A REMARK ON A QUESTION OF M.D. CHOI AND K.R. DAVIDSON

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In [1], M.D. Choi and K.R. Davidson asked the following question:

"If $\mathcal{A}$ and $\mathcal{A}_k$ are similar subalgebras of M_n $(k=1,2,\ldots)$ and $d(\mathcal{A},\mathcal{A}_k)$ tends to 0, does it follow that there are invertible operators S_k such that $\mathcal{A}_k=S_k^{-1}\mathcal{A}S_k$ and $\lim\limits_{k\to\infty}||I-S_k||=0$?"

In the present note we prove that a general affirmative answer to the above question is roughly equivalent with an affirmative answer to the following, involving arbitrary subspaces of the complex n x n matrices M_n (the précise statement is in the remark at the end of the present note).

"If $\mathcal{S}$ is a linear subspace of M_n and A_k, B_k are invertible operators such that $\lim\limits_{k\to\infty} d(A_k^{-1}\mathcal{S}B_k,\mathcal{S})=0$, does it follow that there are invertible operators X_k, Y_k such that $X_k^{-1}\mathcal{S}Y_k=\mathcal{S}$ and $\lim\limits_{k\to\infty} X_kA_k=$ $=\lim\limits_{k\to\infty} Y_kB_k=I$?"

Here the distance $d(.,.)$ between two algebras or subspaces is the Hausdorff distance between their unit balls.

While we don't know what the answer to these questions is, we hope that their equivalence may turn out to be useful in solving them.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S} \subset B(\mathcal{H})$ be a linear subspace. We denote by $\mathcal{A}(\mathcal{S})$ the algebra of all operators of the form $\begin{bmatrix} \lambda I & S \\ 0 & \mu I \end{bmatrix}$ where $\lambda, \mu \in C$, $S \in \mathcal{S}$ and by $\mathcal{A}_o(\mathcal{S})$ its subalgebra $\begin{bmatrix} \lambda I & S \\ 0 & \lambda I \end{bmatrix}$.

If $\mathcal{A}$ is an algebra, its normalizer is defined to be $\mathcal{N}(\mathcal{A}) = \{S \in B(\mathcal{H})$ invertible; $S^{-1}\mathcal{A}S = \mathcal{A}\}$. If $\mathcal{S}$ is a linear subspace, let $\ker \mathcal{S} = \bigcap_{T \in \mathcal{S}} \ker T$.

An algebra $\mathcal{A} \subset B(\mathcal{H})$ is said to have property (A) if for every sequence of invertible operators $\{S_n\} \subset B(\mathcal{H})$ such that $\lim_{n\to\infty} d(S_n^{-1}\mathcal{A}S_n, \mathcal{A}) = 0$ there are $X_n \in \mathcal{N}(\mathcal{A})$ such that $\lim_{n\to\infty} X_n S_n = I$.

A linear subspace $\mathcal{S} \subset B(\mathcal{H})$ is said to have property (B) if for every sequences of invertible operators $\{A_n\}$, $\{B_n\} \subset B(\mathcal{H})$ such that $\lim_{n\to\infty} d(A_n^{-1}\mathcal{S}B_n, \mathcal{S}) = 0$ there are X_n, Y_n invertible operators such that $X_n^{-1}\mathcal{S}Y_n = \mathcal{S}$ and

$$\lim_{n\to\infty} X_n A_n = \lim_{n\to\infty} Y_n B_n = I .$$

<u>PROPOSITION 1</u>. Let $\mathcal{S}, \mathcal{T} \subset B(\mathcal{H})$ be linear subspaces.

(i) If $\ker \mathcal{S} = \ker \mathcal{S}^* = \{0\}$ and $\mathcal{A}_o(\mathcal{S})$ and $\mathcal{A}_o(\mathcal{T})$ are similar (respectively $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{T})$ are similar), then $\ker \mathcal{T} = \ker \mathcal{T}^* = \{0\}$ and every similarity between them is implemented by an operator of the form $\begin{bmatrix} S & X \\ 0 & T \end{bmatrix}$ where $S^{-1}\mathcal{S}T = \mathcal{T}$ (respectively $S^{-1}\mathcal{S}T = \mathcal{T}$ and $X \in S \cdot \mathcal{T}$).

(ii) If $\ker \mathcal{S} = \ker \mathcal{T} = \{0\}$ and $\mathcal{A}_o(\mathcal{S})$ and $\mathcal{A}_o(\mathcal{T})$ are similar (respectively $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{T})$ are similar) then every similarity between them has the preceding form.

Proof. (i) Let $U=\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$ and $U^{-1}=\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$. For $A\in\mathcal{S}$,

$$U^{-1}\begin{bmatrix} \lambda I & A \\ 0 & \lambda I \end{bmatrix}U=\begin{bmatrix} \lambda I+T_1AS_3 & T_1AS_4 \\ T_3AS_3 & \lambda I+T_3AS_4 \end{bmatrix}.$$

It follows that $T_3AS_3=0$ and $T_1AS_3=T_3AS_4=\mu I$ for some $\mu\in C$. If $\mu=0$ then $S_1T_1AS_3+S_2T_3AS_3=0 \Rightarrow AS_3=0$. If $\mu\neq 0$ then $S_3T_1AS_3+S_4T_3AS_3=\mu S_3 \Rightarrow S_3=0$. In both cases, $AS_3=0$ for all $A\in\mathcal{S}$, hence Range $S_3\subset \ker\mathcal{S}=\{0\}$ so $S_3=0$. It follows that $S_4T_4=I$ $\Rightarrow T_3AS_4=T_1AS_3=0 \Rightarrow T_3AS_4T_4=0 \Rightarrow T_3A=0 \Rightarrow A^*T_3^*=0$ for all $A\in\mathcal{S} \Rightarrow$ Range $T_3^*\subset \ker\mathcal{S}^*=\{0\}$ hence $T_3=0$. It follows that $S_1^{-1}=T_1$, $T_4=S_4^{-1}$ hence $S_1^{-1}\mathcal{S}S_4=\mathcal{T}$ and $\ker\mathcal{T}=\ker\mathcal{T}^*=\{0\}$.

(ii) Similarly $S_3=0$ and since $\ker\mathcal{T}=\{0\}$, $T_3=0$ follows by symmetry. For the algebras $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{T})$, notice that any similarity between them induces a similarity between their subalgebras $\mathcal{A}_o(\mathcal{S})$ and $\mathcal{A}_o(\mathcal{T})$. The last assertion $X\in S\cdot\mathcal{T}$ follows easily.

COROLLARY. Let $\mathcal{S}\subset\mathcal{B}(\mathcal{H})$ be a linear subspace, $\ker\mathcal{S}=\{0\}$. Then an operator U belongs to the normalizer of $\mathcal{A}_o(\mathcal{S})$ (respectively $\mathcal{A}(\mathcal{S})$) if and only if $U=\begin{bmatrix} S & X \\ 0 & T \end{bmatrix}$ where $S^{-1}\mathcal{S}T=\mathcal{S}$ (respectively $S^{-1}\mathcal{S}T=\mathcal{S}$ and $X\in S\mathcal{S}$).

LEMMA. Let $\dim\mathcal{H}<\infty$ and let $\mathcal{S}\subset\mathcal{B}(\mathcal{H})$ be a linear subspace such that $\ker\mathcal{S}=\ker\mathcal{S}^*=\{0\}$. If $\{A_n\}, \{B_n\}\subset\mathcal{B}(\mathcal{H})$ are such that $\lim_{n\to\infty} d(A_n\mathcal{S}B_n,\mathcal{S})=0$, then there is $n_o\in N$ such that A_n and B_n are invertible for n greater than n_o.

Proof. It is enough to prove the assertion for $\{A_n\}$ for n sufficiently large.

Let $S_1,\dots,S_p$ be a basis of $\mathcal{S}$. Since $\ker\mathcal{S}^*=\{0\}$ it follows that $\bigvee_{i=1}^{p} \text{Range } S_i = \mathcal{H}$. For each $1 \leq i \leq p$ there is $\{T_{i,n}\}_n \subset \mathcal{B}(\mathcal{H})$ such that $\lim_{n\to\infty} A_n T_{i,n} B_n = S_i$. Now $\lim_{n\to\infty} d(\bigvee_{i=1}^{p} \text{Range } A_n T_{i,n} B_n, \mathcal{H}) = 0$ so $\lim_{n\to\infty} d(\text{Range } A_n, \mathcal{H}) = 0$ hence A_n are invertible for n sufficiently large.

PROPOSITION 2. Let $\dim \mathcal{H} < \infty$ and $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital algebra. Then $\mathcal{A}$ has property (A) if and only if $\mathcal{A}$, as a linear subspace, has property (B).

Proof. " $\Leftarrow$ " Suppose $\lim_{n\to\infty} d(S_n^{-1}\mathcal{A}S_n, \mathcal{A}) = 0$. Then there are invertible operators X_n, Y_n such that $X_n^{-1}\mathcal{A}Y_n = \mathcal{A}$ and $\lim_{n\to\infty} X_n S_n = \lim_{n\to\infty} Y_n S_n = I$. Since $\mathcal{A}$ is unital, there are operators $T_n \in \mathcal{A}$ satisfying $X_n^{-1} T_n Y_n = I$, hence $T_n = X_n Y_n^{-1}$, so T_n is invertible, $X_n^{-1} = Y_n^{-1} T_n^{-1}$ so $\mathcal{A} = Y_n^{-1} T_n^{-1} \mathcal{A} Y_n$. Since $T_n^{-1}\mathcal{A} = \mathcal{A}$, $Y_n^{-1}\mathcal{A}Y_n = \mathcal{A}$ and $\lim_{n\to\infty} Y_n S_n = I$, hence $\mathcal{A}$ has property (A). (Note that $T_n \in \mathcal{A}$ implies $T_n^{-1} \in \mathcal{A}$ for T_n invertible in $\mathcal{B}(\mathcal{H})$ since $\mathcal{H}$ is finite dimensional and $\mathcal{A}$ is unital).

" $\Rightarrow$ " Assume $\mathcal{A}$ has property (A) and let $\{A_n\}$, $\{B_n\}$ be invertible operators such that $\lim_{n\to\infty} d(A_n^{-1}\mathcal{A}B_n, \mathcal{A}) = 0$. Then there are $T_n \in \mathcal{A}$ satisfying $\lim_{n\to\infty} A_n^{-1} T_n B_n = I$. Without loss of generality we may assume that $A_n^{-1} T_n B_n$ are invertible so $A_n^{-1} T_n B_n = U_n$ and $\lim_{n\to\infty} U_n = I$. Then T_n are invertible, $T_n^{-1} \in \mathcal{A}$ and $A_n^{-1} = U_n B_n^{-1} T_n^{-1}$ so that $A_n^{-1}\mathcal{A}B_n = U_n B_n^{-1} T_n^{-1} \mathcal{A} B_n = U_n B_n^{-1} \mathcal{A} B_n$ since $T_n^{-1}\mathcal{A} = \mathcal{A}$. Now $\lim_{n\to\infty} d(U_n B_n^{-1}\mathcal{A}B_n, \mathcal{A}) = 0$ hence $\lim_{n\to\infty} d(B_n^{-1}\mathcal{A}B_n, \mathcal{A}) = 0$ so there are operators X_n, $X_n^{-1}\mathcal{A}X_n = \mathcal{A}$ and $\lim_{n\to\infty} X_n B_n = I$. It follows $A_n^{-1} T_n = U_n B_n^{-1}$ so $\lim_{n\to\infty} A_n^{-1} T_n X_n^{-1} = I$ and it is easy to see that $T_n X_n^{-1} \mathcal{A} X_n = \mathcal{A}$.

PROPOSITION 3. (i) If $\mathcal{S}\subset B(\mathcal{H})$ is a linear subspace, $\ker\mathcal{S}=\{0\}$ and $\mathcal{A}_o(\mathcal{S})$ has property (A) then $\mathcal{S}$ has property (B).

(ii) If $\dim\mathcal{H}<\infty$, $\ker\mathcal{S}=\ker\mathcal{S}^*=\{0\}$ and $\mathcal{S}$ has property (B) then $\mathcal{A}_o(\mathcal{S})$ has property (A).

Proof. (i) Let $\lim_{n\to\infty} d(A_n^{-1}\mathcal{S}B_n,\mathcal{S})=0$. If $S_n=\begin{bmatrix} A_n & 0 \\ 0 & B_n \end{bmatrix}$ then $\lim_{n\to\infty} d(S_n^{-1}\mathcal{A}_o(\mathcal{S})S_n,\mathcal{A}_o(\mathcal{S}))=0$ so there is a sequence $\begin{bmatrix} X_n & Z_n \\ 0 & Y_n \end{bmatrix}$ from the normalizer of $\mathcal{A}_o(\mathcal{S})$ such that $X_n^{-1}\mathcal{S}Y_n=\mathcal{S}$ and $\lim_{n\to\infty}\begin{bmatrix} X_n & Z_n \\ 0 & Y_n \end{bmatrix}\begin{bmatrix} A_n & 0 \\ 0 & B_n \end{bmatrix}=$
$=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Hence $\lim_{n\to\infty} X_nA_n=\lim_{n\to\infty} Y_nB_n=I$ so $\mathcal{S}$ has property (B).

(ii) Let $U_n=\begin{bmatrix} S_{1n} & S_{2n} \\ S_{3n} & S_{4n} \end{bmatrix}$; $U_n^{-1}=\begin{bmatrix} T_{1n} & T_{2n} \\ T_{3n} & T_{4n} \end{bmatrix}$ be such that $\lim_{n\to\infty} d(U_n^{-1}\mathcal{A}_o(\mathcal{S})U_n,\mathcal{A}_o(\mathcal{S}))=0$.

Taking into account that $\dim T_{1n}\mathcal{S}S_{4n}\le\dim\mathcal{S}<\infty$ it follows that $\lim_{n\to\infty} d(T_{1n}\mathcal{S}S_{4n},\mathcal{S})=0$ so, by the preceding lemma, for sufficiently large n, T_{1n} and S_{4n} are invertible. Hence there are A_n, B_n invertible operators, $A_n^{-1}\mathcal{S}B_n=\mathcal{S}$ and $\lim_{n\to\infty} T_{1n}A_n^{-1}=\lim_{n\to\infty} B_nS_{4n}=I$.

Let $U_n'=\begin{bmatrix} (T_{1n})^{-1} & 0 \\ S_{3n} & S_{4n} \end{bmatrix}$. Notice that $U_n'U_n^{-1}=\begin{bmatrix} I & (T_{1n})^{-1}T_{2n} \\ 0 & I \end{bmatrix}\in$
$\in\mathcal{N}(\mathcal{A}_o(\mathcal{S}))$ hence $(U_n')^{-1}\mathcal{A}_o(\mathcal{S})U_n'=U_n^{-1}\mathcal{A}_o(\mathcal{S})U_n$ so we may assume
$U_n=\begin{bmatrix} (T_{1n})^{-1} & 0 \\ S_{3n} & S_{4n} \end{bmatrix}$. Further, $Z_n=\begin{bmatrix} A_n & 0 \\ 0 & B_n \end{bmatrix}\in\mathcal{N}(\mathcal{A}_o(\mathcal{S}))$ so
$(Z_nU_n)^{-1}\mathcal{A}_o(\mathcal{S})Z_nU_n=U_n^{-1}\mathcal{A}_o(\mathcal{S})U_n$ so we may assume $U_n=\begin{bmatrix} V_n & 0 \\ X_n & W_n \end{bmatrix}$,
$U_n^{-1}=\begin{bmatrix} V_n^{-1} & 0 \\ -W_n^{-1}X_nV_n^{-1} & W_n^{-1} \end{bmatrix}$ where $\lim_{n\to\infty} V_n=\lim_{n\to\infty} W_n=I$.

Since $\lim_{n\to\infty} d(U_n^{-1}\mathcal{A}_o(\mathcal{S})U_n, \mathcal{A}_o(\mathcal{S}))=0$ there are $\lambda_n \in C$ and $A_n \in \mathcal{S}$ such that $\lim_{n\to\infty} U_n^{-1}\begin{bmatrix}\lambda_n I & A_n \\ 0 & \lambda_n I\end{bmatrix} U_n = \begin{bmatrix}0 & S \\ 0 & 0\end{bmatrix}$ for a given $S\in\mathcal{S}$.

It follows $\lim_{n\to\infty}\lambda_n=0$ and $\lim_{n\to\infty} V_n^{-1}A_nX_n=0$, so $\lim_{n\to\infty} A_nX_n=0$. We have also $\lim_{n\to\infty} V_n^{-1}A_nW_n=S$ hence $\lim_{n\to\infty} A_n=S$.

Let now $S_1,\dots,S_p$ be a basis of $\mathcal{S}$ and $\sigma=\begin{bmatrix}S_1\\ S_2\\ \vdots \\ S_p\end{bmatrix}\in\mathcal{B}(\mathcal{H},\mathcal{H}^p)$.

As above, there are $\sigma_n\in\mathcal{B}(\mathcal{H},\mathcal{H}^p)$ satisfying $\lim_{n\to\infty}\sigma_n=\sigma$ and $\lim_{n\to\infty}\sigma_nX_n=0$. Since $\ker\sigma=\{0\}$, $\sigma^*\sigma\in\mathcal{B}(\mathcal{H})$ is invertible and $\lim_{n\to\infty}\sigma_n^*\sigma_n= =\sigma^*\sigma$ and $\lim_{n\to\infty}\sigma_n^*\sigma_nX_n=0$, hence $\lim_{n\to\infty}X_n=0$. It follows $\lim_{n\to\infty}U_n=I$, which concludes the proof.

REMARKS. By Propositions 2 and 3 (i) we conclude that the following statements are equivalent.

I. Every unital operator algebra on a finite dimensional Hilbert space has property (A).

II. Every operator algebra of the form $\mathcal{A}_o(\mathcal{S})$ with $\ker\mathcal{S}=\{0\}$ on a finite dimensional Hlbert space has property (A).

III. Every linear space of operators $\mathcal{S}$ on a finite dimensional Hilbert space, with $\ker\mathcal{S}=\{0\}$, has property (B).

Note that Proposition 3 (ii) is a stronger version of III $\Rightarrow$ II.

REFERENCES

[1] M.D. Choi; K.R. Davidson, Perturbations of finite dimensional operator algebras, preprint.