INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

THE RELATIVE HOMOLOGY OF RUNGE PAIRS

by

Nicolae MIHALACHE

PREPRINT SERIES IN MATHEMATICS

No.3/1986

BUCURESTI

THE RELATIVE HOMOLOGY OF RUNGE PAIRS

by

Nicolae MIHALACHE*)

January 1986

*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.

THE RELATIVE HOMOLOGY OF RUNGE PAIRS

§1. INTRODUCTION

The aim of this note is to draw attention on the following fact: the techniques develloped by Hamm in order to study the homotopy type of a q-complete space ([2] and [3]) can also be used to sharpen a classical result of Andreotti-Narashimhan [1, Theorem 1] about the relative homology of Runge pairs. Their theorem is, in turn, a generalization of previous results of Serre [7] and Ramspott-Stein [6]. Namely it will be proved:

"THEOREM 1.1. Let X be a non-degenerate n-dimensional complex space and let $Y \subset X$ be an open, holomorph-convex subspace containing the degeneracy set A of X. Suppose that the pair (X,Y) is Runge. Then:

$$H_r(X,Y;\mathbb{Z})=0 \qquad \text{for} \quad r>n$$

$$H_n(X,Y;\mathbb{Z}) \quad \text{has no torsion."}$$

Recall that given a complex space X and an open subset Y, the pair (X,Y) is called a Runge pair if the restriction map $\mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ has dense image. A non-degenerate space is roughly speaking a proper modification of a Stein space in a discrete set of points. The precise definition will appear in Section 3. One has to keep in mind that Stein spaces and more generally 1-convex spaces are particular cases.

Theorem 1.1 was proved in [1] under the additional assumption that the singular locus of $X \setminus A$ is discrete.

The author wishes to thank dr. M.Coltoiu for several helpful remarks and pointing out a gap in the first version of the paper.

§2. A PARTICULAR CASE

All complex spaces are supposed to be reduced and countable at infinity.

Theorem 1.1 will be a standard consequence of the techniques develloped in [1] and:

"PROPOSITION 2.1. Let X be a n-dimensional Stein space and $\psi : X \to \mathbb{R}$ a C^2, strongly plurisubharmonic exhaustion function. For $\gamma \in \mathbb{R}$ define $\overset{\circ}{X}_\gamma = \{x \in X / \psi(x) < \gamma\}$. Let $\gamma_1 < \gamma_2$ be real numbers. Then:

$$H_r(\overset{\circ}{X}_{\gamma_2}, \overset{\circ}{X}_{\gamma_1}; \mathbb{Z}) = 0 \quad \text{for } r > n$$

$$H_n(\overset{\circ}{X}_{\gamma_2}, \overset{\circ}{X}_{\gamma_1}; \mathbb{Z}) \quad \text{has no torsion ."}$$

Proof :

By the universal coefficient theorem it is enough to verify that :

(1). $$H_r(\overset{\circ}{X}_{\gamma_2}, \overset{\circ}{X}_{\gamma_1}; G) = 0 \quad \text{for } r > n \text{ and any abelian group } G.$$

The proof of (1) is by induction on n=dim X. The case n=0 is obvious. Let S be the singular locus of X and $\mathcal{S}$ a semi--analytic Whitney stratification of (X,S). By Tognoli [8, Theorem 3.4] and Pignoni [5, Theorem 1, Corollary] one may approximate ψ as close as one wants in C^2-topology by a real analytic, strongly plurisubharmonic exhaustion function $\tilde{\psi}: X \to \mathbb{R}$ which is a Morse function on the stratified set $(X, \mathcal{S})$, with distinct critical values.

Using an exhaustion argument one sees that it is enough to prove:

(1') $$H_r(X_{\gamma_2}, X_{\gamma_1}; G) = 0 \quad \text{for } r > n \text{ and any abelian group } G$$

when ψ satisfies these additional assumptions. Here $X_\gamma = \{x \in X / \psi(x) \leq \gamma\}$ for $\gamma \in \mathbb{R}$. By the same reason one may suppose that γ_1, γ_2 are regular values for ψ.

Under these hypotheses on X and ψ, Hamm [3, Lemmas 4-11] has proved that $X_{\gamma_2} \cup S$ has the homotopy type of a topological space obtained from $X_{\gamma_1} \cup S$ by attaching cells of dimensions $\leq n$. The proof uses Morse theory on the singular space X modulo its singular locus S.

In particular (2) $H_r(X_{\gamma_2} \cup S, X_{\gamma_1} \cup S; G) = 0$ for $r > n$ and any abelian group G. For $\gamma \in \mathbb{R}$ denote $S_\gamma = X_\gamma \cap S$.

"LEMMA 2.2. $H_r(X_{\gamma_2}, X_{\gamma_1} \cup S_{\gamma_2}; G) = 0$ for $r > n$ and any abelian group G."

Proof :

By excision one has (3) $H_r(X_{\gamma_2} \cup S_{\gamma_2+\varepsilon}, X_{\gamma_1} \cup S_{\gamma_2+\varepsilon}; G) = 0$ for $\varepsilon > 0$. Take $\varepsilon > 0$ small enough so that $[\gamma_2, \gamma_2 + \varepsilon]$ contains no critical values for ψ. There is a controlled vector field v on a neighborhood of $\psi^{-1}([\gamma_2, \gamma_2 + \varepsilon])$ such that $(d\psi)(v) = -\frac{\partial}{\partial t}$. Using the trajectories of v, one may construct, like in [3, Lemma 4] a retraction $R: X_{\gamma_2} \cup S_{\gamma_2+\varepsilon} \to X_{\gamma_2}$ for the inclusion

$X_{\gamma_2} \hookrightarrow X_{\gamma_2} \cup S_{\gamma_2+\varepsilon}$ (namely if σ is the 1-parameter group generated by v, one may take $R(x)=x$ for $x \in X_{\gamma_2}$ and $R(x)=\sigma(x,\psi(x)-\gamma_2)$ for $x \in S_{\gamma_2+\varepsilon} \setminus X_{\gamma_2}$). From the very definition of R it follows that $R(S_{\gamma_2+\varepsilon}) \subset S_{\gamma_2}$; this shows that $H_r(X_{\gamma_2}, X_{\gamma_1} \cup S_{\gamma_2}; G)$ injects into $H_r(X_{\gamma_2} \cup S_{\gamma_2+\varepsilon}, X_{\gamma_1} \cup S_{\gamma_2+\varepsilon}; G)$ which vanishes by (3). Q.E.D.

The exact sequence of the triple $(X_{\gamma_2}, X_{\gamma_1} \cup S_{\gamma_2}, X_{\gamma_1})$ gives

$$\ldots \to H_{r+1}(X_{\gamma_2}, X_{\gamma_1} \cup S_{\gamma_2}; G) \to H_r(X_{\gamma_1} \cup S_{\gamma_2}, X_{\gamma_1}; G) \to H_r(X_{\gamma_2}, X_{\gamma_1}; G) \to$$
$$\to H_r(X_{\gamma_2}, X_{\gamma_1} \cup S_{\gamma_2}; G) \to \ldots$$

Therefore (4) $H_r(X_{\gamma_2}, X_{\gamma_1}; G) \cong H_r(X_{\gamma_1} \cup S_{\gamma_2}, X_{\gamma_1}; G)$ for $r > n$ and any abelian group G, by Lemma 2.2.

"LEMMA 2.3. $H_r(X_{\gamma_1} \cup S_{\gamma_2}, X_{\gamma_1}; G) \cong H_r(S_{\gamma_2}, S_{\gamma_1}; G)$ for any r."

Proof:

The pair of semi-analytic sets $(X_{\gamma_1}, S_{\gamma_1})$ can be triangulated in such a way that it becomes a polyhedral pair. Therefore there exists a neighborhood T of S_{γ_1} in X_{γ_1} together with a strong deformation retraction $\tilde{R}: T \times I \to T$ for the inclusion $S_{\gamma_1} \hookrightarrow T$. By excision with $V = X_{\gamma_1} \setminus T$ one has:

(5) $$H_r(X_{\gamma_1} \cup S_{\gamma_2}, X_{\gamma_1}; G) \cong H_r(T \cup S_{\gamma_2}, T; G)$$

But obviously $\tilde{R}$ extends to a strong defomration retraction $\hat{R}: (T \cup S_{\gamma_2}) \times I \to T \cup S_{\gamma_2}$ for the inclusion $S_{\gamma_2} \hookrightarrow T \cup S_{\gamma_2}$. Consequently $H_r(T \cup S_{\gamma_2}, T; G) \cong H_r(S_{\gamma_2}, S_{\gamma_1}; G)$. By (5) this concludes the proof of Lemma 2.

The following Lemma will end the proof of Proposition 2.1.

"LEMMA 2.4. $H_r(S_{\gamma_2}, S_{\gamma_1}; G) \cong H_r(\overset{\circ}{S}_{\gamma_2}, \overset{\circ}{S}_{\gamma_1}; G)$ for $r>1$ and any abelian group G."

Above $\overset{\circ}{S}_\gamma = \overset{\circ}{X}_\gamma \cap S$ for $\gamma \in \mathbb{R}$.

Indeed, by the induction hypothesis one obtains $H_r(S_{\gamma_2}, S_{\gamma_1}; G)=0$ for $r>n$; this combined with (4) and Lemma 2.3 proves (1').

PROOF OF LEMMA 2.4. Denote $\varphi = \psi|_S$. Then φ is a real analytic strongly plurisubharmonic exhaustion function on the Stein space S. Moreover φ is a Morse function (with respect to the induced Whitney stratification on S), it has distinct critical values and γ_1, γ_2 are also regular values for φ.

Let $\gamma \in \mathbb{R}$ a regular value for φ. Then, there exists $\varepsilon > 0$ small enough and a homeomorphism $\varphi^{-1}((\gamma-\varepsilon, \gamma]) \simeq \varphi^{-1}(\gamma) \times (-\varepsilon, 0]$. This can be seen from the proof of Thom's first Isotopy Lemma. By excision one obtains:

(6) $$H_r(S_\gamma, \overset{\circ}{S}_\gamma; G)=0 \quad \text{for } r>0$$

From the exact sequence of the triple $(S_{\gamma_2}, S_{\gamma_1}, \overset{\circ}{S}_{\gamma_1})$ and (6) one has: (7) $H_r(S_{\gamma_2}, \overset{\circ}{S}_{\gamma_1}; G) \cong H_r(S_{\gamma_2}, S_{\gamma_1}; G)$ for $r>1$.

Finally, from the exact sequence of the triple $(S_{\gamma_2}, \overset{\circ}{S}_{\gamma_2}, \overset{\circ}{S}_{\gamma_1})$ and (6) one has: (8) $H_r(\overset{\circ}{S}_{\gamma_2}, \overset{\circ}{S}_{\gamma_1}; G) \cong H_r(S_{\gamma_2}, \overset{\circ}{S}_{\gamma_1}; G)$ for $r>0$.

Now (7) together with (8) give the desired isomorphism.

Q.E.D.

" COROLLARY 2.5. Under the assumptions of 1.2.:

$H_r(X,\overset{\circ}{X}_\gamma;\mathbb{Z})=0$ for $r>n$ and

$H_n(X,\overset{\circ}{X}_\gamma;Z)$ has not torsion, for any real number γ ".

This follows from the fact that $(X,\overset{\circ}{X}_\gamma)$ can be exhausted with pairs of type $(\overset{\circ}{X}_\eta,\overset{\circ}{X}_\gamma)$ and the proof of Theorem 1.2.

REMARK. Proposition 2.1 and Corollary 2.5 can be generalized to the case of q-complete spaces with the same proof. The reason is that given X a q-complete space and $\psi : X \to \mathbb{R}$ a real analytic strongly pseudoconvex exhaustion function (satisfying the additional assumptions stated in the proof of Theorem 1.2) then actually Hamm [3] proves that $X_{\gamma_2} \cup S$ has the homotopy type of a topological space obtained from $X_{\gamma_1} \cup S$ by attaching cells of dimensions $n+q$. The statements are left to the reader.

§3. PROOF OF THEOREM 1.1

A complex space X is called non-degenerate if there is an analytic set $A \subset X$ (the degeneracy set of X), a Stein space $\tilde{X}$ and a proper holomorphic map $p:X \to \tilde{X}$ such that: i) $\dim_x A > 0$ for any $x \in A$; ii) p induces a biholomorphism $X \setminus A \xrightarrow{\sim} \tilde{X} \setminus \tilde{A}$ (where $\tilde{A}=p(A)$) and $p_* \mathcal{O}_X = \mathcal{O}_{\tilde{X}}$; iii) $\tilde{A}$ is discrete. In particular such a space is holomorph-convex. If $\tilde{A}$ is finite (equivalently A is compact) X is called 1-convex and A is its exceptional set. This notation will be kept through out this Section.

For the basic properties of Runge pairs we refer to [1, Preliminaries].

To prove the theorem it is enough to show that $H_r(X,Y;G)=0$ for $r>n$ and any abelian group G.

The first step is the reduction to the case when X is Stein and Y is a relatively compact open Stein subset such that the pair (X,Y) is Runge.

Since $\tilde{A}$ is discrete one can choose a sequence $\{X_\nu\}$ of open Stein and Runge subsets of X such that $\tilde{X}_\nu \subset\subset \tilde{X}_{\nu+1}$, $\bigcup_\nu \tilde{X}_\nu=\tilde{X}$ and $\partial \tilde{X}_\nu \cap \tilde{A}=\emptyset$. Let $X_\nu=p^{-1}(\tilde{X}_\nu)$ and $Y_\nu=Y\cap X_\nu$. Then X_ν is 1-convex (with exceptional set $A\cap X_\nu$), Y_ν is an open holomorph-convex space, the pair (X_ν,Y_ν) is Runge and moreover $\{(X_\nu,Y_\nu)\}_\nu$ exhaust (X,Y). So X may be supposed 1-convex. Now one can choose an increasing sequence $\{\tilde{Y}_\nu\}$ of open, relatively compact Stein and Runge subsets of $\tilde{X}$ containing $\tilde{A}$ (which is now finite) and such that $\bigcup_\nu \tilde{Y}_\nu=p(Y)$. Consequently Y may be supposed relatively compact. Finally, by excision and the fact p induces a biholomorphism $X\setminus A\simeq\tilde{X}\setminus\tilde{A}$ one has $H_r(X,Y;G)\cong H_r(\tilde{X},p(Y);G)$. This concludes the first step.

When X is Stein and (X,Y) is a Runge pair, given a compact subset K of Y one can produce a real analytic strongly plurisubharmonic function $\psi:X\to\mathbb{R}$ such that, say, $K\subset\{x\in X/\ \psi(x)<1/2\}\subset Y$. This can be done by slightly modifying the proof that any Stein space carries a real analytic strongly plurisubharmonic exhaustion function (see e.g. [4]). It is clear now that the pair (X,Y) can be exhausted with pairs of the form $(X_{\gamma_\nu^2},X_{\gamma_\nu^1})$ corresponding to triples $(\psi_\nu, \gamma_\nu^1, \gamma_\nu^2)$ where $\psi_\nu : X \to \mathbb{R}$ are (possibly different) real analytic, strongly plurisubharmonic exhaustion functions for X and $\gamma_\nu^1 < \gamma_\nu^2$ are real numbers.

Using Proposition 2.1 this concludes at once the proof of Theorem 1.1.

REMARK. The corollaries of Theorem 1 in [1] can be strenghtened using Theorem 1.1 instead. (However some of these corollaries follow directly from Hamm's results). Again the statements are left to the reader.

REFERENCES

1. Andreotti, A.; Narashimhan, R.: A topological property of Runge pairs, Ann. of Math., 76, 3, 499-509 (1962).
2. Hamm, H.: Zum Homotopietyp Steinscher Räume, J.Reine Angev. Math., 338, 121-135 (1983).
3. Hamm, H.: Zum Homotopietyp q-vollständiger Räume (to appear).
4. Narasimhan, R.: The Levi problem for complex spaces I. Math. Ann. 142, 355-365 (1961).
5. Pignioni, R.: Density and Stability of Morse Functions on a Stratified Space, Ann. Scuola Norm. Sup. Pisa (4) 4, 592-608 (1979).
6. Ramspott, K.J.; Stein, K.: Über Rungesche Paare komplexes Mannigfaltigkeiten. Math. Ann. 145, 444-463 (1962).
7. Serre, J.-P.: Une propriété topologique des domaines de Ruge. Proc. Amer. Math. Soc. 6, 133-134 (1955).
8. Tognoli, A.: Proprieta globali degli spazi analitici reali, Ann. Math. Pura Appl. (4) 75, 143-218 (1967).