OPERATIONS ON CERTAIN NON-COMMUTATIVE OPERATOR-VALUED RANDOM VARIABLES

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An example motivating the study of the addition of free pairs of "non-commutative operator-valued random variables" is provided by the computation of spectra of convolution operators on free groups.

Let G be the (non-commutative) free group on two generators g_1, g_2 and let λ denote the left regular representation on $\ell^2(G)$ To compute spectra of convolution operators

$$Y = \sum_{g \in G} c_g \ \lambda(g)$$

with $c_g \neq 0$ only for finitely many $g \in G$ it suffices to be able to decide whether such Y is invertible. This in turn is equivale to deciding whether a certain operator

$$X = \sum_{k \in \mathbb{Z}} (\alpha_k \otimes \lambda(g_1^k) + \beta_k \otimes \lambda(g_2^k))$$

where $\alpha_{-k} = \alpha_k^*$, $\beta_{-k} = \beta_k^*$ are n x n matrices, is invertible. If n=1, i.e. if the matrices are scalars, then the spectrum of X can be computed using our results on the addition of free pairs of non-commutative random variables [8]. Thus the computation of the spectrum of Y is reduced to a generalization of the additio of free pairs of non-commutative random variables to the case of "matrix-valued non-commutative random variables".

The present paper deals with the extension of our previou work ([8],[9]) on addition and multiplication of free pairs of non-commutative random variables to, what might be called, the operator-valued case. This means that the field of complex numbers is replaced by an operator algebra, the free products are with amalgamation over this algebra and the specified states are replaced by specified conditional expectations. Also the natural frame-work of operator algebras with dual algebraic structure ([10]) for the considered operations in the "scalar" case has a corresponding extension to the "operator-valued" case.

Though our results are meant for applications to operator algebras and spectral theory, most of our considerations will be in a purely algebraic context, since we shall be mainly concerned with finding the formulae for computing the operation on the distributions of the random-variables. Concerning distributions of operator-valued non-commutative random-variables, let us only say that since the scalars $\mathbb{C}$ are replaced by an operator algebra B, the moments of the variable X are the expectation valued of monomials of the form $Xb_1Xb_2\dots Xb_{n-1}X$. It is an important fact for the computation of spectra that the addition of free pairs of B-valued random variables gives an operation among the symmetric parts of the distributions i.e. among the expectation values of monomilas of the form $bXbX\dots bXb$. For the symmetric distributions the addition formulae closely resemble those in the scalar case with the generating series viewed as germs of maps $\mathbb{C} \longrightarrow \mathbb{C}$ replaced by germs of maps $B \longrightarrow B$.

The paper has seven sections.

The first section discusses free families of non-commutative B-valued random variables and distributions of such random variables.

The second and third section deal with the algebras A(M) and the canonical form of a random variable with a given distribution.This is the analogue for the B-valued case of the special Toeplitz operators which we used in the scalar case for studying the addition of free pairs of noncommutative random variables.

The fourth section gives the solution to the addition problem for the symmetric parts of distributions of B-valued random variables.It is obtained by studying the differential equation for semigroups with respect to addition.The final formulae closely resemble those in the scalar case.

The fifth section deals with the differential equation for semigroups with respect to the multiplicative operation.

The sixth section presents the application to the computation of spectra of convolution operators on free groups.

Section seven is a brief outline of the necessary adaptations to make the operations on B-valued random variables fit in a framework of dual algebraic structures as in the scalar case.

1. B-valued noncommutative random-variables

1.1. Throughout B will denote a fixed unital algebra over $\mathbb{C}$ (this choice of the base field is inessential). Let A be another unital algebra over $\mathbb{C}$ containing B as a subalgebra (with the same unit) and let $\varphi : A \longrightarrow B$ be a conditional expectation i.e. a linear map such that $\varphi(b_1 a b_2) = b_1 \varphi(a) b_2$ if $b_1, b_2 \in B, a \in A$ and $\varphi(b) = b$ if $b \in B$. An element $a \in A$, will be viewed as a B-valued random variable.

1.2. Definition. Let (A, φ) be as in 1.1 and let $B \subset A_i \subset A$ $(i \in I)$ be subalgebras. The family $(A_i)_{i \in I}$ will be called free if

$$\varphi(a_1 a_2 \ldots a_n) = 0$$

whenever $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \ldots \neq i_n$ and $\varphi(a_j) = 0$ for $1 \leq j \leq n$. A family of subsets $X_i \subset A$ (elements $a_i \in A$) where $i \in I$ will be called free if the family of subalgebras A_i generated by $B \cup X_i$ (respectively $B \cup \{a_i\}$) is free.

Free families of subalgebras arise in the C^*-algebraic context (in which case the conditional expectations are of norm one) from reduced free products with amalgamation (see § 5 in [7]).

1.3. Proposition. Let (A, φ) be as in 1.1 and let $B \subset A_i \subset A$ $(i \in I)$ be subalgebras such that A is generated by $\bigcup_{i \in I} A_i$ and $(A_i)_{i \in I}$ is a free family. Then φ is completely determined by the $\varphi_i = \varphi | A_i$ $(i \in I)$.

Proof. By linearity it is sufficient to prove that we may compute $\varphi(a_1 \dots a_n)$ whenever $a_j \in A_{i_j}$ $(1 \le j \le n)$. We shall proceed by induction on be the least non-negative integer such that $\varphi_{i_j}(a_j)=0$ if $k < j$ and $i_{k+1} \ne i_{k+2} \ne \dots \ne i_n$. If $k=0$ then $\varphi(a_1 \dots a_n)=0$. Assume our assertion has been established up to a certain k. Then for k+1 if $i_k \ne i_{k+1}$ we have

$$\varphi(a_1 \dots a_n) = \varphi(a_1 \dots a_k(\varphi_{i_{k+1}}(a_{k+1})a_{k+2})a_{k+3} \dots a_n) +$$
$$+ \varphi(a_1 \dots a_k a'_{k+1} a_{k+2} \dots a_n) \text{ where } a'_{k+1} = a_{k+1} - \varphi_{i_{k+1}}(a_{k+1}),$$

so that the induction hypothesis applies.

If $i_k = i_{k+1}$ then we write

$$\varphi(a_1 \dots a_n) = \varphi(a_1 \dots a_{k-1}(a_k a_{k+1} - \varphi_{i_k}(a_k a_{k+1}))a_{k+2} \dots a_n) +$$
$$+ (a_1 \dots a_{k-1} \varphi_{i_k}(a_k a_{k+1}) a_{k+2} \dots a_n)$$

which is again a reduction to the induction hypothesis.

Q.E.D.

1.4. The algebra freely generated by B and an intedetrminate X will be denoted by $B\langle X\rangle$. Let (A,φ) be as in 1.1 and $a \in A$ a B-valued random variable. The distribution of a is the conditional expectation $\mu_a : B\langle X\rangle \longrightarrow B$ defined by $\mu_a = \varphi \circ \tau_a$ where $\tau_a : B\langle X\rangle \longrightarrow A$ is the unique homomorphism such that $\tau_a(b)=b$ for $b \in B$ and $\tau_a(X)=a$. Quantities such as $\mu_a(b_0 X b_1 X \dots b_{n-1} X b_n)$ will be called moments. The set of all conditional expectations $\mu : B\langle X\rangle \longrightarrow B$ will be denoted by Σ_B.

1.5. Let $\mathfrak{S}_n$ denote the symmetric group and let

$$S_n(b_1 \dots b_n) = \sum_{\sigma \in \mathfrak{S}_n} b_{\sigma(1)} X b_{\sigma(2)} \dots X b_{\sigma(n)}$$

$S_1(b)=b$ and $S_0=1$. Let further

$$SB\langle X\rangle = \ell.s.\{S_n(b,\dots,b) \mid b\in B, n\geqslant 1\} =$$

$$= \ell.s.\{S_n(b_1,\dots,b_n) \mid b_j\in B, n\geqslant 1, n\geqslant j\geqslant 1\}$$

$$\widetilde{SB}\langle X\rangle = \mathbb{C}X + X(SB\langle X\rangle)X$$

where "$\ell.s.$" denotes the vector space spanned by the given set.

Lemma. We have

$$B(SB\langle X\rangle)B = B + B(\widetilde{SB}\langle X\rangle)B.$$

Proof. The inclusion $\subset$ is obvious. To prove the converse remark that if $n\geqslant k+1, n\geqslant 3$, we have

$$S_n(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots 1) - (n-k)kbXS_{n-2}(\underbrace{b,\dots b}_{(k-1)\text{-times}},1,\dots 1)X - (n-k)kXS_{n-2}(\underbrace{b,\dots,b}_{(k-1)\text{ times}},$$
$$1,\dots,1)Xb - k(k-1)bXS_{n-2}(\underbrace{b,\dots,b}_{(k-2)\text{-times}},1,\dots)Xb =$$
$$\overbrace{=X}^{(n-k)(n-k-1)}S_{n-2}(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1)X.$$

Taking into account that

$$n(n-1)XS_{n-2}(1,\dots,1)X = S_n(1,\dots,1)$$

the preceding recurrence relation applied for $k=1,\dots,n-2$ can be used to prove inductively that for $n\geqslant 3$ and $1\leq k\leq n-2$ we have

$$XS_{n-2}(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1)X \in B(SB\langle X\rangle)B$$

Also

$$X = 2^{-1}S_2(1,1) \in B(SB\langle X\rangle)B.$$

Q.E.D.

The above lemma implies that if $\mathcal{M}\in\Sigma_B$ then

$\mu \mid SB\langle X\rangle$ is completely determied by $\mu \mid \tilde{S}B\langle X\rangle$ and conversely $\mu \mid \tilde{S}B\langle X\rangle$ is completely determined by $\mu \mid SB\langle X\rangle$ We shall denote by $S\Sigma_B$ the set

$$S\Sigma_B = \{ (\mu \mid B(SB\langle X\rangle)B) \mid \mu \in \Sigma_B \}$$

and we shall write $S\mu = \mu \mid B(SB\langle X\rangle)B$ if $\mu \in \Sigma_B$. If $a \in A$ is a random variable, then $S\mu_a$ will be called the symmetric distribution of a and quantities of the type $\mu_a(S(b_1,\ldots,b_n))$ or $\mu_a(XS(b_1,\ldots,b_n)X)$ will be called symmetric moments of a.

1.6. If $\{a_1,a_2\} \subset A$ is a free pair of B-valued random variables then it follows from Proposition 1.3 that $\mu_{a_1+a_2}$ and $\mu_{a_1a_2}$ depend only on μ_{a_1} and μ_{a_2}. For any given $\mu_1,\ldots,\mu_n \in \Sigma_B$ one can find a free family $\{a_1,\ldots,a_n\}$ of random-variables in some (A,φ) such that $\mu_{a_j} = \mu_j$. We shall not give an ad-hoc proof for this here since it will follow from our rasults on the canonical form of a random variable. This implies that there are well-defined operations, $\boxplus$ and $\boxtimes$ on Σ_B such that if $\{a_1,a_2\}$ is a free pair then

$$\mu_{a_1+a_2} = \mu_{a_1} \boxplus \mu_{a_2}$$

$$\mu_{a_1a_2} = \mu_{a_1} \boxtimes \mu_{a_2}$$

This gives two semigroup structures on Σ_B.

2. The algebra A(M)

2.1. Let M be a right B-module and let $\mathcal{X}_n(M)=\mathcal{L}(M^{\otimes n},B)$ be the n-linear B-valued maps of $M\times\cdots\times M$ into B (the $\otimes$ and linearity are over $\mathbb{C}$) and $\mathcal{X}_0(M)=B$. Let further $\mathcal{X}(M)=\bigoplus_{n\geq 0}\mathcal{X}_n(M)$ with its natural right B-module structure. If $\xi\in\mathcal{X}_n(M)$ we define the endomorphism $\lambda(\xi)$ of the right B-module $\mathcal{X}(M)$ by:

$$\lambda(\xi)\eta\in\mathcal{X}_{n+k}(M)$$

$$(\lambda(\xi)\eta)(m_1\otimes\ldots\otimes m_{n+k})=$$

$$=\eta(m_{n+1}\,\xi(m_1\otimes\cdots\otimes m_n)\otimes m_{n+2}\otimes\cdots\otimes m_{n+k})$$

if $\deg\eta=k>0$ where deg refers to the obvious grading of $\mathcal{X}(M)$ and

$$\lambda(\xi)\eta=\xi\eta$$

if $\deg\eta=0$ i.e. $\eta\in B$. We also define $\lambda^*(m)$, where $m\in M$, by:

$$\lambda^*(m)\eta=0 \text{ if } \deg\eta=0$$

$$\deg\lambda^*(m)\eta=\deg\eta-1$$

$$(\lambda^*(m)\eta)(m_1\otimes\ldots\otimes m_{k-1})=\eta(m\otimes m_1\otimes\cdots\otimes m_{k-1})$$

if $\deg\eta=k>0$.

A(M) is the algebra of endomorphisms of the right B-module $\mathcal{X}(M)$ generated by

$$\{\lambda(\xi)\mid\xi\in\mathcal{X}_n(M),n\geq 0\}\cup\{\lambda^*(m)\mid m\in M\}$$

Endowing A(M) with the natural grading corresponding to its action on $\mathcal{X}(M)$ we have $\deg\lambda(\xi)=\deg\xi$ and $\deg\lambda^*(m)=-1$

2.2. It is easy to check that the following equalities hold

$$\lambda(\xi_1)\,\lambda(\xi_2) = \lambda(\lambda(\xi_1)\,\xi_2)$$

$$\lambda^*(m)\ \lambda(\xi) = \lambda(\lambda^*(m)\ \xi) \text{ if } \deg \xi > 0$$

$$\lambda^*(m)\ \lambda(\xi) = \lambda^*(m\ \xi) \text{ if } \deg \xi = 0$$

2.3. We define a linear map

$$\gamma : \Big(\bigoplus_{n\geq 0} \mathfrak{X}_n(M)\Big) \otimes \Big(\bigoplus_{k\geq 0} M^{\otimes k}\Big) \longrightarrow A(M)$$

by

$$\gamma(\xi \otimes (m_1 \otimes \cdots \otimes m_k)) = \lambda(\xi)\ \lambda^*(m_1) \ldots \lambda^*(m_k)$$

<u>Lemma</u>. γ <u>is a bijection</u>.

<u>Proof</u>. Clearly the range of γ contains the $\lambda(\xi)$'s and the $\lambda^*(m)$'s and using the relations 2.2 we easily infer that the range of γ is an algebra, so that γ is onto.

For the injectivity let

$$\alpha = \sum_{k_0 \leq k \leq k_1} \ \sum_{\iota \in I_k} \xi_{\iota,k} \otimes \nu_{\iota,k} \neq 0$$

where $\xi_{\iota,k} \in \bigoplus_{n\geq 0} \mathfrak{X}_n(M)$ and $\nu_{\iota,k} \in M^{\otimes k}$ for $\iota \in I_k$. Since $\alpha \neq 0$ we may assume the $\nu_{\iota,k}$'s are linearly independent and the $\xi_{\iota,k}$'s are non-zero. The fixing $\iota_0 \in I_{k_0}$ there is $\eta \in \mathfrak{X}_{k_0}(M)$ such that $\eta(\nu_{\iota_0,k_0}) = 1 \in B$ and $\eta(\nu_{\iota,k_0}) = 0$ for $\iota \in I_{k_0} \setminus \{\iota_0\}$

Let $\eta' \in \mathfrak{X}_{k_0}(M)$ be defined by

$$\eta'(m_1 \otimes \ldots \otimes m_{k_0}) = \eta(m_{k_0} \otimes \ldots \otimes m_1).$$

We have

$$\gamma(\alpha)\eta' = \gamma\Big(\sum_{\iota \in I_{k_0}} \xi_{\iota,k_0} \otimes \nu_{\iota,k_0}\Big)\eta' =$$

$$= \sum_{\iota \in I_{k_0}} \lambda(\xi_{\iota,k_0})\, \eta(\nu_{\iota,k_0}) = \lambda(\xi_{\iota_0,k_0})1 =$$

$$= \xi_{\iota_0,k_0} \neq 0$$

Q.E.D.

2.4. B identifies via $\lambda : \mathcal{X}_0(M) \simeq B \longrightarrow A(M)$ with a subalgebra of A(M) and there is a linear map $\varepsilon_M : A(M) \to B$ defined by $\varepsilon_M(\gamma(\xi_n \otimes \nu_k)) = 0$ if $n+k > 0$ where $\xi_n \in \mathcal{X}_n(M)$, $\nu_k \in M^{\otimes k}$ and $\varepsilon_M(\gamma(\xi_0 \otimes \eta_0)) =$ $= \gamma(\xi_0 \otimes \eta_0) = \xi_0 \nu_0 \in B$ if $\xi_0 \in \mathcal{X}_0(M) = B$ and $\nu_0 \in M^{\otimes 0} \simeq \mathbb{C}$. It is easily seen that ε_M is a conditional expectation i.e. that $\varepsilon_M(\lambda(b_1)a\ \lambda(b_2)) = b_1\ \varepsilon_M(a)b_2$ and $\varepsilon_M(\lambda(b)) = b$.

2.5. Remark. If $B = \mathbb{C}$ and $M = \mathbb{C}^n$ then $A(\mathbb{C}^n)$ is isomorphic with a certain dense subalgebra of an extension of the C^*-algebra O_n of Cuntz [3] realized on the Fock space for Boltzmann statistics ([6], [5], [4]).

2.6. It will be useful to consider a larger algebra $\overline{A}(M) \supset A(M)$ acting on $\overline{\mathcal{X}}(M) = \prod_{n \geq 0} \mathcal{X}_n(M)$ such that there is a bijection

$$\overline{\gamma} : \left(\prod_{n \geq 0} \mathcal{X}_n(M)\right) \otimes \left(\bigoplus_{k \geq 0} M^{\otimes k}\right) \longrightarrow \overline{A}(M)$$

extending γ and the multiplication of the formal sums which consitute A(M) is also determined by the formulae 2.2. The obvious extension of ε_M to $\overline{A}(M)$ will be denoted also by ε_M.

We have for $T \in \overline{A}(M)$

$$\varepsilon_M(T) = (T\,1)_o$$

where $1 \in B = \mathcal{X}_o(M) \subset \overline{\mathcal{X}}(M)$ and $(.)_o$ denotes the component of degree zero. Note also that along the same lines as in the proof of Lemma 2.3 it is easy to show that the representation of $A(M)$ on $\overline{\mathcal{X}}(M)$ is faithful.

2,7, If $M = M_1 \oplus M_2$ there are injections

$$\chi_j : \Big(\prod_{n\geq 0} \mathcal{X}_n(M_j)\Big) \otimes \Big(\bigoplus_{k\geq 0} M_j^{\otimes k}\Big) \longrightarrow \Big(\prod_{n\geq 0} \mathcal{X}_n(M)\Big) \otimes \Big(\bigoplus_{k\geq 0} M^{\otimes k}\Big)$$

given by

$$\chi_j\big((\xi_n)_{n\geq 0} \otimes \nu_k\big) = (\xi_n \circ pr_j^{\otimes n}) \otimes (i_j^{\otimes k}\, \nu_k)$$

where $i_j : M_j \hookrightarrow M$ are the natural inclusions and $pr_j : M \longrightarrow M_j$ the projections onto the two summands and $\xi_o \circ pr_j^{\otimes 0}$ means just ξ_o. Since the relations 2.2 determine the multiplication in the algebras $\overline{A}(.)$ it is easy to check that the maps $h_j : \overline{A}(M_j) \longrightarrow \overline{A}(M)$ such that $h_j \circ \overline{\gamma} = \overline{\gamma} \circ \chi_j$ are homomorphisms. Moreover we have $h_j(\lambda(b)) = \lambda(b)$ for $b \in B = \mathcal{X}_0(M_j) = \mathcal{X}_o(M)$ and $\varepsilon_M \circ h_j = \varepsilon_{M_j}$.

<u>Proposition. If</u> $M = M_1 \oplus M_2$ <u>then with</u> h_1, h_2 <u>as above, the pair of subalgebras</u> $(h_j(\overline{A}(M_j)))_{j=1,2}$ <u>is</u> B-<u>free in</u> $(\overline{A}(M), \varepsilon_M)$.

<u>Proof</u>. Write

$$\overline{\mathcal{X}}(M) = \Gamma_1 \oplus \Gamma_2 \oplus B$$

where

$$\Gamma_j = \prod_{n\geq 1} \mathcal{L}(M_j \otimes M^{\otimes(n-1)}, B)$$

with $\mathcal{L}(M_j \otimes M^{\otimes(n-1)}, B)$ identified with a subspace of $\mathcal{L}(M^{\otimes n}, B)$ via $\eta_n \rightsquigarrow \eta_n \circ (pr_j \otimes id_M \otimes \cdots \otimes id_M)$.

If $T \in h_1(\overline{A}(M_1))$ and $\varepsilon_M(T) = 0$ then $T(\Gamma_2 \oplus B) \subset \Gamma_1$. Also the analogue of this with 1 and 2 interchanged holds. This easily implies our assertion. For instance if $T_j \in h_1(\overline{A}(M_1))$ and $S_j \in h_2(\overline{A}(M_j))$ and $\varepsilon_M(T_j) = \varepsilon_M(S_j) = 0$ then $T_1 1 \in \Gamma_1$, $S_1 T_1 1 \in \Gamma_2$ and continuing in this way we get $S_n T_n \ldots S_1 T_1 1 \in \Gamma_2$ so that

$$\varepsilon_M(S_n T_n \ldots S_1 T_1) = 0$$

Q.E.D.

2.8. If $M = B^m$ we shall denote $\overline{A}(M)$ by $\overline{A}(m)$ and ε_M by ε_m ,

3. The canonical form

3.1. Elements $a \in \bar{A}(1)$ of the form

$$a = \lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n)$$

where $\xi_n \in \mathcal{X}_n(B)$, will be called canonical.

Proposition. Given a distribution $\mu \in \sum_B$ there is a unique canonical element

$$a = \lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n)$$

such that $\mu_a = \mu$

Proof. We have $\mu_a(X) = \varepsilon_1(a) = \xi_0$ so that we must put $\xi_0 = \mu(X)$. If $n > 0$ we have

$$\varepsilon_1(a\ \lambda(b_1)a\ \lambda(b_2)\dots a\ \lambda(b_n)a) =$$

$$= \varepsilon_1(\lambda^*(1)\ \lambda(b_1)\lambda^*(1)\ \lambda(b_2)\dots\ \lambda^*(1)\ \lambda(b_n)\ \lambda(\xi_n)) +$$

$+E_n(\xi_0, \dots, \xi_{n-1})(b_1 \otimes \dots \otimes b_n)$ where $E_n(\xi_0, \dots, \xi_{n-1}) \in \mathcal{L}(B^{\otimes n}, B)$ depends only on $\xi_0, \dots, \xi_{n-1}$. Remark that

$$\varepsilon_1(\lambda^*(1)\ \lambda(b_1)\dots\ \lambda^*(1)\ \lambda(b_n)\ \lambda(\xi_n)) = \xi_n(b_n \otimes \dots \otimes b_1).$$

We infer that ξ_n satisfies $\xi_n(b_n \otimes \dots \otimes b_1) =$
$= \mu(Xb_1Xb_2\dots Xb_nX) - E_n(\xi_0, \dots, \xi_{n-1})(b_1 \otimes \dots \otimes b_n)$ which determines ξ_n inductively.

Q.E.D.

The canonical element a in the above proposition will be called the canonical form of a random variable with distribution μ and we shall write $\xi_n = R_{n+1}(\mu)$.

3.2. Proposition. *Let*

$$a_k = \lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_{n,k})$$

$k=1,2,3$ *be canonical elements. Then* $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2}$ *if and only if*

$$\xi_{n,3} = \xi_{n,1} + \xi_{n,2}$$

for all $n \geq 0$.

Proof. In view of the uniqueness of the canonical form it will be sufficient to prove that if $\xi_{n,3} = \xi_{n,1} + \xi_{n,2}$ for all $n \geq 0$ then $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2}$. Passing to $\bar{A}(2)$ we have in view of 2.7 that $h_1(a_1)+h_2(a_2)$ has distribution $\mu_{a_1} \boxplus \mu_{a_2}$.

Let

$$Y = h_1(a_1) + h_1(a_2) = \lambda^*(1 \oplus 1) + \sum_{n\geq 0} \lambda(\xi_{n,1} \circ pr_1 +$$

$$+ \xi_{n,2} \circ pr_2)$$

Expanding

$$\varepsilon_2(Y \lambda(b_1) Y \ldots Y \lambda(b_n) Y) \text{ and}$$

$$\varepsilon_1(a_3 \lambda(b_1) a_3 \ldots a_3 \lambda(b_n) a_3)$$

our assertion is obtained from the following remark. Let

$$\varepsilon_2(S_1 \lambda(b_1) S_2 \ldots S_n \lambda(b_n) S_{n+1})$$

where each S_j is an element of one of the following forms

$$\lambda^*(1 \oplus 1), \quad \lambda(\beta_n \circ pr_1^{\otimes n}) \text{ or } \lambda(\beta_n \circ pr_2^{\otimes n})$$

Then replacing S_j by S'_j where S'_j is obtained from S_j by replacing $\lambda^*(1 \oplus 1)$ by $\lambda^*(1)$, $\lambda(\beta_n \circ pr_k^{\otimes n})$ $(k=1,2)$

by $\lambda(\beta_n)$ it is easy to see that

$$\varepsilon_1(S'_1 \ \lambda(b_1)S'_2 \ldots S'_n \ \lambda(b_n)S'_{n+1}) =$$

$$= \varepsilon_2(S_1 \ \lambda(b_1)S_2 \ldots S_n \ \lambda(b_n)S_{n+1}).$$

Q.E.D.

Thus we have proved that

$$R_n(\mu_1 \boxplus \mu_2) = R_n(\mu_1) + R_n(\mu_2)$$

for all $n \geqslant 1$ and $\mu_j \in \Sigma_B$, $j=1.2.$

4. The differential equation for $\boxplus$

4.1. Lemma. *Let $T \in \bar{A}(1)$ and let $\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n) \in \bar{A}(1)$ be a canonical element. Then if $Y(\alpha) = h_1(\lambda^*(1) + \alpha \sum_{n\geq 0} \lambda(\xi_n))$, we have*

$$\frac{d}{d\alpha} \varepsilon_2(\lambda(b))(Y(\alpha) + h_2(T))\lambda(b))^m)\Big|_{\alpha=0} =$$

$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\dots+k_{n+1}=m-n-1 \\ k_0 \geq 0, \dots, k_{n+1} \geq 0}} \varepsilon_1(\lambda(b)(T\lambda(b))^{k_0} \lambda(\xi_n((b(\varepsilon_1((T\lambda(b))^{k_n}))) \otimes$$

$$\dots \otimes (b\xi_1((T\lambda(b))^{k_1})))\lambda(b)(T\lambda(b))^{k_{n+1}})$$ *where* $\alpha \in \mathbb{C}$ *and* $b \in B$.

Proof. The expression the derivative of which must be computed is a polynomial in $\alpha \in \mathbb{C}$ with coefficients in B, which shows also the sense in which this derivative should be unterstood.

We have

$$Y(\alpha) = \lambda^*(1 \oplus 0) + \alpha \sum_{n\geq 0} \lambda(\xi'_n) \quad \text{where } \xi'_n = \xi_n \circ pr_1^{\otimes n}.$$ Let $\eta_n = \lambda(\xi'_n)b$, $S = h_2(T\lambda(b))$ and $X(\alpha) = \lambda^*(b \oplus 0) + \alpha \sum_{n\geq 0} \lambda(\eta_n)$.

We have

$$\varepsilon_2(\lambda(b)((Y(\alpha) + h_2(T))\lambda(b))^m) =$$
$$= \varepsilon_2(\lambda(b)(S + X(\alpha))^m) \text{ and hence}$$

$$\frac{d}{d\alpha} \varepsilon_2(\lambda(b)((Y(\alpha) + h_2(T))\lambda(b))^m\Big|_{\alpha=0} =$$
$$= \sum_{j=0}^{m-1} \varepsilon_2(\lambda(b)(S + \lambda^*(b \oplus 0))^j \sum_{n\geq 0} \lambda(\eta_n)(S + \lambda^*(b \oplus 0))^{m-1-j})$$

For the computation which follows one should keep in mind that $\varepsilon_M(R) = (R1)_0$ and the proof of Proposition 2.7. We have

$$\sum_{j=0}^{m-1} \varepsilon_2(\lambda(b)(S+\lambda^*(b\oplus 0))^j \sum_{n\geq 0}\lambda(\eta_n)(S+\lambda^*(b\oplus 0))^{m-1-j}) =$$

$$= \sum_{j=0}^{m-1}\sum_{n=0}^{j}\sum_{\substack{n+k_0+\dots+k_n=j\\ k_0\geq 0,\dots,k_n\geq 0}} b\,\varepsilon_2(S^{k_0}\lambda^*(b\oplus 0)\dots S^{k_{n-1}}\lambda^*(b\oplus 0)S^{k_n}\lambda(\eta_n)S^{m-1-j}) =$$

$$= \sum_{n=0}^{m-1}\sum_{\substack{k_0+\dots+k_{n+1}=m-n-1\\ k_0\geq 0,\dots,k_{n+1}\geq 0}} b\,\varepsilon_2(S^{k_0}\lambda^*(b\oplus 0)\dots S^{k_{n-1}}\lambda^*(b\oplus 0)S^{k_n}\lambda(\eta_n)S^{k_{n+1}}) =$$

$$= \sum_{n=0}^{m-1}\sum_{\substack{k_0+\dots+k_{n+1}=m-n-1\\ k_0\geq 0,\dots,k_{n+1}\geq 0}} b\,\varepsilon_2(S^{k_0}\lambda(\eta_n(((b\oplus 0)\,\xi_2(S^{k_n})\otimes\dots$$

$$\dots\otimes((b\oplus 0)\,\varepsilon_2(S^{k_1})))S^{k_{n+1}})) =$$

$$= \sum_{n=0}^{m-1}\sum_{\substack{k_0+k_{n+1}=m-n-1\\ k_0\geq 0,\dots,k_{n+1}\geq 0}} \varepsilon_1(\lambda(b)(T\lambda(b))^{k_0}\lambda(\xi_n((b\,\varepsilon_1((T\lambda(b))^{k_n})\otimes$$

$$\dots\otimes(b\,\varepsilon_1((T\lambda(b))^{k_1}))))\lambda(b)(T\lambda(b))^{k_{n+1}}).$$

Q.E.D.

4.2. It is easy to see that the same computation yields the more general formula

$$\frac{d}{d\alpha}\,\varepsilon_2(\lambda(b_0).(Y(\alpha)+h_2(T))\lambda(b_1)\dots\lambda(b_{m-1})(Y(\alpha)+h_2(T))\lambda(b_m))\Big|_{\alpha=0} =$$

$$= \sum_{n=0}^{m-1}\sum_{\substack{k_0+\dots+k_{n+1}=m-n-1\\ k_0\geq 0\dots k_{n+1}\geq 0}} \varepsilon_1(\lambda(b_0)T\lambda(b_1)\dots T\lambda(b_{k_0})$$

$$\lambda(\xi_n(\varepsilon_1(\lambda(b_{k_0+\dots+k_{n-1}+n})T\dots\lambda(b_{k_0+\dots+k_n+n-1})T\lambda(b_{k_0+\dots+k_n+n}))\otimes$$

$$\dots\otimes\varepsilon_1(\lambda(b_{k_0+1})T\dots\lambda(b_{k_0+k_1+1})T\ \lambda(b_{k_0+k_1+2})))$$

$$\lambda(b_{k_0+\dots+k_n+n+1})T\dots\lambda(b_{k_0+\dots+k_{n+1}+n})T\ \lambda(b_{k_0+\dots+k_{n+1}+n+1}))$$

where $b_0,\dots,b_m \in B$.

4.3. Before passing to the differential equations we have to discuss certain formal series which are the analogue of formal power series when maps $\mathbb{C} \to \mathbb{C}$ are replaced by maps $B \to B$.

Let $S\mathcal{X}_n(B) \subset \mathcal{X}_n(B)$ be the subspace of symmetric n-linear maps i.e. $\xi_n(b_1 \otimes \dots \otimes b_n) = \xi_n(b_{\sigma(1)} \otimes \dots \otimes b_{\sigma(n)})$ for all $\sigma \in \mathfrak{S}_n$. If $\eta \in \mathcal{X}_n(B)$ we denote by $S\eta \in S\mathcal{X}_n(B)$ the element such that $S\eta(b^{\otimes n}) = \eta(b^{\otimes n})$. Elements of $S\mathcal{X}(B) = \prod_{n\geq 0} S\mathcal{X}_n(B)$ will be written

$$\sum_{n\geq 0} \xi_n$$

$S\mathcal{X}(B)$ is a ring with multiplication such that $(\xi_m\xi_n)(b^{\otimes(m+n)}) = \xi_m(b^{\otimes m})\,\xi_n(b^{\otimes n})$. $S\mathcal{X}(B)$ has a natural filtering given by the powers of the ideal formed by elements of the form $\sum_{n\geq 1} \xi_n$. If $\varphi = \sum_{n\geq 0} \xi_n$ and $\psi = \sum_{n\geq 1} \eta_n$ then the composition $\varphi \circ \psi$ is easily seen to be well defined.

The differential of $\varphi \in S\mathcal{X}(B)$ is an element of $\prod_{n\geq 0} S\mathcal{L}(B^{\otimes n}, \mathcal{L}(B,B))$ where $S\mathcal{L}(B^{\otimes n}, E)$ denotes the symmetric n-linear E-valued maps. If $\varphi = \sum_{n\geq 0} \xi_n$ then the differential is

$$D\varphi = \sum_{n\geq 1} D\xi_n$$

where $D\xi_n \in \mathcal{L}(B^{\otimes(n-1)}, \mathcal{L}(B,B))$ is such that $(D\xi_n(b^{\otimes(n-1)}))(\beta) =$

$$= \sum_{k=0}^{n-1} \xi_n(\underbrace{b\otimes\ldots\otimes b}_{k\text{-times}} \otimes \beta \otimes b \otimes \ldots \otimes b)$$

We shall write formally also $\varphi(b) = \sum_{n\geq 0} \xi_n(b^{\otimes n})$, $(D\varphi)(b)[\beta]$ or $(D_b\varphi)(b)[\beta]$ and $\varphi(\psi(b))$.

If $\varphi = \sum_{n\geq 0} \xi_n$ and the ξ_n's depend on a parameter then the derivative of φ with respect to this parameter is meant component wise.

4.4. If $\mu \in S\Sigma_B$ we consider the formal series

$$G_\mu(b) = \sum_{n\geq 0} (b(Xb)^n).$$

It will be also useful to consider

$$\Gamma_\mu(b) = \sum_{n\geq 0} \mu((Xb)^n X)$$

so that

$$G_\mu(b) = b + b\Gamma_\mu(b)b.$$

If $\mu \in \Sigma_B$ we shall write also G_μ for $G_{S\mu}$ and Γ_μ for $\Gamma_{S\mu}$. Also if μ is a distribution μ_T we shall write G_T and Γ_T instead of G_{μ_T} and Γ_{μ_T}.

4.5. <u>Proposition</u>. <u>Let</u> $T \in \bar{A}(1)$ <u>and let</u> $Y(\alpha) = h_1(\lambda^*(1) + \alpha\sum_{n\geq 0}\lambda(\xi_n)) \in h_1(\bar{A}(1))$. <u>Let</u> $T(\alpha) = Y(\alpha) + h_2(T)$. <u>Then we have</u> $G_{T(0)} = G_T$ <u>and</u>

$$\frac{\partial}{\partial\alpha} G_{T(\alpha)}(b) = (D_b G_{T(\alpha)})(b)\left[b\, \boxminus(G_{T(\alpha)}(b))b\right], \text{ \underline{where}}$$

$$\boxminus(b) = \sum_{n\geq 0} \xi_n(b^{\otimes n})$$

Proof. If $\alpha=0$ the equality of the terms which are of degree m+1 in b in the differential equation is precisely the equality established in Lemma 4.1. The general case can be reduced to the case $\alpha=0$ since in view of 2.7 and 3.2 we have

$$\mu_{T(\alpha)}=\mu_T \boxplus \mu_{Y(\alpha)}=\mu_{T(\alpha_0)}\boxplus\mu_{Y(\alpha-\alpha_0)}$$

Q.E.D.

4.6. <u>Corollary. Let</u> $\mu\in\Sigma_B$ <u>and</u> $a=\lambda^*(1)+\sum_{n\geq 0}\lambda(\xi_n)$ <u>be the canonical element with distribution</u> μ <u>. Then the</u> $(S\xi_n)_{n\geq 0}$ <u>depend only on</u> $S\mu$ <u>and conversely</u> $S\mu$ <u>depends only on the</u> $(S\xi_n)_{n\geq 0}$. <u>In particular if</u> $\mu_1,\mu_2\in\Sigma_B$ <u>then</u> $S(\mu_1\boxplus\mu_2)$ <u>is completely determined by</u> $S\mu_1$ <u>and</u> $S\mu_2$.

Proof. Let $a(\alpha)=\lambda^*(1)+\alpha\sum_{n\geq 0}\lambda(\xi_n)$.

We have

$$\frac{d}{d\alpha}\varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^m)=bS\xi_{m-1}(b\otimes\ldots\otimes b)b+$$

$$+F(S\xi_j,\ \varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^j)b,0\leq j\leq m-2)$$

where F is a "polynomial" of the quantities on which it depends. These differential equations with initial condition

$\varepsilon_1(\lambda(b)(a(0)\lambda(b))^m)=0$ if $m\geq 1$ can be solved recurrently and we obtain that

$$\varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^m)=$$

$$=\alpha bS\xi_{m-1}(b\otimes\ldots\otimes b)b+$$

$$+P(\alpha,b,S\xi_j,0\leq j\leq m-2)$$

where P is "polynomials". Taking $\alpha=1$ we see that $S\mu$ completely determines the $(S\xi_n)_{n\geq 0}$ and also that conversely the $(S\xi_n)_{n\geq 0}$ completely determine $S\mu$. The assertion concerning $S(\mu_1 \boxplus \mu_2)$ follows now from 3.2.

Q.E.D.

4.7. The differential equation in 4.5 immeditely implies the following fact: if $\mu_1, \mu_2 \in S\Sigma_B$ and $\mu(\alpha) \in S\Sigma_B$ is such that $SR_n(\mu(\alpha)) = SR_n(\mu_1) + \alpha SR_n(\mu_2)$ then

$$\frac{\partial}{\partial\alpha} G_{\mu(\alpha)}(b) = (D_b G_{\mu(\alpha)})(b)\left[b\,\Xi\,(G_{\mu(\alpha)}(b))b\right] \text{ where } \Xi = \sum_{n\geq 0}\xi_n$$

where $\xi_n = SR_{n+1}(\mu_2)$. Interpreting this equation as a system of ordinary differential equations as in 4.6 we see that with the initial condition $G_{\mu(0)} = G_{\mu_1}$ we have $G_{\mu(1)} = G_{\mu_1 \boxplus \mu_2}$ which is completely determined by the differential equation.

4.8. We shall now assume B is a Banach algebra and Γ_μ and hence G_μ is an analytic function in some neighborhood of $0 \in B$. This implies that the symmetric moments of μ viewed as n-linear maps $B^n \longrightarrow B$ are continuous and the formal series defining G_μ is absolutely convergent in some neighborhood of 0. For instance if T is a B-valued random-variable $T \in A$ where A is also a Banach algebra, $A \supset B$ with a continuous conditional expectation $\varphi : A \longrightarrow B$ then $G_T(b) = \sum_{n\geq 0} \varphi(b(Tb)^n) = \varphi(b(1-Tb)^{-1})$ satisfies these assumptions.

For the lemma which follows we shall denote by $\mathcal{M}$ the set of germs at $0 \in B$ of analytic B-valued maps and we shall use the notation F^{-1} only for multiplicative inverses, not for inverses with respect to composition.

Lemma. Let $\Gamma, G \in \mathcal{M}$ be such that $G(b)=b+b\,\Gamma(b)b$ near 0.

(i) If $K \in \mathcal{M}$ is such that $K(G(b))=G(K(b))=b$ near 0, then there is $Q \in \mathcal{M}$ such that $K(b)=b+bQ(b)b$.

(ii) There is $R \in \mathcal{M}$ such that for some neighborhood V of $0 \in B$ we have $(K(b))^{-1}=b^{-1}+R(b)$ if $b \in V \cap GL(B)$. R is unique.

Proof. (i) If $\|b\|$ is small enough, we have

$$b=G(b)(1+\Gamma(b)b)^{-1}=G(b)(1+\Gamma(K(G(b)))K(G(b)))^{-1}$$

so that there is $H \in \mathcal{M}$ for which

$$K(b)=bH(b)$$

Similarly there is $J \in \mathcal{M}$ so that

$$K(b)=J(b)b$$

We have

$$b= G(b)-b\,\Gamma(b)b=G(b)-G(b)H(G(b))\,\Gamma(K(G(b)))J(G(b))G(b)$$ so

$Q(b)=-H(b)\,\Gamma(K(b))J(b)$ will do.

(ii) Choosing V small enough, if $b \in V \cap GL(B)$ we have

$$K(b)^{-1}=b^{-1}(1+bQ(b))^{-1}=$$

$$=b^{-1}Q(b)(1+bQ(b))^{-1}.$$

The uniqueness of R is easily seen from the fact that $R|(V \cap GL(B))$ determines the germ of R at 0.

Q.E.D.

4.9. Theorem. Assume B is a Banach algebra and $\mu \in S\Sigma_B$ is analytic in some neighborhood of $0 \in B$. Let K and R be germs of B-valued analytic functions at $0 \in B$ such that

$$K(G_\mu(b))=G_\mu(K(b))=b$$

<u>and</u>

$$K(b)^{-1}=b^{-1}+R(b)$$

<u>for</u> $b\in GL(B)$ <u>in some neighborhood of</u> 0.<u>Then we have</u>

$$R(b)=\sum_{n\geq 0} SR_{n+1}(\mu)(b^{\otimes n})$$

<u>where the</u> $SR_{n+1}(\mu)$ <u>are given by the canonical element with distribution</u> μ .

<u>Proof.</u>Let

$$K(\alpha,b)=(b^{-1}+\alpha R(b))^{-1}=(b^{-1}(1+\alpha bR(b)))^{-1}=(1+\alpha bR(b))^{-1}b$$

which for $0\leq\alpha\leq 1$ makes sense in some fixed neighborhbod of 0,the last equality making the invertibility of b superflous. There is a neighborhood of 0 independent of $0\leq\alpha\leq 1$ for which $K(\alpha,b)$ has an inverse (with respect to composition) $K(\alpha,G(\alpha,b))=G(\alpha,K(\alpha,b))=b$.

We have

$$\frac{d}{d\alpha}G(\alpha,K(\alpha,b))=0$$

which with $b_1=K(\alpha,b)$ gives

$$0=\frac{\partial}{\partial\alpha}G(\alpha,b_1)+(D_bG)(\alpha,b_1)\left[-b_1R(b)b_1\right]=$$
$$=\frac{\partial}{\partial\alpha}G(\alpha,b_1)-(D_bG)(\alpha,b_1)\left[b_1R(G(\alpha,b_1))b_1\right].$$

Moreover $G(0,b)=b$.Thus defining $\mu(\alpha)\in S\Sigma_B$ by $G_{\mu(\alpha)}(b)=G(\alpha,b)$ we have that $\frac{\partial}{\partial\alpha}G_{\mu(\alpha)}(b)=$ $=(D_bG_{\mu(\alpha)})(b)\left[bR(G_{\mu(\alpha)}(b))\,b\right]$ and $\mu(0)$ is the distribution of the 0 random-variable.In view of 4.7 this implies that G is the generating series for a symmetric distribution for which the corresponding symmetric parts of the components of the corresponding canonical element yield the series R(b). Thus we have

$$R(b)= \sum_{n\geq 0} SR_{n+1}(\mu(1))(b^{\otimes n}).$$

On the other hand $G_{\mu(1)} = G_{\mu}$ so that $\mu(1)=\mu$

Q.E.D.

4.1o. We have chosen in 4.8 and 4.9 to work in the Banach algebra context where we work with genuine functions since this is the situation for the applications to computations of spectra. On the other hand the reader will not find it difficult to transpose Lemma 4.8 and Theorem 4.9 in the framework of formal series and general B where similar statements hold.

4.11. Thus in the B-valued case the computation of $\mu_1 \boxplus \mu_2$, $\mu_j \in S\Sigma_B$ is done as in the scalar case: one forms G_{μ_j}, then the inverse K_{μ_j} then the multiplicative inverses $b^{-1}+R_{\mu_j}$. Then $R_{\mu_1\boxplus\mu_2} = R_{\mu_1} + R_{\mu_2}$ and from $R_{\mu_1\boxplus\mu_2}$ one goes back to $K_{\mu_1\boxplus\mu_2}$ and $G_{\mu_1\boxplus\mu_2}$.

5. The differential equation for $\boxtimes$

5.1. Lemma. *Let* $T_1,\dots,T_m \in \bar{A}(1)$, *let* $a(\tau) = \lambda^*(1) + \lambda(1) + \tau \sum_{n\geq 1} \lambda(\xi_n) \in \bar{A}(1)$ *be a canonical element and let* $Y(\tau) = h_1(a(\tau))$. *Then we have*

$$\frac{d}{d\tau} \varepsilon_2 (Y(\tau)h_2(T_1)Y(\tau)h_2(T_2)\dots Y(\tau)h_2(T_m))\Big|_{\tau=0} =$$

$$= \sum_{p\geq 1} \sum_{\substack{j_0+\dots+j_{p+1}=m \\ j_0 \geq 0 \\ j_1\geq 1,\dots,j_{p+1}\geq 1}} \varepsilon_1 (T_1\dots T_{j_0} \lambda(\xi_p (\cdot \varepsilon_1(T_{j_0+\dots+j_{p-1}+1} \dots T_{j_0+\dots+j_p}) \otimes \dots$$

$$\dots \otimes \varepsilon_1 (T_{j_0+1}\dots T_{j_0+j_1})))T_{j_0+\dots j_p+1}\dots T_{j_0+\dots+j_{p+1}}).$$

Proof. For the computation it will be convenient to put $\xi'_p = \xi_p \circ pr_1^{\otimes p}$ so that $\lambda(\xi'_p) = h_1(\lambda(\xi_p))$ and $S_k = h_2(T_k)$.

We have

$$\frac{d}{d\tau} \varepsilon_2(Y(\tau)S_1\dots Y(\tau)S_m)\Big|_{\tau=0} =$$

$$= \sum_{j=1}^{m} \varepsilon_2(Y(0)S_1\dots Y(0)S_{m-j} \sum_{p\geq 1} \lambda(\xi'_p) S_{m-j+1} Y(0)S_{m-j+2}\dots Y(0)S_m) =$$

$$= \sum_{j=1}^{m} \varepsilon_2((\lambda^*(1\oplus 0) + \lambda(1))S_1\dots(\lambda^*(1\oplus 0) + \lambda(1))S_{m-j} \sum_{p\geq 1} \lambda(\xi'_p) S_{m-j+1}\dots S_m) =$$

$$= \sum_{j=1}^{m} \sum_{p\geq 1} \sum_{\substack{j_0+\dots+j_p=m-j \\ j_0\geq 0 \\ j_1\geq 1,\dots,j_p\geq 1}} \varepsilon_2 (S_1\dots S_{j_0} \lambda^*(1\oplus 0)S_{j_0+1}\dots S_{j_0+j_1} \lambda^*(1\oplus 0)\dots$$

$$\lambda^*(1\oplus 0)S_{j_0+\dots+j_{p-1}+1}\dots S_{j_0+\dots+j_p} \lambda(\xi'_p) S_{j_0+\dots+j_p+1}\dots S_m) =$$

$$= \sum_{p\geq 1} \sum_{\substack{j_0+\dots+j_{p+1}=m \\ j_0\geq 0 \\ j_1\geq 1,\dots,j_{p+1}\geq 1}} \varepsilon_2(S_1\dots S_{j_0}\lambda(\xi_p(\varepsilon_2(S_{j_0+\dots+j_{p-1}+1}\dots S_{j_0+\dots+j_p})\otimes$$

$$\dots\otimes\varepsilon_2(S_{j_0+1}\dots S_{j_0+j_1})))S_{j_0+\dots+j_p+1}\dots S_m)=$$

$$= \sum_{p\geq 1} \sum_{\substack{j_0+\dots+j_{p+1}=m \\ j_0\geq 0 \\ j_1\geq 1,\dots,j_{p+1}\geq 1}} \varepsilon_1(T_1\dots T_{j_0}\lambda(\xi_p(\varepsilon_1(T_{j_0+\dots+j_{p-1}+1}\dots T_{j_0+\dots+j_p})\otimes$$

$$\dots\otimes\varepsilon_1(T_{j_0+1}\dots T_{j_0+j_1})))T_{j_0+\dots+j_p+1}\dots T_{j_0+\dots j_{p+1}}).$$

Q.E.D.

5.2. Corollary. *Let* $T\in\bar{A}(1)$, *let*

$$a(\tau)=\lambda^*(1)+\lambda(1)+\tau\sum_{n\geq 1}\lambda(\xi_n)\in\bar{A}(1)$$

be a canonical element and let $Y(\tau)=h_1(a(\tau))$. *Then we have*

$$\frac{d}{d\tau}\varepsilon_2((\lambda(b)Y(\tau)h_2(T))^m)\Big|_{\tau=0}=$$

$$=\sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_{p+1}=m \\ j_0\geq 0 \\ j_1\geq 1,\dots,j_{p+1}\geq 1}}\varepsilon_1\Big(\lambda(b)(T\lambda(b))^{j_0}\lambda(\xi_p(\varepsilon_1((T\lambda(b))^{j_p})\otimes\dots$$

$$\otimes\varepsilon_1((T\lambda(b))^{j_1}))T(\lambda(b)T)^{j_{p+1}-1})$$

where $b\in B$ *and* $\tau\in\mathbb{C}$.

5.3. Proposition. *Let* $T\in\bar{A}(1)$ *and let*

$$Y(\tau)=h_1(\lambda^*(1)+\lambda(1)+\tau\sum_{n\geq 1}\lambda(\xi_n))\in\overline{A}(1).$$

Let $T(\tau)=Y(\tau)h_2(T)$. Then we have

$$\frac{d}{d\tau}(b\ \Gamma_{T(\tau)}(b))\Big|_{\tau=0}=(D_b(b\ \Gamma_T))(b)\ [\ b(\textcircled{H}(\Gamma_T(b)b)$$

where $\textcircled{H}(b)=\sum_{n\geq 1}\xi_n(b^{\otimes n})$

Proof. The proposition is obtained from Corollary 5.2 by looking at the terms which are degree m in b.

Q.E.D.

Note: Section 5 is incomplete in this preliminary version of our article. In the complete version it will be shown that the differential equations provide a way for studying the operation $\boxtimes$. In particular we will show the existence of a map, called free exponential

$$\text{fexp}: \prod_{n\geq 1} \mathcal{X}_n(B) \to \Sigma_{1,B}$$

where $\Sigma_{1,B} = \{\mu \in \Sigma_B \mid \mu(X) = 1\}$. This map is the exponential map for the infinite dimensional Lie group $(\Sigma_{1,B}, \boxtimes)$. It will be shown that fexp is bijective. This map will appear as a solution at $z = 1$ to the system of differential equations

$$\frac{d}{d}\mu_z(Xb_1 \ldots Xn_m) = \sum_{p\geq 1} \sum_{\substack{j_0+\ldots+j_{p+1}=m \\ j_1\geq 1, \ldots, j_{p+1}\geq 1}} \mu_z(Xb_1 \ldots Xb_{j_0} \xi_p(\mu_z(Xb_{j_0+\ldots+j_{p-1}+1} \ldots Xb_{j_0+\ldots+j_p}) \otimes \ldots$$

$$\ldots \otimes \mu_z(Xb_{j_0+1} \ldots Xb_{j_0+j_1}))) Xb_{j_0+\ldots+j_p+1} \ldots Xb_{j_0+\ldots+j_{p+1}})$$

derived from Lemma 5.1.

It will be also shown that the symmetric part of $\text{fexp}((\xi_n)_{n\geq 1})$ depends only on the symmetric parts of the ξ_n's and also conversely the symmetric part of the ξ_n's is uniquely determined by the symmetric parts of $\text{fexp}((\xi_n)_{n\geq 1})$. The connecting equation for the symmetric parts is obtained from Corollary 5.3. This equation with initial condition a gi $\mathfrak{h}\Gamma_\mu$, $\mu \in \Sigma_{1,B}$ will give for $z = 1$, $\text{fexp}\,\xi \boxtimes \mu$.

6. Computation of spectra

As we mentioned in the introduction the results concerning the operation $\boxplus$ provide a method for computing spectra of left convolution operators in $\ell^2(G)$ where $G=\mathbb{Z}*\mathbb{Z}$ is the free group on two generators g_1, g_2. Actually the same ideas provide a method for dealing with more complicated groups obtained by taking free products with amalgamation. We shall however stick here to the case of $\mathbb{Z}*\mathbb{Z}$ since we think this particular example will suffice to explain our approach.

6.1 Let $Y=\sum_{g\in G} c_g \lambda(g)$ where $c_g \in \mathbb{C}$, $c_g \neq 0$ only for finitely many $g\in G$ and where λ is the left regular representation of G on $\ell^2(G)$. To compute the spectrum of Y we have to provide a method for deciding whether $Y-zI$ is invertible for a given $z\in\mathbb{C}$. Since $Y-zI$ is of the same form as Y we may state our problem as deciding whether Y is invertible.

6.2. We recall one of the standard algebraic tricks with matrices.

Let A be a ring and let $c_0,\dots,c_n$ and $u_1,\dots,u_n$ be elements in A. Let further $y=c_0+\sum_{k=1}^{n} c_k u_k\cdots u_1$ and $y_p=c_p+\sum_{k=p+1}^{n} c_k u_k\cdots u_{p+1}$.

Then in the ring $\mathcal{M}_{n+1}(A)$ of $(n+1)\times(n+1)$ matrices over A we have:

$$\begin{pmatrix} 1 & y_1 & \cdots & y_n \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}\begin{pmatrix} y & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & 0 \\ -u_1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ -u_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix}$$

This identity shows that y is invertible if and only if the matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ -u_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix}$$

is invertible.

6.3. Let $C_r^*(G)$ be the reduced C^*-algebra of G. An application of 6.2 to the element Y in 6.1 with $A=C_r^*(G)$, $y=Y$, $c_j \in \mathbb{C}$, $u_j \in \{\lambda(g_1), \lambda(g_2), \lambda(g_1^{-1}), \lambda(g_2^{-1})\}$ shows that given Y there is $q \in \mathbb{N}$ and there are $\alpha_j, \beta_j \in \mathcal{M}_q(\mathbb{C})$, $(j=\pm 1)$ and $\gamma \in \mathcal{M}_q(\mathbb{C})$ with α_j, β_j and q depending on $\{g \in G \mid c_g \neq 0\}$ and γ a first order polynomial function of the c_g, such that:

(Y invertible) $\Leftrightarrow$ $(\alpha_{-1} \otimes \lambda(g_1^{-1}) + \alpha_1 \otimes \lambda(g_1) + \beta_{-1} \otimes \lambda(g_2^{-1}) + \beta_1 \otimes \lambda(g_2) + \gamma \otimes 1$ invertible).

6.4. It will be convenient to make one further matrix transformation so as to be in the self-adjoint case. With the notations of 6.3 put

$a = \alpha_{-1} \otimes \lambda(g_1^{-1}) + \alpha_1 \otimes \lambda(g_1) + \gamma \otimes 1$

$b = \beta_{-1} \otimes \lambda(g_2^{-1}) + \beta_1 \otimes \lambda(g_2)$.

Then we have

$$(a+b \text{ invertible}) \Longleftrightarrow \left(\begin{pmatrix} 0 & a+b \\ a^*+b^* & 0 \end{pmatrix} \text{ invertible}\right).$$

So defining $X_1 = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$ we have

$$(Y \text{ invertible}) \Longleftrightarrow (X_1+X_2 \text{ invertible}).$$

Moreover

$$X_j = X_j^* \in \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*(\lambda(g_j)) \subset \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*_{red}(G) \qquad (j=1,2)$$

and if $\{g \in G \mid c_g \neq 0\}$ is a fixed finite set then X_2 is constant. and X_1 is a first order polynomial in c_g and $\overline{c_g}$ $(g \in G)$. Also only $\lambda(g_j^{\pm 1})$ appear in the expression of X_j.

6.5. Let $B = \mathcal{M}_{2q}(\mathbb{C})$, $A = \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*_{red}(G)$, $A_j = \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*(\lambda(g_j)) \subset A$ and $\varphi : A \to \mathcal{M}_{2q}(\mathbb{C})$ the conditional expectation $\varphi = id \otimes \tau$ where τ is the canonical trace on $C^*_{red}(G)$. Then $\{A_1, A_2\}$ is a free pair of subalgebras in (A, φ) and hence $\{X_1, X_2\}$ is a free pair of $\mathcal{M}_{2q}(\mathbb{C})$-valued random variables. It is especially easy to compute G_{X_j} since $A_j \simeq \mathcal{M}_{2q}(\mathbb{C}) \otimes C(\mathbb{T})$. Using the results of section 4 we have a method for computing $G_{X_1+X_2}(b) = \varphi(b(I-(X_1+X_2)b)^{-1})$ for $b \in \mathcal{M}_{2q}(\mathbb{C}) \otimes I \subset A$. Note that $Tr_{2q} \circ \varphi$ is faithful on A so that taking $b = zI_{2q} \otimes I$, $z \in \mathbb{C}$ and $Tr_{2q}(G_{X_1+X_2}(zI_{2q} \otimes I))$ gives us the generating series for the moments of X_1+X_2 with respect to a faithful trace on A. Solving this moment problem one gets the spectrum of X_1+X_2 and hence the possibility of deciding whether X_1+X_2 is invertible.

7. Dual algebraic structures

This section deals with the necessary adaptations that have to be performed in our considerations in [10] in order to fit the B-valued case.

7.1. The basic idea is to replace the category of unital pro-C*-algebras in [10] by some other category. Corresponding to the two cases: the purely algebraic one when B is just a unital algebra over $\mathbb{C}$ and the C*-algebraic once when B is a unital C*-algebra, we will consider the categories $\mathcal{C}_B$ and respectively $\mathcal{C}_B^*$.

$\mathcal{C}_B$ is the category of unital algebras A over $\mathbb{C}$ containing B as a subalgebra $B \hookrightarrow A$, the inclusion being unital and the morphisms are homomorphisms for which the diagrams

$$\begin{tikzcd} A_1 \arrow[rr] & & A_2 \\ & B \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

are commutative.

$\mathcal{C}_B^*$ is the category of B-pro-C*-algebras, i.e. unital C*-algebras $(A, \|\ \|)$ with $1 \in B \subset A$ and endowed with a family of C*-seminorms $(\|\ \|_\alpha)_{\alpha \in I}$ indexed by some directed set I so that $\|b\|_\alpha = \|b\|$ if $b \in B$ and $\alpha \leq \beta \Rightarrow \|x\|_\alpha \leq \|x\|_\beta$, $\|x\| = \sup_{\alpha \in I} \|x\|_\alpha$ if $x \in A$ and moreover

$$A_1 = \varprojlim_{\alpha \in I} A_{\alpha 1}$$

where the subscript 1 is for the unit ball and A_α is the quotient of A by the ideal annihilated by $\|\ \|_\alpha$. Morphisms in $\mathcal{C}_B^*$ are morphisms of unital pro-C*-algebras (see 1.4 in [10]) $A \to A'$ making the diagram

$$\begin{tikzcd} A \arrow[rr] & & A' \\ & B \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

commutative.

7.2. A dual algebraic structure is an algebraic structure in a category as defined in Ch. IV §1 of [2]. We examined in [10] what a dual group structure means in the category of unital pro-C*-algebras. In $\mathcal{C}_B$ and $\mathcal{C}_B^*$ we have a similar situation.

Let μ, j, χ be the binary, unary and nullary operations defining the dual group structure on A. Here

$$\begin{tikzcd} A \arrow[rr, "\chi"] & & B \\ & B \arrow[ul] \arrow[ur, "id_B"'] & \end{tikzcd}$$

is commutative.

Also the free products with amalgamation over $\mathbb{C}$ have to be replaced by free products with amalgamation over B. Thus $\mu: A \longrightarrow A \underset{B}{*} A$.

If $A \in \mathcal{C}_B^*$, this free product is defined as follows: it is the inverse limit of the C*-algebraic free product with amalgamation $A_\alpha \underset{B}{*} A_\alpha$.

7.3. If $A \in \mathcal{C}_B$ the state space of A denoted by S(A) is the set of conditional expectations $\varphi: A \to B$. If $A \in \mathcal{C}_B^*$ then S(A) is the set of conditional expectations $\varphi: A \longrightarrow B$ such that $\|\varphi(a)\| \leq \|a\|_\alpha$ for one of the seminorms of A.

If $\varphi_j \in S(A_j)$ $(j=1,2)$ then there is an unique $\varphi \in S(A_1 \underset{B}{*} A_2)$ such that $\varphi(a_1 \ldots a_n) = 0$ whenever $a_k \in A_{j(k)}$, $j(k) \in \{1,2\}$,

$\varphi_{j(k)}(a_k)=0$, $j(k)\neq j(k+1)$ $(1\leq k\leq n-1)$ and $\varphi|A_j=\varphi_j$.

Uniqueness of φ follows from 1.3 both in the $\mathcal{C}_B$ and $\mathcal{C}_B^*$ cases. Existence of φ in the $\mathcal{C}_B^*$ case is obtained from §5 of [7].

The existence of φ in the $\mathcal{C}_B$-case is seen as follows. Let $\mathring{A}_j=\mathrm{Ker}\,\varphi_j$ and $D_n=\{(i_1,\ldots,i_n)\mid i_j\in\{1,2\},\ i_k\neq i_{k+1},\ 1\leq j\leq n,\ 1\leq k\leq n-1\}$. Then we have

$$A_1 \underset{B}{*} A_2 \simeq B \oplus \bigoplus_{n\geq 0}\ \bigoplus_{(i_1,\ldots,i_n)\in D_n} \mathring{A}_{i_1} \otimes_B \mathring{A}_{i_2} \otimes_B \cdots \otimes_B \mathring{A}_{i_n}$$

and we define φ as the projection onto the B-summand.

We shall denote by $\varphi_1 * \varphi_2$.

7.4. If (A,μ,j,χ) is a dual group in $\mathcal{C}_B$ or $\mathcal{C}_B^*$ (actually dual semigroup would suffice) and if $\varphi_1,\varphi_2\in S(A)$ then

$(\varphi_1,\varphi_2) \rightsquigarrow \varphi_1 \textcircled{\mu} \varphi_2=(\varphi_1 * \varphi_2)\circ\mu$ defines a semigroup structure on $S(A)$ with unit χ.

7.5. In $\mathcal{C}_B$ there is a dual group structure on $B\langle X\rangle$ defined by

$$B\langle X\rangle \xrightarrow{\mu} B\langle X\rangle *_B B\langle X\rangle \simeq B\langle X_1,X_2\rangle$$

$$\mu(X)=X_1+X_2, \qquad j(X)=-X$$

and $\chi(b_oXb_1\ldots b_{n-1}Xb_n)=0$ if $n\geq 1$ and $\chi(b)=b$. Then $S(B\langle X\rangle)=\Sigma_B$ and $\textcircled{\mu}$ is $\boxplus$.

Similarly in $\mathcal{C}_B^*$ a corresponding dual group is $A=\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B$ (with the notations of 5.1 [10]) and since $(\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B) *_B (\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B)=$ $=(\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} \mathbb{R}_{\mathbb{C},nc}) *_{\mathbb{C}} B$ we define μ from the dual operation of $\mathbb{R}_{\mathbb{C},nc}$.

It is easy to construct similar examples for other dual groups considered in [10] by taking free products with B.

7.6. There are also examples of a somewhat different nature involving tensor products. For instance let B[X] be the polynomials in X with coefficients in B (X and $b \in B$ commute) and let

$$\mu(X)=X_1+X_2$$

where X_j are the two images of X in $B[X] *_B B[X]$, $j(X)=-X$ and $\chi(bX^n)=0$ if $n>0$, $\chi(b)=b$.

REFERENCES

[1] W.B. Arveson, The harmonic analysis of automorphism groups, in Operator Algebras and Applications (ed. R.V. Kadison), Proc. Symp. Pure Math. 38, Amer. Math. Soc., Providence 1981.

[2] I. Bucur, A. Deleanu, Introduction to the theory of categories and functors, New York, J. Wiley, 1968.

[3] J. Cuntz, Simple C*-algebras generated by isometries, Comm. Math. Phys., 57 (1977), 173-185.

[4] D. Evans, On O_n, Publ. Res. Inst. Math. Sci., 16(1980), 915-927.

[5] W. Paschke, N. Salinas, Matrix algebras over O_n, Michigan Math. J., 26(1979), 3-12.

[6] M. Pimsner, S. Popa, The Ext-groups of some C*-algebras considered by J. Cuntz, Rev. Roumaine Math. Pures Appl., 23 (1978), 1069-1076.

[7] D. Voiculescu, Symmetries of some reduced free product C*-algebras, in Operator Algebras and their connections with Topology and Ergodic Theory, Springer Lecture Notes, vol. 1132.

[8] D. Voiculescu, Addition of certain non-commuting random variables, J. Functional Analysis 1986.

[9] D. Voiculescu, Multiplication of certain non-commuting random variables, INCREST preprint no. 76/1985.

[10] D. Voiculescu, Dual algebraic structure on operator algebras related to free products, INCREST preprint no. 77/1985.