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by

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By A.BUIUM

0.Terminology. We use standard Hopf algebra terminology from [Sw]. For convenience we recall some of the basic conventions. Fields are always commutative while groups are generally not. Algebras are associative with unit but not necessarily commutative. Hopf algebras have antipodes and are neither commutative nor cocommutative in general. When we say that a Hopf K-algebra (K a field) is finitely generated we mean it is finitely generated as a K-algebra. By a group K-algebra we understand a Hopf algebra of the form $KG=\bigoplus_{g\in G} Kg$ (where G is a group and KG is equipped with the standard Hopf structure as in [Sw]p.54). Finally we fix throughout the paper an algebraically closed ground field k of characteristic zero; all fields under consideration will be assumed to contain k.

1. Introduction. Recall that the category of commutative finitely generated Hopf algebras is equivalent to the category of affine algebraic groups. Under this equivalence, commutative finitely generated group algebras correspond to diagonalizable groups (d-groups, see [Hu]p.105). So it is a general hope that basic properties of d-groups have their non-commutative

(i.e..Hopf) analog in other words that they extend to properties of group algebras. We will be concerned here with rigidity properties. A trivial example of such a property is that the automorphism group of a group algebra KG equals the automorphism group of G itself; this is the analog of a well known (trivial) property of d-groups [Hu] p.105.

The aim of this paper is to prove non-commutative analogs of some deeper rigidity properties of d-groups. Roughly speaking the properties we have in mind are the following:

(1.1) There are no non-trivial families of d-groups inside a given affine algebraic group.

(1.2) There are no non-trivial families of d-group actions on a given projective scheme.

Intuitively, by a trivial family of d-groups (respectively of actions of a d-group) we understand here a family which "generically" is obtained by conjugating a fixed d-group (respectively a fixed action) by a variable element of the given group (respectively with a variable automorphism of the given projective scheme). Property (1.1) follows for instance from the conjugacy of maximal tori in affine algebraic groups [Hu] p.135 while property (1.2) is a consequence of (1.1) applied to the automorphism group of our projective scheme on which a polarisation has been fixed.

Precise statements of our non-commutative analogs of (1.1) and (1.2) will be given in the next section. Note that in the non-commutative case it isnot anymore clear that the analog of (1.1) already implies the analog of (1.2) (see (7.4) below).

3. Statement of the results. Before stating our non-commutative analog of (1.1) let's recall some facts about coconjugacy in Hopf algebras. Let K be a field and H be a Hopf K-algebra with comultiplication Δ and antipode S. For any K-algebra map $u \in Alg_K(H,K)$ write $u^S = u \circ S \in Alg_K(H,K)$ and define $C(u): H \longrightarrow H$ to be the map $C(u) = (u^S \otimes 1 \otimes u)\Delta_2$ where $\Delta_2 : H \longrightarrow H \otimes_K H \otimes_K H$ is the natural comultiplication $\Delta_2 = (1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ and $u^S \otimes 1 \otimes u : H \otimes_K H \otimes_K H \longrightarrow H$ takes $a \otimes b \otimes c$ into $u^S(a)u(c)b$; $C(u)$ is easily seen to be a Hopf K-algebra automorphism and will be called coinner K-automorphism of H.

Now two Hopf ideals I and J in H are said to be coconjugate over some field extension $\tilde{K}$ of K if there exists a coinner $\tilde{K}$-automorphism σ of $H \otimes_K \tilde{K}$ such that $\sigma(I \otimes_K \tilde{K}) = J \otimes_K \tilde{K}$ (here we view $H \otimes_K \tilde{K}$ as a Hopf K-algebra in the natural way).

Here is our analog of (1.1):

(2.1) THEOREM. Let H_o be a finitely generated Hopf k-algebra K a field extension of k, $H = H_o \otimes K$ and J a Hopf ideal in H such that H/J is a group algebra. Then there exists a Hopf ideal J_o in H_o such that J and $J_o \otimes K$ are coconjugate over some field extension of K.

Intuitively we should view H/J as a family of group algebras "coembedded" in H_o and with parameter space Spec K. Our theorem says that any such family must be coconjugate to a "trivial" one.

To state our non-commutative analog of (1.2) recall first some facts about coactions. Let H be a Hopf K-algebra with commultiplication Δ and counit ε and let A be a K-algebra. By a H-coaction on A we mean a H-comodule algebra structure on A (see [Ra] p.324) i.e. a K-algebra map $\alpha : A \longrightarrow H \otimes_K A$

such that $(\varepsilon\otimes 1)\alpha = 1_A$ and $(\Delta\otimes 1)\alpha = (1\otimes\alpha)\alpha : A \to H\otimes_K H\otimes_K A$. Given two coactions $\alpha : A \to H\otimes_K A$ and $\alpha' : A \to H'\otimes_K A$ we say that α and α' commute if $(1\otimes\alpha)\alpha' = (\tau\otimes 1)(1\otimes\alpha')\alpha : A \longrightarrow H'\otimes_K H\otimes_K A$ where $\tau : H\otimes_K H' \to H'\otimes_K H$ is the twist map.

Put $T = K\mathbb{Z} = K[t, t^{-1}]$ = Hopf algebra of the 1-dimensional torus. A T-coaction on A, $\alpha : A \longrightarrow T\otimes_K A = A[t, t^{-1}]$ is called positive if $\alpha(A) \subset A[t]$ and the image of $\alpha(A)$ in $A = A[t]/(t)$ equals K. A H-coaction on A will be called projective if it commutes with some positive T-coaction on A.

Finally let $\alpha : A \to H\otimes_K A$ and $\beta : B \to H\otimes_K B$ be two coactions as above; we say they are equivalent over some field extension $\tilde{K}$ of K if there is a $\tilde{K}$-isomorphism of algebras $f : \tilde{A} = A\otimes_K \tilde{K} \to \tilde{B} = B\otimes_K \tilde{K}$ such that $(1\otimes f)\tilde{\alpha} = \tilde{\beta} f$ where $\tilde{\alpha} : \tilde{A} \longrightarrow H\otimes_K \tilde{A}$ and $\tilde{\beta} : \tilde{B} \longrightarrow H\otimes_K \tilde{K}$ are the naturally induced maps. Here is our analog of (1.2):

(2.2) THEOREM. Let H_o be a group k-algebra, A_o a finitely generated k-algebra, K a field extension of k and $H = H_o\otimes K$, $A = A_o\otimes K$. Then for any projective H-coaction $\alpha : A \to H\otimes_K A$ there is a finitely generated k-algebra B_o and a H_o-coaction $\beta_o : B_o \to H_o\otimes B_o$ such that if we put $B = B_o\otimes K$ then α and $\beta = \beta_o\otimes 1 : B \to H\otimes_K B$ are equivalent over some field extension of K.

Intuitively the above theorem says that any family of projective coactions of a group algebra is equivalent with a "constant" family.

The proofs of Theorems (2.1) and (2.2) are similar in spirit but independent of eachother. First step will be to prove some "infinitesimal" versions of our statements; this will be done by certain tricks with derivations. The second step will be the to "integrate"; this will be done by applying a version of Kolchin's existence theorem for Picard-Vessiot extensions [Kol] pp.420-421. The use of Kolchin's theorem is a somewhat unexpected feature of our proofs but on the other hand our method is quite general and can be applied in various other situations (see [Bu]).

The plan of the paper is the following. Sections 3 and 4 are devoted to the proof of Theorem (2.1); sections 5 and 6 are devoted to the proof of Theorem (2.2); in section 7 we make further comments and discuss some open questions.

3. Infinitesimal rigidity of Hopf ideals. First some more terminology and conventions. For any K-linear space V we put $V^* = \mathrm{Hom}_K(V,K)$ the linear dual of V. For $u \in V^*$ and $x \in V$ we write $\langle u,x \rangle$ instead of $u(x)$. For any K-linear map $f:V \longrightarrow W$ we denote by $f^*:W^* \longrightarrow V^*$ its usual transpose so $\langle f^*u,x \rangle = \langle u,fx \rangle$ for all $x \in V, u \in W^*$. If $(u_i)_i$ is a family of elements in V^* and if for all $x \in V$ there are at most finitely many indices i such that $\langle u_i,x \rangle \neq 0$ then $\sum u_i$ is a well defined element in V^*; we shall consider several times such infinite sums.

If $f:A \longrightarrow B$ is a K-algebra map and L is a subfield of K then a map $d:A \longrightarrow B$ is called an L-f-derivation from A to B if it is L-linear and $d(xy)=d(x)f(y)+f(x)d(y)$ for all $x,y \in A$; if f is the identity we say that d is an L-derivation on A. Whenever $d:A \longrightarrow B$ is an L-f-derivation as above with $d(K) \subset K$, the L-linear map $d \otimes f + f \otimes d : A \otimes_L A \longrightarrow B \otimes_L B$ sends $\ker(A \otimes_L A \longrightarrow A \otimes_K A)$ into $\ker(B \otimes_L B \longrightarrow B \otimes_K B)$ hence it induces an L-$f \otimes f$-derivation from $A \otimes_K A$ to $B \otimes_K B$ which we still denote by $d \otimes f + f \otimes d$ (note that neither $d \otimes f$ nor $f \otimes d$ are well defined maps from $A \otimes_K A$ to $B \otimes_K B$ while their sum is).

Finally for any Hopf algebra H we denote by $\Delta_H, \varepsilon_H, S_H$ (or simply by Δ, ε, S) the comultiplication, counit and antipode on H; if $x \in H$ we write $\Delta x = \sum_i x_{1i} \otimes x_{2i}$. Recall from [Sw] p.9 that for a Hopf K-algebra as above the dual H^* has a natural K-algebra structure with multiplication given by $H^*\otimes_K H^* \subset (H\otimes_K H)^* \xrightarrow{\Delta^*} H^*$ and unit element given by ε_H.

From now on, throughout the present section as well as in the next one, we keep notations from Theorem (2.1) and denote by $r:H \longrightarrow H/J=KG$ the natural projection (where G is some group).

Start with any k-derivation δ on K and define k-derivations d_H and d_{KG} on $H=H_o\otimes K$ and $KG=kG\otimes K$ by $d_H(x\otimes a)=x\otimes\delta a$ and $d_{KG}(y\otimes b)=y\otimes\delta b$ respectively. The following obvious formulae hold:

(3.1) LEMMA. $(1\otimes d_{KG}+d_{KG}\otimes 1)\Delta_{KG}=\Delta_{KG}d_{KG}$ and

$$(1\otimes d_H+d_H\otimes 1)\Delta_H=\Delta_H d_H.$$

(3.2) LEMMA. The map $\partial:=d_{KG}r-rd_H$ is a K-r-derivation from H to KG and the following formula holds:

$$(\partial\otimes r+r\otimes\partial)\Delta_H=\Delta_{KG}\partial .$$

Proof. Use (3.1) and the equality $(r\otimes r)\Delta_H=\Delta_{KG}r:H\longrightarrow KG\otimes_K KG$.

(3.3) LEMMA. The map ∂^* is a K-r^*-derivation from $(KG)^*$ to H^*.

Proof. For all $u,v\in(KG)^*$ and $x\in H$ we have:

$$\langle\partial^*(uv),x\rangle=\langle uv,\partial x\rangle=\langle u\otimes v,\Delta_{KG}\partial x\rangle \quad \text{by (3.1)}$$

$$=\langle u\otimes v,(\partial\otimes r+r\otimes\partial)\Delta_H x\rangle=$$

$$=\langle(\partial^*u)(r^*v)+(r^*u)(\partial^*v),x\rangle \quad \text{QED.}$$

Now for any $g\in G$ let $u_g\in(KG)^*$ be defined by the projection map $u_g:KG=\bigoplus_h Kh \longrightarrow Kg=K$ onto the g-component and let $w_g\in H^*$ be defined by $w_g=r^*u_g=u_g\circ r$.

(3.4) LEMMA. The following relations hold in the algebra $(KG)^*$:

$$u_g u_g = u_g \quad \text{for all } g \in G$$

$$u_g u_h = 0 \quad \text{for } g,h \in G,\ g \neq h$$

$$\sum_g u_g = 1$$

hence the corresponding relations will hold for the w_g's in H^*.

Proof. It is a trivial explicit computation.

(3.5) LEMMA. The sum $\theta := \sum_g w_g(\partial^* u_g)$ is a well defined element in the algebra H^* and, viewed as a map from H to K it is a K-$\mathcal{E}$-derivation.

Proof. For $x \in H$ and $g \in G$ we have :

$$\langle w_g(\partial^* u_g), x\rangle = \langle w_g \otimes \partial^* u_g, \Delta_H x\rangle = \sum_i \langle w_g, x_{1i}\rangle \langle u_g, \partial x_{2i}\rangle$$

which vanishes for all but finitely many g's so θ is well defined. To prove that it is a K-$\mathcal{E}$-derivation remark first that for all $x,y \in H$ and $g,h \in G$ we have

$$\langle w_g, xy\rangle = \sum_f \langle w_f, x\rangle \langle w_{f^{-1}g}, y\rangle \quad \text{and}$$

$$\sum_i \langle w_g, x_{1i}\rangle \langle w_h, x_{2i}\rangle = \langle w_g \otimes w_h, \Delta_H x\rangle = \langle w_g w_h, x\rangle =$$

$$= \begin{cases} \langle w_g, x\rangle & \text{if } g=h \\ 0 & \text{if } g \neq h \end{cases}$$

Using the above formulae we get:

$$\langle \theta, xy\rangle = \sum_{i,j,g} \langle w_g, x_{1i}y_{1j}\rangle \langle u_g, \partial(x_{2i}y_{2j})\rangle =$$

$$= \sum_{i,j,g,f,h} \langle w_f, x_{1i}\rangle\langle w_{f^{-1}g}, y_{1j}\rangle\langle u_h, \partial x_{2i}\rangle\langle w_{h^{-1}g}, y_{2j}\rangle +$$

$$+ \sum_{i,j,g,f,h} \langle w_f, x_{1i}\rangle\langle w_{f^{-1}g}, y_{1j}\rangle\langle w_h, x_{2i}\rangle\langle u_{h^{-1}g}, \partial y_{2j}\rangle =$$

$$= \sum_{i,g,f} \langle w_f, x_{1i}\rangle\langle u_f, \partial x_{2i}\rangle\langle w_{f^{-1}g}, y\rangle +$$

$$+ \sum_{j,g,f} \langle w_{f^{-1}g}, y_{1j}\rangle\langle u_{f^{-1}g}, \partial y_{2j}\rangle\langle w_f, x\rangle =$$

$$= \langle \theta, x\rangle\langle \varepsilon, y\rangle + \langle \varepsilon, x\rangle\langle \theta, y\rangle \quad \text{QED.}$$

(3.6) LEMMA. The map $E:=(\varepsilon\otimes 1\otimes\theta - \theta\otimes 1\otimes\varepsilon)\Delta_3$ is a K-derivation on H and for all $u\in H^*$ we have $E^*u=u\theta-\theta u$.

Proof. To see that E is a K-derivation on H it is sufficient to check that $1+zE: H\otimes_K K[z] \longrightarrow H\otimes_K K[z]$ is a $K[z]$-algebra map, where $K[z]=K\oplus Kz$, $z^2=0$. But $1+zE=((\varepsilon - z\theta)\otimes 1\otimes(\varepsilon + z\theta))\Delta_3$ and (3.5) easily implies that $\varepsilon+z\theta$, $\varepsilon - z\theta$ are $K[z]$-algebra maps from $H\otimes_K K[z]$ to $K[z]$; consequently $1+zE$ is also an algebra map. To compute E^*u note that $(\varepsilon\otimes 1\otimes\theta - \theta\otimes 1\otimes\varepsilon)^*u = \varepsilon\otimes u\otimes\theta - \theta\otimes u\otimes\varepsilon \in H^*\otimes_K H^*\otimes_K H^*$ hence $E^*u=u\theta-\theta u$, QED.

Finally our infinitesimal rigidity is expressed by the following

(3.7) LEMMA. If $D:=d_H+E$ then D is a k-derivation on H which agrees with δ on K and such that $D(J)\subset J$.

Proof. Everything is clear except $D(J)\subset J$. First note that by (3.4) we have $\sum u_g u_g=1$ hence by applying ∂^* we get by (3.3) that $\sum(\partial^* u_g)w_g + \sum w_g(\partial^* u_g)=0$. Consequently for all $g\in G$ we have:

$$E^*w_g = w_g\theta - \theta w_g = \sum_h w_g w_h(\partial^* u_h) + \sum_h (\partial^* u_h) w_h w_g =$$

$$= w_g(\partial^* u_g) + (\partial^* u_g) w_g = \partial^*(u_g u_g) = \partial^* u_g .$$

Now take any $x \in J$ and let's prove that $Dx \in J$. It is sufficient to check that $\langle w_g, Dx\rangle = 0$ for all $g \in G$. But

$$\langle w_g, Dx\rangle = \langle w_g, d_H x\rangle + \langle w_g, Ex\rangle = \langle w_g, d_H x\rangle + \langle E^* w_g, x\rangle =$$

$$= \langle w_g, d_H x\rangle + \langle \partial^* u_g, x\rangle = \langle u_g, r d_H x\rangle + \langle u_g, \partial x\rangle =$$

$$= \langle u_g, d_{KG} r x\rangle = 0 \quad \text{QED.}$$

4. Actual rigidity of Hopf ideals. We shall keep notations from Theorem (2.1).

(4.1) Consider the functor $\mathcal{H}_o^{alg}$: {commutative k-algebras} $\longrightarrow$ {groups} defined by $\mathcal{H}_o^{alg}(B) = Alg_k(H_o, B) = Alg_B(H_o \otimes B, B)$; here $Alg_k(H_o, B)$ is a group with multiplication induced by the multiplication in the convolution algebra $Hom_k(H_o, B)$, see [Sw]p.82 . Recall also that if $u \in Alg_k(H_o, B)$ then its group inverse is given by $u^S := u \circ S$. On the other hand $\mathcal{H}_o^{alg}$ is obviously representable by some finitely generated k-algebra H_o^{alg} which becomes an affine algebraic group over k. Now by [Sw] p.47 there is a finite dimensional sub-coalgebra V_o of H_o such that H_o is generated by V_o as a k-algebra. Fix a k-basis $x_1, \ldots, x_n$ in V_o and consider the standard functor $\mathcal{G}l_n$: {commutative k-algebras} $\longrightarrow$ {groups} , viewing $\mathcal{G}l_n(B)$ as identified with $Aut_B(V_o \otimes B)$ via our fixed basis in V_o. Of course $\mathcal{G}l_n$ is representable by the standard algebraic group GL_n (considered as a scheme over k). There is a functorial homomorphism $\mathcal{C} : \mathcal{H}_o^{alg} \to \mathcal{G}l_n$ defined as follows: for any $u \in \mathcal{H}_o^{alg}(B)$, look at u as a map from $H_o \otimes B$ to

B and let $\mathcal{C}(B)(u) \in Aut_B(V_o \otimes B)$ be the restriction of $(u^S \otimes 1 \otimes u)\Delta_2 : H_o \otimes B \to (H_o \otimes B) \otimes_B (H_o \otimes B) \otimes_B (H_o \otimes B) \longrightarrow H_o \otimes B$ to $V_o \otimes B$ (which takes $V_o \otimes B$ into itself because V_o is a sub-coalgebra). We won't check here that $\mathcal{C}(B)$ is a group homomorphism but only note that the most convenient way of doing it is to use the fact that the B-module transpose (i.e. the image via $Hom_B(-,B)$) of $(u^S \otimes 1 \otimes u)\Delta_2 : H_o \otimes B \longrightarrow H_o \otimes B$ is the map from $Hom_k(H_o,B)$ to itself defined by $v \longmapsto u^{-1}vu$; in other words the B-module transpose of a coinner automorphism is a "genuine" inner automorphism of the convolution algebra. On the ptner hand the maps $v \longmapsto u^{-1}vu$ clearly behave nicely with respect to group operation. Representability yelds a morphism of algebraic groups $H_o^{alg} \longrightarrow GL_n$; let Γ be its image and let γ be the Lie algebra of Γ viewed as a Lie subalgebra of $g\ell_n$.

Now if D is as in (3.7) then we have $D(V_o \otimes K) \subset V_o \otimes K$ (once again because V_o is a sub-coalgebra). Let $a_{ij} \in K$ be such that $Dx_i = \sum a_{ij}x_j$. We claim that $a=(a_{ij}) \in \gamma(K)$. Indeed since $Dx_i = Ex_i$ it is sufficient to prove that the restriction of $1+zE$ to $V_o \otimes K[z]$ (where $z^2=0$) belongs to $\Gamma(K[z])$. But as already noted in the proof of (3.6) we have $\mathcal{E}+z\theta \in \mathcal{H}_o^{alg}(K[z])$ and $1+zE = \mathcal{C}(K[z])(\mathcal{E}+z\theta) \in \Gamma(K[z])$ so our claim is proved.

So far we worked with a single k-derivation δ on K and constructed a k-derivation D and a matrix $a=(a_{ij}) \in \gamma(K)$. Now choose a family $(\delta_p)_p$ of k-derivations on K such that if $x \in K$ is such that $\delta_p x=0$ for all p then $x \in k$ (such a family exists because k is algebraically closed of characteristic zero) and denote by D_p and $a_p=(a_{pij}) \in \gamma(K)$ the corresponding derivations and matrices. At this point we need the following version of Kolchin's theorem mentioned in section 2 (notations are not necessarily the ones used above !):

(4.2) THEOREM. Let Γ be a closed subgroup of GL_n (viewed as an algebraic group over k), let K be a field extension of k, $(\delta_p)_p$ a family of k-derivations on K such that if $x \in K$ is such that $\delta_p x=0$ for all p then $x \in k$ and let $(a_p)_p$ be a family of matrices in $\gamma(K)$ where γ is the Lie algebra of Γ (viewed as a Lie subalgebra of $g\ell_n$). Then there exist a finitely generated field extension $\tilde{K}$ of K and derivations $(\tilde{\delta}_p)_p$ on $\tilde{K}$ extending the derivations δ_p and there exists a matrix $\sigma \in \Gamma(\tilde{K})$ such that:

1) If $x \in \tilde{K}$ is such that $\tilde{\delta}_p x=0$ for all p then $x \in k$ and
2) $\tilde{\delta}_p \sigma = -\sigma a_p$ for all p.

A proof of (4.2) is given in [Bu] (note that there is a weaker version of (4.2) in [NW] p.982 which is not suited for our purpose). For convenience we briefly sketch an argument for (4.2). Put $B=K[T_{ij}]$, where i,j run from 1 to n and define k-derivations on B, extending the derivations δ_p and still denoted by δ_p by putting $\delta_p T_{ij} = -\sum_s T_{is} a_{psj}$. One checks that the defining prime ideal $I \subset B$ of the identity component of Γ is δ_p-invariant for all p. Then one picks a prime ideal Q in B which is maximal among the ideals containing I and which are δ_p-invariant for all p. Finally one checks that (4.2) follows with $\tilde{K}=B_Q/QB_Q$ and $\sigma=(\sigma_{ij})$, σ_{ij}=image of T_{ij} in $\tilde{K}$, $\tilde{\delta}_p = \delta_p \bmod Q$.

(4.3) Let us apply Theorem (4.2) to the situation described in (4.1) namely to our specific Γ, γ, a_p. Let $\tilde{K}$, $\tilde{\delta}_p$ and σ be as in Theorem (4.2); note that after replacing $\tilde{K}$ by some finite extension of it we may assume that $\sigma = \mathscr{C}(\tilde{K})(u)$ for some $u \in H_o^{alg}(\tilde{K})$. Let $\tilde{D}_p$ be the unique k-derivation on $\tilde{H}=H \otimes_K \tilde{K}$ which agrees with D_p on $H \otimes 1$ and with $\tilde{\delta}_p$ on $\tilde{K}$. Moreover let $\tilde{d}_p$ be the k-derivation on $\tilde{H}$ defined by $\tilde{d}_p(x \otimes a)=x \otimes \tilde{\delta}_p a$ for $x \in H_o$, $a \in \tilde{K}$ and put $\tilde{J}=J \otimes_K \tilde{K}$, $J_o=(\sigma^{-1}\tilde{J}) \cap H_o$.

(4.4) LEMMA. $\tilde{d}_p = \sigma^{-1}\tilde{D}_p\sigma$ for all p. In particular $\tilde{d}_p(\sigma^{-1}\tilde{J}) \subset \subset \sigma^{-1}\tilde{J}$ for all p.

Proof. Since $\tilde{d}_p$ and $\sigma^{-1}\tilde{D}_p\sigma$ agree on $1 \otimes \tilde{K}$, they will agree everywhere provided $\tilde{D}_p\sigma$ vanishes on $x_1, \ldots, x_n$. Now

$$\tilde{D}_p(\sigma(x_i)) = \tilde{D}_p\left(\sum_j \sigma_{ij}x_j\right) = \sum_m (\tilde{\delta}_p\sigma_{im})x_m + \sum_j \sigma_{ij}D_p x_j =$$

$$= \sum_m (\tilde{\delta}_p\sigma_{im})x_m + \sum_{j,m} \sigma_{ij}a_{pjm}x_m = 0$$

so the first statement of the lemma is checked. To check the second statement note that by (3.7) $D_p(J) \subset J$ hence $\tilde{D}_p(\tilde{J}) \subset \tilde{J}$ hence $(\sigma^{-1}\tilde{D}_p\sigma)(\sigma^{-1}\tilde{J}) \subset \sigma^{-1}\tilde{J}$. QED.

Our Theorem (2.1) will be proved if we prove the following:

(4.5) LEMMA. 1) $\sigma^{-1}\tilde{J} = J_o \otimes \tilde{K}$.

2) J_o is a Hopf ideal in H_o.

Proof. Statement 1) follows in a way similar to [Bu] p.55 or [MD], Appendix. For convenience recall the argument. Suppose $(\sigma^{-1}\tilde{J}) \setminus (J_o \otimes \tilde{K}) \neq \emptyset$ choose a basis $(f_q)_q$ of the k-linear space H_o and choose an element $f = \sum c_q f_q \in (\sigma^{-1}\tilde{J}) \setminus (J_o \otimes \tilde{K})$, $c_q \in \tilde{K}$, for which $N = \text{card}\{q; c_q \neq 0\}$ is minimal. We may assume that at least one of the c_q's equals 1. There are indices p_o and q_o such that $c := \tilde{\delta}_{p_o} c_{q_o} \neq 0$. By minimality of N we have $f - c_{q_o} c^{-1}\tilde{d}_{p_o} f \in J_o \otimes \tilde{K}$ and $\tilde{d}_{p_o} f \in J_o \otimes \tilde{K}$ hence $f \in J_o \otimes \tilde{K}$, contradiction.

Statement 2) follows from 1) in the following way: first J_o is of course an ideal in H_o and $S(J_o) \subset J_o$. To prove that it is a coideal note that J_o goes to zero under the map

$$H_o \longrightarrow \tilde{H} \longrightarrow \tilde{H} \otimes_{\tilde{K}} \tilde{H} \longrightarrow (\tilde{H}/\sigma^{-1}\tilde{J}) \otimes_{\tilde{K}} (\tilde{H}/\sigma^{-1}\tilde{J}) = (H_o/J_o) \otimes (H_o/J_o) \otimes \tilde{K}$$

hence $\Delta_{H_o}(J_o) \subset H_o \otimes J_o + J_o \otimes H_o$ and we are done.

5. Infinitesimal rigidity of coactions. This section and the next one are devoted to the proof of Theorem (2.2).

(5.1) Start by recalling that there is dictionary allowing to translate statements about coactions of groups algebras KG into statements about G-gradations and conversely. Recall that by a G-gradation on a K-algebra A we mean a family $(A_g)_{g\in G}$ of K-linear subspaces in A such that $A=\bigoplus A_g$ and $A_g A_h \subset A_{gh}$ for all $g,h \in G$. If we are given a G-gradation as above one can construct a H=KG-coaction $\alpha : A \longrightarrow H\otimes_K A$ by putting $\alpha(a)=\sum g\otimes a_g$ where $a=\sum a_g$, $a_g \in A_g$; conversely for any H=KG-coaction $\alpha : A \longrightarrow H\otimes_K A$ define a G-gradation on A by putting $A_g = \alpha_g(A)$ where α_g is the composed map $A \longrightarrow H\otimes_K A = \bigoplus_h Ah \longrightarrow Ag=A$. The two constructions above are inverse to eachother.

If $(A_g)_g$ is a G-gradation on A the elements of $\bigcup A_g$ are called G-homogenous and we write $g=\deg(x)$ whenever $x\in A_g$.

It is trivial to checkthat:

(5.1.1) A T-coaction on A is positive if and only if the corresponding $\mathbb{Z}$-gradation $A=\bigoplus A_m$ is positive in the sense that $A_o=K$ and $A_m=0$ for $m<0$.

(5.1.2) A H-coaction $\alpha : A \longrightarrow H\otimes_K A$ commutes with a H'-coaction α' (where H=KG, H'=KG') if and only if $\alpha_g \alpha'_{g'} = \alpha'_{g'} \alpha_g$ for all $g\in G$, $g' \in G'$; otherwise stated, if and only if the family of linear subspaces $A_{gg'} := A_g \cap A_{g'}$ defines a $G\times G'$-gradation on A (here we denoted by gg' the element $(g,g') \in G\times G'$; we have then $A_g = \bigoplus_{g'} A_{gg'}$, $A_{g'} = \bigoplus_{g} A_{gg'}$).

(5.1.3) A H-coaction $\alpha : A \longrightarrow H\otimes_K A$ is equivalent over $\tilde{K}$ to $\beta_o \otimes 1 : B = B_o \otimes K \longrightarrow H\otimes_K B$ for some H_o=kG-coaction $\beta_o : B_o \longrightarrow H_o \otimes B_o$

if and only if the corresponding gradation descends to k over $\tilde{K}$ in the following sense: there exists a k-algebra B_o, a G-gradation $(B_{og})_g$ on B_o and a $\tilde{K}$-isomorphism of algebras $f: A\otimes_K\tilde{K} \longrightarrow B_o\otimes\tilde{K}$ such that $f(A_g\otimes_K\tilde{K})=B_{og}\otimes\tilde{K}$ for all $g\in G$.

Now it is easy to see that in order to prove Theorem (2.2) it is sufficient to prove the following statement:

(5.2) THEOREM. Let K be a field extension of k, A_o a finitely generated k-algebra, $A=A_o\otimes K$, $\pi: G\longrightarrow \mathbb{Z}$ a group homomorphism and $(A_g)_g$ a G-gradation on A such that the induced $\mathbb{Z}$-gradation $(A_m)_m$ on A ($A_m=\bigoplus_{\pi(g)=m} A_g$) is positive. Then $(A_g)_g$ descends to k over some field extension of K.

Indeed to the H-coaction in our Theorem (2.2) there corresponds a C-gradation $(A_c)_{c\in C}$ (where H=KC). By the above dictionary there is a $\mathbb{Z}$-gradation $(A_m)_m$ on A such that the linear subspaces $A_{cm}=A_c\cap A_m$ define a $C\times\mathbb{Z}$-gradation. Put $G=C\times\mathbb{Z}$ and let $\pi: G\longrightarrow\mathbb{Z}$ be the second projection. By Theorem (5.4) the gradation $(A_{cm})_{cm}$ descends to k over some field extension of K, hence so does $(A_c)_c$.

So we concentrate ourselves on proving (5.2). Once for all choose G-homogenous generators $x_1,\dots,x_n$ of the K-algebra A such that if $\deg(x_i)=g_i$ then $\pi(g_i)>0$. Let $R=K\{X_1,\dots,X_n\}$ be the free K-algebra on variables $X_1,\dots,X_n$ (so R=tensor K-algebra on $KX_1\oplus\dots\oplus KX_n$) and denote the product of any two elements $a,b\in R$ by $a\otimes b$ rather than by ab. There is a unique G-gradation on R such that $\deg(X_i)=g_i$ and clearly $\dim_K R_g$ is finite for all $g\in G$. Moreover put $J=\ker(R\longrightarrow A,\ X_i\longmapsto x_i)$; clearly J is generated by G-homogenous elements. We will keep these notations throughout this section and the next one.

So far we just translated our coaction problem into a gradation problem. Now we go into our "infinitesimal rigidity". Let δ be any k-derivation on K, define a k-derivation d on $A=A_o\otimes K$ by $d(a\otimes b)=a\otimes\delta b$ and let $\alpha_g : A\longrightarrow A$ be the K-linear maps defined by the coaction corresponding to our G-gradation.

(5.3) LEMMA. The sum $\nabla=\sum_g \alpha_g d\alpha_g$ is a well defined k-derivation on A.

Proof. For all $x,y\in A$ we have

$$\nabla(xy)=\sum_g \alpha_g d\alpha_g(xy) = \sum_g \alpha_g d(\sum_{hf=g} \alpha_h(x)\alpha_f(y)) =$$

$$= \sum_{h,f} \alpha_{hf}(d(\alpha_h(x))\alpha_f(y)) + \sum_{h,f} \alpha_{hf}(\alpha_h(x)d(\alpha_f(y))) =$$

$$= \sum_{h,f} (\alpha_h d\alpha_h)(x)\alpha_f(y) + \sum_{h,f} \alpha_h(x)((\alpha_f d\alpha_f)(y)) =$$

$$= (\nabla x)y + x(\nabla y), \quad \text{QED.}$$

Our infinitesimal rigidity is expressed by the following:

(5.4) LEMMA. There is a k-derivation D on R which agrees with δ on K and such that $D(J)\subset J$ and $D(R_g)\subset R_g$ for all g.

Proof. Note that ∇ in (5.3) agrees with δ on K and sends A_g into itself for all $g\in G$. Since R is free ∇ lifts to some D enjoying the desired properties.

6. Actual rigidity of coactions. Let $R_o=k\{X_1,\dots,X_n\}$ and consider on R_o the natural G-gradation induced by $g_1,\dots,g_n$ as in section 5. Put $r=\max\{\pi(g_1),\dots,\pi(g_n)\}$ and let $\mathcal{M}$ be the set of monomials M in R_o for which $\pi(\deg(M))\leq r$ (where by a monomial we mean of course an element of R_o of the form $X_{i_1}\otimes\dots\otimes X_{i_q}$

$q \geq 1$). Clearly $\mathfrak{M}$ is finite. Let V_o be the k-linear subspace of R_o spaned by $\mathfrak{M}$ and let v its dimension. Now consider the functors $\mathcal{F}$, $\mathcal{G}\ell_v$: $\{$commutative k-algebras$\} \to \{$groups$\}$, where $\mathcal{F}(B)=$ =group of B-algebra automorphisms of $B\{X_1,\dots,X_n\}$ preserving the B-modules $R_{og}\otimes B$ for all g and $\mathcal{G}\ell_v(B)=GL_v(B)=Aut_B(V_o\otimes B)$ where the last identification is made by choosing $\mathfrak{M}$ as a basis for V_o. There is an obvious injective restriction map $m: \mathcal{F}\to\mathcal{G}\ell_v$. On the other hand $\mathcal{F}$ is easily seen to be representable by some affine algebraic group over k, call it F. So there is a closed immersion $F \longrightarrow GL_v$ whose image will be denoted by Γ; the Lie algebra of Γ will be denoted by γ. In particular for any field L and any L-derivation E on $L\{X_1,\dots,X_n\}$ preserving the G-gradation ,the image of E in $g\ell_v(L)$ belongs to $\gamma(L)$:for if $z^2=0$ then $1+zE\in\mathcal{F}(L[z])$ hence $m(L[z])(1+zE)\in\Gamma(L[z])$.

Now choose a family $(\delta_p)_p$ of k-derivations on K as in section 4 and denote by $(D_p)_p$ the corresponding family of derivations constructed in Lemma (5.4). Since $D_p(V_o\otimes K)\subset V_o\otimes K$ we may write $D_pM=\sum_{N\in\mathfrak{M}} a_{pMN}N$, $a_{pMN}\in K$, for all $M\in\mathfrak{M}$. We claim that the matrix $a_p=(a_{pMN})$ belongs to $\gamma(K)$. Indeed let d_p be the unique k-derivation on R which agrees with δ_p on K and vanishes on all monomials. Then D_p-d_p is a K-derivation on R preserving the G-gradation hence its image in $g\ell_v$ (which is precisely the matrix a_p) belongs to $\gamma(K)$ and our claim is proved. Now apply the same machinery as in section 4, namely use Kolchin's theorem to find some finitely generated field extension $\tilde{K}$ of K and some matrix $\sigma\in\Gamma(\tilde{K})$ such that if $\tilde{J}=J\otimes_K\tilde{K}$ and if $J_o=(\sigma^{-1}\tilde{J})\cap R_o$ then $\sigma^{-1}\tilde{J}=J_o\otimes\tilde{K}$. Putting $B_o=R_o/J_o$ we see that B_o has a natural G-gradation inducing our original gradation on $A\otimes_K\tilde{K}$; this closes the proof of (5.2) and hence of Theorem (2.2) too.

7. Complements and open questions.

(7.1) Our proofs show that coconjugacy in Theorem (2.1) and equivalence in Theorem (2.2) hold over a finitely generated (rather than arbitrary) field extension of K. It would be interesting to knowwhether they hold over a finite (rather than finitely generated) extension of K. We can prove this is the case for Theorem (2.2)by using some standard specialisation arguments. On the other hand, for Theorem (2.1) specialisation arguments do not seem to work without some additional finitness assumptions.

(7.2) Along the lines of Theorem (2.1) it would be interesting for instance to dispose of a non-commutative analog for "conjugacy of maximal tori in an affine algebraic group". The analog of maximal tori snould be perhaps the minimal prime Hopf ideals for which the corresponding quotient is a group algebra.

(7.3) One is tempted to conjecture that Theorem (2.2) holds without the projectivity assumption. Indeed, we used this assumption essentially in order to force the automorphisms we were looking for to form a nice algebraic group. This still holds if one replaces "projectivity" by some other finitness conditions. That no conditions at all are needed is suggested by what happens in the commutative case where there is some evidence for our conjecture. Indeed in this case a d-group action on A means roughly speaking a group homomorphism (with some algebraic features) from our d-group to Aut(SpecA); but now, as remarked by Białynicki-Birula the latter automorphism group has a "universal" structure of direct limit of affine algebraic groups, so again we should be led in this case to "nice algebraic groups" as in our proof.

(7.4) An interesting question is to decide representability of the functors $\mathcal{H}_A$: {finitely generated Hopf K-algebras} → {sets} defined by $\mathcal{H}_A(H)$=set of all H-coactions on a given finitely generated K-algebra A(possibly with some additional properties e.g. commuting with some given positive T-coaction).

(7.5) We close by noting that antipodes are not at all essential to our work. Indeed both Theorems (2.1) and (2.2) still hold (with identical proofs) if one replaces Hopf algebras by bialgebras, Hopf ideals by bi-ideals and group algebras by cancellative monoid algebras (where a monoid M is called cancellative if $x=y$ whenever $x,y,z \in M$ and either $xz=yz$ or $zx=zy$). In this more general context the concept of coconjugacy must be defined as follows. Let H be a K-bialgebra, let H^* be the dual algebra, H^{alg} the set of all elements of H^* which are K-algebra maps (H^{alg} is then a monoid with respect to the multiplication induced from H^*) and H^{inv} the group of invertible elements of the monoid H^{alg} (in the Hopf case $H^{inv}=H^{alg}$). By a coinner automorphism of H we mean a map $H \longrightarrow H$ of the form $(u^{-1} \otimes 1 \otimes u)\Delta_2$ with $u \in H^{inv}$. Now coconjugacy is defined exactly as in section 2, using the above concept of coinner automorphism.

We should emphasize that cancellativity is necessary to go through the computations from (3.5) and (5.3).

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