretization properties for the continuous solution of problem (P) on approximation subspaces.

REFERENCES

1 BANK, R.E., "A comparison of two multi-level iterative methods for nonsymmetric and indefinite elliptic finite element equations", SIAM J.Numer.Anal.18/1981.

2 BRANDT, A., "Algebraic multigrid theory: the symmetric case"- a preliminary version, 1983.

3 HACKBUSCH, W., "Multigrid Convergence Theory", Multigrid Methods, Proc.of the Conf.Köln-Ponz, November 1981, Lecture Notes in Math. 960, Springer-Verlag, 1982.

4 MAITRE, J.-F. and MUSY,F., "Multigrid Methods: Convergence Theory in a Variational Framework" - SIAM J.Numer.Anal.21/1984.

5 MANDEL, J., "Multigrid Convergence for Nonsymmetric, Indefinite Variational Problems and One Smoothing Step" - presented at the Second Copper Mountain Conference on Multigrid Methods, Copper Mountain, Colorado, April 1985.

6 McCORMICK, S. and RUGE, J., "Multigrid methods for variational problems" - SIAM J.Numer Anal.,19/1982.

7 McCORMICK, S., "Multigrid methods for variational problems: general theory for the V-cycle" - submitted to: SIAM J.Num. Anal.

8 NECAS, J., "Les methodes directes en theorie des equations elliptiques" - Editions de l'Academie Tchechoslovaque des Sciences, Prague, 1967.

9 NICOLAIDES, R.A., "On the l^2 convergence of an algorithm for solving finite element equations" - Math.of Comp., vol.31, No.140/1977.

10 STÜBEN , K. and TROTENBERG, U., "Multigrid Methods: Fundamental Algorithms, Model Problem Analysis and Applications" - Multigrid Methods, Proc. of the Conf. Köln-Porz., November 1981, Lecture Notes in Math.960, Springer-Verlag 1982.