INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

STRONGLY MODULAR AND STRONGLY LATTICIAL CLASSES OF OPERATORS

by

Dan Tudor VUZA

PREPRINT SERIES IN MATHEMATICS

No. 43/1986

BUCURESTI

STRONGLY MODULAR AND STRONGLY LATTICIAL

CLASSES OF OPERATORS

by

Dan Tudor VUZA*)

July 1986

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd. Pacii 220, 79622 Bucharest, ROMANIA

STRONGLY MODULAR AND STRONGLY

LATTICIAL CLASSES OF OPERATORS

Dan Tudor VUZA

0. Introduction

The work on the theory of principal modules was initiated by the author in 1980 by a paper circulated as an INCREST preprint (Preprint Series Math. INCREST 62 (1980), published as [18], [19]). One of its purposes was to provide an unified framework for the proof of the results in the circle of ideas which arouse with the Dodds - Fremlin theorem : recall that in 1979, P. Dodds and D.H. Fremlin published in [5] their famous result asserting that if E,F are Banach lattices such that E' and F have order continuous norms and if U,V:E $\longrightarrow$ F are linear operators such that $0\leq U\leq V$, then the compacity of V implies the compacity of U. Since then, many results of this type have been given for various classes of operators; their common purpose was to prove that, under certain assumptions, a given class of linear operators has an order ideal - type property (i.e. $0\leq U\leq V$ and V in the class imply U in the class). As examples, the following classes beside compact operators have been investigated from this point of view : kernel operators (A.R. Schep [15]), the closed algebraic ideal generated by a regular operator (H. Leinfelder [9], C.D. Aliprantis and O. Burkinshaw [2], B. de Pagter [12], D. Vuza [23], [27]), operators defined by M - tensor products (D. Vuza [20], [22]), Dunford - Pettis operators (N.J. Kalton and P. Saab [8]).

The theory of principal modules ([17]-[25], [27]) offers an unified method for the proof of all above mentioned results, the main tool being provided by the following permanence theorem which was proved for the first time in 1981 in [17] (see also [20] - [23]) :

THEOREM 0.1. Let E be a principal A-module and let F be an order complete principal B-module such that its topology is order continuous. Then the space $L_r^c(E,F)$ of all order bounded linear operators from E to F with continuous modulus is a principal $A\overline{\otimes}B$-module for the solid strong topology.

(See §1 for the notations and definitions used in this statement).

A common feature of theorem 0.1 and of most of the above mentioned results is that order continuity of the topology of the Riesz space F is needed; when this assumption is dropped, theorem 0.1 fai to be true and so happens with most of the theorems of Dodds - Fremlin type, as it was shown by examples by various authors. Nevertheless, a variant of theorem 0.1 holds even in such a situation provided if we replace the large space $L_r'(E,F)$ by certain smaller classes of operators; one such example is offered by the class of oru-compact operators, as it was pointed out by the author in [25]. However, there are many other examples of such classes; it is the purpose of the present paper to give a systematic approach of them by developing the theory of the so-called strongly latticial classes of operators.

There is another advantage which can be taken from the introduction of strongly latticial classes. In general (i.e., when F is not order complete), the space $L_r(E,F)$ of all regular operators from E to F is not a Riesz space; nevertheless, any strongly latticial class is a Riesz space and therefore, our theory provides some methods for producing Riesz spaces of operators acting between Riesz spaces which are not necessarily order complete. In fact (see §4), a number of classical Banach lattices of operators acting between Banach lattices which are not necessarily order complete, are strongly latticial classes.

The paper is divided into five sections. After a preliminaries section, §2 provides the module theoretic background for the whole study; the notions of A-principal element and A-principal subspace in an A-module are introduced there. Their study is motivated by the fact that there is a tight connection (as shown in §3) between A-principality and operators with strong moduli; consequently, the theory of modules over f-algebras turns to be a useful instrument for the investigation of strongly modular and strongly latticial classes which is undertaken in §3. After introducing the notion of strong modulus of an operator, strongly modular classes are defined as vector spaces of operators having strong moduli; strongly latticial classes are strongly modular classes closed and strongly latticial classes is indicated; in particular, it is proved that maximal strongly modular classes are latticial. The extension of theorem 0.1 to the case of strongly latticial classes as well as a monotone approximation theorem are proved.

In §4, a list of concrete examples of strongly modular and strongly latticial classes is presented and a partial answer to the following question is given: when the whole space $L_r(E,F)$ is a strongly latticial class? It is shown that order continuity of the topology of F is a necessary and sufficient condition when E is arbitrary, but there are some special E for which $L_r(E,F)$ is strongly latticial even if F does not satisfy the above condition.

Finally, §5 offers a group of applications of the general theory by proving theorems of Dodds - Fremlin type for some classes of operators already mentioned at the beginning of the introduction in the situation when we drop the order continuity hypothesis on F but instead we confine ourselves to operators contained in a given strongly latticial class. The section closes with a permanence of principality theorem for M - tensor products.

1. Preliminaries

a) Notations.

$\overline{M}$: the closure of the subset M of a topological space.

$U|G$: the restriction of a map $U:E \longrightarrow F$ to a subset G of E.

1_E : the identity map on a set E.

B_E : the unit ball of the normed vector space E.

E' : the dual of the topological vector space E.

The notations $\langle f,x \rangle$ and $f(x)$ will alternatively be used for the value of a linear form f on E at $x \in E$.

b) Riesz spaces.

Let E be a Riesz space. For any $x \in E$ we shall denote by E_x the principal order ideal generated by x. The seminorm $\| \ \|_x$ on E_x is defined by

$$\|y\|_x = \inf \{\alpha \mid \alpha \in \mathbb{R}_+ , |y| \leq \alpha |x| \} ;$$

it is a norm in case E is Archimedean. It induces a norm on the quotient Riesz is an AM-space with strong order unit which will be denoted by $\overline{E}_x$; we shall consider $\overline{E}_x$ as topologized by the norm $\| \|_u$, u being the canonical image of $|x|$.

E is called relatively uniformly complete if it is Archimedean and every principal order ideal E_x is complete for $\| \|_x$.

Let f be a positive linear form on E. The seminorm $x \longmapsto f(|x|)$ induces a norm on the quotient Riesz space E/N_f, where N_f is the null space of this seminorm; the norm completion of E/N_f is an AL-space which will be denoted by (E,f).

To every $x \in E_+$ we attach the set $D_E(x)$ (or simply D(x) if no confusion can arise) formed by all systems $(x_1,\ldots,x_n)$ (n is varying over $\mathbb{N}$) with the properties that $x_i \in E_+$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} x_i = x$. The following preorder relation is defined on D(x) : $(x_1,\ldots,x_m) \leq (y_1,\ldots,y_n)$ if there is a partition $(P_i)_{1 \leq i \leq m}$ of $\{1,\ldots,n\}$ such that $x_i = \sum_{j \in P_i} y_j$ for $1 \leq i \leq m$. The Riesz decomposition property ensures that D(x) is upwards directed for the preorder relation so defined.

The Riesz space of all order bounded linear forms on E will be denoted by $E^{\sim}$.

Whenever E is Archimedean, $\hat{E}$ will be its Dedekind extension.

A majorizing Riesz subspace of E is a Riesz subspace F with the property that for every $x \in E$ there is $y \in F$ such that $|x| \leq y$.

c) Topological Riesz spaces.

The phrase "topological Riesz space" will be employed to design a Riesz space endowed with a locally solid topology.

Let E be a Riesz space and let F be a majorizing Riesz subspace of E endowed with a locally solid topology τ. By the canonical extension of τ to E we mean the topology on E having the solid hulls of the τ-neighborhoods of 0 as a basis for 0. In particular, whenever E is an Archimedean topological Riesz space, we shall always consider on $\hat{E}$ the topology obtained by canonical extension.

A subset M in a topological Riesz space E is called order precompact if for any neighborhood W of 0 in E there is $y \in E_+$ such that $(|x| - y)_+ \in W$ whenever $x \in M$.

THEOREM 1.1. [7] A subset of an AL-space is relatively weakly compact iff it is order precompact.

A positive element u in the topological Riesz space E is called quasi interior if E_u is dense in E.

For every Riesz subspace G in $E^{\sim}$, $|\sigma|(E,G)$ will be the topology on E defined by the seminorms $x \mapsto g(|x|)$ for $g \in G_+$.

A locally solid topology on a Riesz space is called order continuous if every net decreasing to 0 converges to 0; a Banach lattice is said to have order continuous norm if its norm topology is order continuous. We shall make use of the following characterization of Banach lattices with order continuous norm:

THEOREM 1.2. [5] For every Banach lattice E the following are true:

i) E has order continuous norm iff $B_{E'}$ is order precompact for $|\sigma|(E',E)$.

ii) E' has order continuous norm iff B_E is order precompact for $|\sigma|(E,E')$.

d) Spaces of linear operators.

For any normed vector spaces E and F, L(E,F) will be the space of all continuous linear operators from E to F equipped with the usual operator norm. If $U \in L(E,F)$, then $U' \in L(F',E')$ will be its transpose.

Let E,F be Riesz spaces. A linear operator $U:E \to F$ is called regular if it can be written as a difference of two positive linear operators. The ordered vector space of all regular operators from E to F will be denoted by $L_r(E,F)$; the order relation on it is as usual, $U \geq 0$ meaning that $U(E_+) \subset F_+$.

In case E and F are topological Riesz spaces, $L_r^c(E,F)$ (respectively $L_r^*(E,F)$) will be the subspace of $L_r(E,F)$ generated by all positive continuous linear operators (respectively all positive linear operators which are continuous on every order bounded subset of E).

Let E be a Riesz space, let F be a topological Riesz space, let $\mathcal{M}$ be a collection of subsets of E_+ and let L be a directed subspace of $L_r(E,F)$ such that U(M) is topologically bounded whenever $U \in L$ and $M \in \mathcal{M}$. The solid $\mathcal{M}$-topology on L has $\{\mathcal{V}_{M,W} \mid M \in \mathcal{M}$, W neighborhood of 0 in F$\}$ as a basis for 0, $\mathcal{V}_{M,W}$ being the set of those $U \in L$ for which there is $V \in L$ such that $U \in [-V,V]$ and $V(M) \subset W$. In particular, taking $\mathcal{M} = \{\{x\} \mid x \in E_+\}$ we obtain the solid strong topology on $L_r(E,F)$; taking $\mathcal{M}$ to be the collection of all order precompact subsets of E_+ (E being also a topological Riesz space) we obtain the solid order precompact topology on $L_r^{\circ}(E,F)$.

Let E be a Riesz space and F be an Archimedean Riesz space. For any $U \in L_r(E,F)$ and any $x \in E_+$ we have $U(N_x) = \{0\}$; consequently, U induces a regular operator from E_x/N_x into F. As $\hat{F}$ is relatively uniformly complete, this operator can be uniquely extended to a regular operator from $\overline{E}_x$ into $\hat{F}$, which will be denoted by U_x .

Recall that for any Banach lattices E,F we have $L_r(E,F) = L_r^{\circ}(E,F)$. The regular norm on $L_r(E,F)$ is given by

$$\|U\|_r = \inf \{\|V\| \mid U \in [-V,V]\}$$

where $\| \ \|$ denotes the operator norm.

For any Banach lattice E, $\check{E}$ (respectively $\overline{\check{E}}$) will be the order ideal generated by E in E'' (respectively the norm closure of $\check{E}$ in E''). If F is another Banach lattice and if $U \in L_r(E,F)$, then $\check{U} \in L_r(\check{E},\check{F})$ (respectively $\overline{\check{U}} \in L_r(\overline{\check{E}},\overline{\check{F}})$) will be the operators defined by the corresponding restrictions of U''.

e) <u>f-algebras</u>.

Recall that an f-algebra is a Riesz space A endowed with a structure of algebra such that $A_+A_+ \subset A_+$ and $ac \wedge b = ca \wedge b = 0$ whenever $a,b,c \in A_+$ and $a \wedge b = 0$. In this paper, however, the word f-algebra will be exclusively employed to design an Archimedean f-algebra A admitting an element $e \in A_+$ as an algebraic unit as well as a strong order unit; such an f-algebra will always be considered as normed by $\| \ \|_e$, the subscript e being therefore omitted.

By an f-subalgebra of A we shall mean a Riesz subspace of A closed under multiplication and containing e.

As examples of f-algebras which will be used in the following we mention :

$C(X)$: the continuous real functions on the compact space X.

$B(X)$: the bounded real Baire functions on the compact space X.

$Z(E)$: the center of the Archimedean Riesz space E (that is, the set of all operators $U \in L_r(E,E)$ such that $U \in [-\alpha 1_E, \alpha 1_E]$ for some $\alpha \in \mathbb{R}_+$).

Throughout the paper, the letters A and B will denote f-algebras.

Let $A \otimes B$ be the

defined by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$). The tensor product $A \overline{\otimes} B$ in the sense of D.H. Fremlin [6] is a Riesz space containing $A \otimes B$ as a vector subspace. By using the universality property of $A \overline{\otimes} B$ (theorem 4.3 in [6]) one can see that there is a unique structure of f-algebra on $A \overline{\otimes} B$ such that $A \otimes B$ becomes a subalgebra. If e_1 (respectively e_2) denotes the unit of A (respectively B), then $e_1 \otimes e_2$ is the unit of $A \overline{\otimes} B$.

For the theory of f-algebras we refer to [3].

f) A-modules.

Let A be an f-algebra. By a Riesz A-module (or simply, A-module if no confusion can arise) we shall mean an Archimedean Riesz space E endowed with a structure of algebraic module over A such that $A_+ E_+ \subset E_+$. An important fact about A-modules is that the relation $|ax| = |a||x|$ holds for every $a \in A$ and $x \in E$ (see [23]).

For any $M \subset A$ and any x in the A-module E, Mx will be the set $\{ ax \mid a \in M \}$.

For any Archimedean Riesz space E, the map $(S,x) \longmapsto S(x)$ from $Z(E) \times E$ into E defines a structure of Riesz Z(E)-module on E.

Let E be an Archimedean Riesz space which is an A-module and also a B-module; using again theorem 4.2 in [6] one can see that there is a unique structure of $A \overline{\otimes} B$-module on E such that $(a \otimes b)x = a(bx)$ for $a \in A$, $b \in B$ and $x \in E$.

Whenever E is a Riesz A-module and F is a Riesz B-module, $L_r(E,F)$ will be considered as an $A \overline{\otimes} B$-module by defining $(a \otimes b)U$ to be the operator $x \longmapsto bU(ax)$.

g) Approximation lemmas.

LEMMA 1.1. Let F be a dense Riesz subspace of the topological Riesz space E, let $x \in F_+$ and let $(x_1, \ldots, x_n) \in D_E(x)$. Then for every neighborhood W of 0 in E there is $(y_1, \ldots, y_n) \in D_F(x)$ such that $x_i - y_i \in W$ for $1 \leq i \leq n$.

PROOF. Let $(x_{i\delta}) \subset F$ be such that that $x_{i\delta} \longrightarrow x_i$. Define inductively

$$y_{1\delta} = (x_{1\delta})_+ \wedge x,$$

$$y_{i\delta} = (x_{i\delta})_+ \wedge \Big(x - \sum_{j=1}^{i-1} y_{j\delta}\Big), \quad 1 < i < n,$$

$$y_{n\delta} = x - \sum_{j=1}^{n-1} y_{j\delta}$$

Then $(y_{1\delta},\dots,y_{m\delta}) \in D_F(x)$ and $y_{i\delta} \longrightarrow x_i$.

LEMMA 1.2. Let $c \in A\bar{\otimes}B$ be such that $|c| \leq e_1 \otimes e_2$. Then for every $\varepsilon > 0$ there are $a_1,\dots,a_n \in A$ and $b_1,\dots,b_n \in B$ with the following properties:

i) $(a_1,\dots,a_n) \in D(e_1)$.

ii) $|b_i| \leq e_2$ for $1 \leq i \leq n$.

iii) $\| c - \sum_{i=1}^{n} a_i \otimes b_i \| \leq \varepsilon$.

If in addition $c \geq 0$ we may assume that $b_i \geq 0$ for $1 \leq i \leq n$.

PROOF. Using well known representation theorems we may assume that A (respectively B) is a dense f-subalgebra in C(X) (respectively C(Y)) for some compact spaces X,Y. Then $A\bar{\otimes}B$ can be identified with the Riesz subspace of $C(X \times Y)$ generated by $A \otimes B$ (see [6]). Let $c \in A\bar{\otimes}B$ be such that $|c| \leq e_1 \otimes e_2$. By compactity we can find an open covering $(G_i)_{1 \leq i \leq n}$ of X such that $\sup_{t \in Y} |c(s,t) - c(s',t)| \leq \varepsilon/2$ whenever $s,s' \in G_i$ and $1 \leq i \leq n$. Let $(\varphi_i)_{1 \leq i \leq n}$ be a partition of unit subordonated to the covering (G_i) (that is, $(\varphi_1,\dots,\varphi_n) \in D_{C(X)}(e_1)$ and $\operatorname{supp} \varphi_i \subset G_i$). Choose a point $s_i \in G_i$ and let $\psi_i \in C(Y)$ be defined by $\psi_i(t) = c(s_i,t)$. We have

$$\sup_{s \in X,\ t \in Y} \left| c(s,t) - \sum_{i=1}^{n} \varphi_i(s)\,\psi_i(t) \right| \leq \varepsilon/2 .$$

For any $\eta > 0$ we can find by lemma 1.1 $(a_1,\dots,a_n) \in D_A(e_1)$ such that $\sup_{1 \leq i \leq n} \| a_i - \varphi_i \| \leq \eta$; we can also find $b_i \in B$ verifying $|b_i| \leq e_2$ (and $b_i \geq 0$ if $c \geq 0$) and $\sup_{1 \leq i \leq n} \| b_i - \psi_i \| \leq \eta$. Choosing η to be small enough we may achieve that iii) holds.

2. Principal subspaces in A-modules

Throughout the section, the letter E will denote a topological Riesz space which is also a Riesz A-module. The unit of A will be denoted by e.

The notion of A-disjointness will be introduced through a set of equivalent conditions :

PROPOSITION 2.1. For every $x_1, x_2 \in E$ the following are equivalent :

i) For every neighborhood W of 0 in E there is $a \in A$ such that $x_1 - ax_1 \in W$ and $ax_2 \in W$.

ii) The same as i) but with $a \in [0,e]$.

iii) For every neighborhood W of 0 in E there are $a_1, a_2 \in [0,e]$ such that

$a_1 \wedge a_2 = 0$ and $x_i - a_i x_i \in W$, $i = 1,2$.

PROOF.

i) $\Longrightarrow$ ii) Let $W' \subset W$ be a solid neighborhood of 0. By hypothesis there is $a \in A$ such that $x_1 - ax_1 \in W'$ and $ax_2 \in W'$. Let $b = |a| \wedge e$. The relations

$$|x_1 - bx_1| = |e-b||x_1| \leq |e-a||x_1| = |x_1 - ax_1| \;,$$
$$|bx_2| = b|x_2| \leq |a||x_2| = |ax_2|$$

show that $x_1 - bx_1 \in W$ and $bx_2 \in W$.

ii) $\Longrightarrow$ iii) Let $W' \subset W$ be a solid neighborhood of 0. By hypothesis there is $a \in [0,e]$ such that $x_1 - ax_1 \in 2^{-1}W'$ and $ax_2 \in 2^{-1}W'$. Put $a_1 = (2a-e)_+$, $a_2 = (2a-e)_-$. We have $a_1, a_2 \in [0,e]$, $a_1 \wedge a_2 = 0$ and

$$|x_1 - a_1 x_1| = (e \wedge 2(e-a))|x_1| \leq 2(e-a)|x_1| = 2|x_1 - ax_1|,$$
$$|x_2 - a_2 x_2| = (e \wedge 2a)|x_2| \leq 2a|x_2| = 2|ax_2| \;;$$

consequently, $x_i - a_i x_i \in W$ ($i = 1,2$).

iii) $\Longrightarrow$ i) Let W be a solid neighborhood of 0. By hypothesis there are $a_1, a_2 \in [0,e]$ such that $a_1 \wedge a_2 = 0$ and $x_i - a_i x_i \in W$ ($i = 1,2$). The relation

$$|a_1 x_2| = |a_1(x_2 - a_2 x_2)| \leq |x_2 - a_2 x_2|$$

implies that $a_1 x_2 \in W$; as also $x_1 - a_1 x_1 \in W$, the conclusion follows.

We say that two elements x_1, x_2 in E are A-disjoint if they satisfy the conditions i) - iii) in proposition 2.1. It follows from the above proposition that x_1, x_2 are A-disjoint iff $|x_1|$ and $|x_2|$ are.

PROPOSITION 2.2.

i) The set of all couples (x_1, x_2) such that x_1 is A-disjoint from x_2 is closed in $E \times E$.

ii) The set of all $y \in E$ which are A-disjoint from a given $x \in E$ is a closed order ideal in E.

PROOF.

i) Let (x_1, x_2) belong to the closure of the set under consideration, let W be a neighborhood of 0 in E and let W' be a solid neighborhood of 0 such that $W' + W' \subset W$. There are A-disjoint elements y_1, y_2 such that $x_i - y_i \in W'$ ($i = 1,2$).

As y_1, y_2 [illegible]

From the relations

$$|x_1 - ax_1| \leq |x_1 - y_1| + |y_1 - ay_1| ,$$
$$|ax_2| \leq |x_2 - y_2| + |ay_2|$$

it follows that $x_1 - ax_1 \in W$ and $ax_2 \in W$; consequently, x_1 and x_2 are A-disjoint.

ii) Let N be the set under consideration. By i), it is closed; to see that it is an order ideal, it suffices to prove that $|y_1| + |y_2| \in N$ whenever $y_1, y_2 \in N$. There are nets (a_δ), $(b_\delta) \subset [0,e]$ such that $x - a_\delta x \longrightarrow 0$, $a_\delta y_1 \longrightarrow 0$, $x - b_\delta x \longrightarrow 0$, $b_\delta y_2 \longrightarrow 0$. If $c_\delta = a_\delta \wedge b_\delta$ then $c_\delta(|y_1| + |y_2|) \longrightarrow 0$ and

$$|x - c_\delta x| = (e - a_\delta)|x| \vee (e - b_\delta)|x| \longrightarrow 0$$

which implies the desired conclusion.

PROPOSITION 2.3. If the topology of E is separated then A-disjoint elements in E are order disjoint.

PROOF. Let x_1, x_2 be A-disjoint and let $x = |x_1| \wedge |x_2|$. For any solid neighborhood W of 0 in E there is $a \in [0,e]$ such that $|x_1 - ax_1| + |ax_2| \in W$. It follows then from the relation

$$x = ex \leq (e-a)|x_1| + a|x_2|$$

that $x \in W$; as W is arbitrary, $x = 0$.

The notion of a principal module was introduced in [18] (see also [17], [21]-[23]) : the topological Riesz A-module E is called principal if Ax is dense in E_x for every $x \in E$. This is equivalent to require $[0,e]x$ to be dense in $[0,x]$ for every $x \in E_+$ or $[-e,e]x$ to be dense in $[-|x|,|x|]$ for every $x \in E$. Using the notion of A-disjointness we can reformulate theorem 2.2 in [23] as follows :

THEOREM 2.1. A topological Riesz A-module E is principal iff order disjoint elements in E are A-disjoint.

Any Banach lattice with a quasi interior element is a principal module over its center.

We introduce now the notion of a principal element.

PROPOSITION 2.4. For every $x \in E$ the following are equivalent:

i) $x \in \overline{A|x|}$.

ii) $x \in \overline{[-e,e]|x|}$.

iii) $|x| \in \overline{[-e,e]x}$.

iv) x_+ and x_- are A-disjoint.

PROOF.

i) $\Longrightarrow$ ii) If $a_\delta|x| \longrightarrow x$ and $b_\delta = (a_\delta \wedge e)\vee(-e)$, then $b_\delta \in [-e,e]$ and $b_\delta|x| \longrightarrow x$.

ii) $\Longrightarrow$ i) Obvious.

ii) $\Longrightarrow$ iv) If $a_\delta|x| \longrightarrow x$ with $a_\delta \in [-e,e]$, then we also have $(a_\delta)_+|x| \longrightarrow$ $\longrightarrow x_+$ and $(a_\delta)_-|x| \longrightarrow x_-$. It follows that

$$(a_\delta)_+x_- = (a_\delta)_+(x_- - (a_\delta)_-|x|) \longrightarrow 0 \ ;$$

consequently,

$$(a_\delta)_+x_+ = (a_\delta)_+|x| - (a_\delta)_+x_- \longrightarrow x_+$$

which proves the A-disjointness of x_+ and x_- .

iii) $\Longrightarrow$ iv) If $a_\delta x \longrightarrow |x|$ with $a_\delta \in [-e,e]$, then we also have $|a_\delta||x| \longrightarrow$ $\longrightarrow |x|$; thus,

$$(a_\delta)_+x_+ + (a_\delta)_-x_- = 2^{-1}(|a_\delta||x| + a_\delta x) \longrightarrow x_+ + x_- \ .$$

The above relation together with $(a_\delta)_+x_+ \in [0,x_+]$, $(a_\delta)_-x_- \in [0,x_-]$ imply $(a_\delta)_+x_+ \longrightarrow$ $\longrightarrow x_+$ and $(a_\delta)_-x_- \longrightarrow x_-$; hence x_+ and x_- are A-disjoint.

iv) $\Longrightarrow$ ii) and iii) Let (a_δ), $(b_\delta) \in [0,e]$ be such that $a_\delta \wedge b_\delta = 0$, $a_\delta x_+ \longrightarrow x_+$, $b_\delta x_- \longrightarrow x_-$. We have

$$a_\delta x_- = a_\delta(x_- - b_\delta x_-) \longrightarrow 0 \ ,$$
$$b_\delta x_+ = b_\delta(x_+ - a_\delta x_+) \longrightarrow 0 \ .$$

Therefore, $a_\delta - b_\delta \in [-e,e]$, $(a_\delta - b_\delta)|x| \longrightarrow x$ and $(a_\delta - b_\delta)x \longrightarrow |x|$.

We say that an element in E is A-principal if it satisfies the conditions i) - iv) in proposition 2.4. We denote by $Pr_A(E)$ the set of all A-principal elements in E. Clearly $Pr_A(E)$ is closed under multiplication by elements in A and contains $E_+ \cup (-E_+)$; in general, it is not closed under addition.

PROPOSITION 2.5. $Pr_A(E)$ is closed.

PROOF. Follows from proposition 2.4 iv) and proposition 2.2 i).

PROPOSITION 2.6. Let $x,y \in Pr_A(E)$ be A-disjoint. Then $x + y \in Pr_A(E)$.

PROOF. Factoring, if necessary, by the intersection of all neighborhoods of 0 we may assume that the topology of E is separated. Then proposition 2.3 gives $|x|\wedge|y| = 0$; consequently, $(x + y)_+ = x_+ + y_+$ and $(x + y)_- = x_- + y_-$. By proposition 2.2, $x_+ + y_+$ is A-disjoint from $x_- + y_-$; hence $x + y \in Pr_A(E)$.

As $Pr_A(E)$ is generally not closed under addition, it makes sense to introduce the following notions :

An A-principal subspace of E is a vector subspace of E contained in $Pr_A(E)$.

An A-principal Riesz subspace of E is an A-principal subspace which is also a Riesz subspace.

PROPOSITION 2.7. The closure of an A-principal (Riesz) subspace is an A-principal (Riesz) subspace.

PROOF. Follows from proposition 2.5.

PROPOSITION 2.8. Order disjoint elements in an A-principal Riesz subspace are A-disjoint.

PROOF. Let F be an A-principal Riesz subspace of E and let $x, y \in F$ be order disjoint. If $z = |x| - |y|$ then $z \in F$, $|x| = z_+$ and $|y| = z_-$. As z is A-principal it follows that x and y are A-disjoint.

THEOREM 2.2. Every A-principal Riesz subspace which is also an A-submodule is a principal A-module (for the induced structures).

PROOF. Follows from theorem 2.1 and proposition 2.8.

The following proposition offers the possibility to express the A-principality of an element in terms of principality with respect to a certain σ-order complete f-algebra.

PROPOSITION 2.9. Let X be a compact space, let C be an f-subalgebra of C(X) and let τ be a locally solid topology on C(X). Denote by B the f-subalgebra of B(X) consisting of those functions which are measurable with respect to the σ-algebra generated by the functions in C (that is, the smallest σ-algebra on X with respect to which each $f \in C$ is measurable). Consider C(X) as a topological Riesz C-module for the topology τ and B(X) as a topological Riesz B-module for the canonical extension of τ (the structures of modules being defined by pointwise multiplication). Then we have $Pr_C(C(X)) = Pr_B(B(X)) \cap C(X)$.

PROOF. Clearly $Pr_C(C(X)) \subset Pr_B(B(X)) \cap C(X)$. To prove the converse, let $x \in Pr_B(B(X)) \cap C(X)$; we have to show that $x \in \overline{C|x|}$. To this purpose, let W be a τ-neighborhood of 0 in C(X). We can find a solid τ-neighborhood W' of 0 and $\varepsilon > 0$ such that $W' + \varepsilon[-u,u] \subset W$ (u being the unit of the f-algebra C(X) and the order interval $[-u,u]$ being considered in C(X)). The solid hull of W' taken in B(X) is a neighborhood of 0 for the canonical extension of τ to B(X) ; consequently, as x is B-principal, there is $b_0 \in B$ and $y \in W'$ such that $|x - b_0|x|| \leq y$. The proof will be complete if we show that $x - C|x|$ intersects $[-y,y] + \varepsilon[-u,u]$. Suppose the contrary; we have then two convex subsets in C(X) with void intersection, one of them having a non void interior for the topology defined by the sup norm on C(X). We can therefore find a nonzero linear functional f on C(X) continuous for the sup norm such that

$$\sup f([-y,y] + \varepsilon[-u,u]) = \alpha \leq \inf f(x - C|x|).$$

The relation $\alpha \leq \inf f(x - C|x|)$ implies that $\alpha \leq f(x)$ and $f(C|x|) = \{0\}$. Let μ be the Radon measure associated with f. The set $\{ b \mid b \in B(X), \int b|x| d\mu = 0 \}$ is a vector subspace of B(X) which is closed under taking pointwise limits of order bounded sequences and contains the subalgebra C; consequently, the monotone class theorem (theorem 20 in ch. I of [11]) implies that $\int b|x| d\mu = 0$ for any $b \in B$. On the other side, the relation $\sup f([-y,y] + \varepsilon[-u,u]) \leq \alpha$ implies $|f|(y) + \varepsilon|f|(u) \leq \alpha$ as $[-y,y] + \varepsilon[-u,u]$ is a solid subset of C(X). Therefore, observing that $|\mu|$ is associated with $|f|$, we have

$$\alpha \leq f(x) = \int (x - b_0|x|) d\mu \leq \int |x - b_0|x|| d|\mu| \leq \int y d|\mu| =$$
$$= |f|(y) < |f|(y) + \varepsilon |f|(u) \leq \alpha$$

as $f \neq 0$ implies $|f|(u) > 0$. The contradiction so obtained concludes the proof.

THEOREM 2.3. The A-submodule generated by an A-principal subspace is again an A-principal subspace.

PROOF. Let F be an A-principal subspace of E. We have to show that $\sum_{i=1}^{m} a_i x_i$ is A-principal for any $a_1, \ldots, a_m \in A$ and $x_1, \ldots, x_m \in F$. Let $x = \sum_{i=1}^{m} |x_i|$. There is a compact space X and an order isomorphism J of E_x onto a dense Riesz subspace of C(X) such that J(x) equals the unit u of the f-algebra C(X). As E_x is an A-submodule, one can define, by continuous extension a structure of A-module on C(X). We have

$$(av)(bw) = (ab)(vw), \quad a,b \in A, \quad v,w \in C(X) \qquad (1)$$

which is a consequence of the well known fact that $Z(C(X))$ is isomorphic to $C(X)$. From the above relation it is seen that Au is an f-subalgebra of $C(X)$ and

$$Pr_A(E_x) = J^{-1}(Pr_{Au}(C(X)) \cap J(E_x)),$$

the topology τ on $C(X)$ being the canonical extension of the image by J of the restriction to E_x of the topology of E. By proposition 2.9,

$$Pr_{Au}(C(X)) = Pr_B(B(X)) \cap C(X)$$

where B is obtained from Au as indicated and the topology τ' on $B(X)$ is the canonical extension of τ. Hence, all we have to do is to show that $\sum_{i=1}^{m} (a_i u)J(x_i) \in Pr_B(B(X))$. As B is σ-order complete, there are sequences (b_{in}) of linear combinations of idempotent functions in B such that $\| a_i u - b_{in} \| \longrightarrow 0$ as $n \longrightarrow \infty$ $(1 \leq i \leq m)$. As $\sum_{i=1}^{m} b_{in} J(x_i) \longrightarrow \sum_{i=1}^{m} (a_i u)J(x_i)$ for τ' and $Pr_B(B(X))$ is closed for τ', the proof will be complete if we show that $\sum_{i=1}^{m} b_i J(x_i) \in$ $\in Pr_B(B(X))$ whenever the b_i's are linear combinations of idempotent functions in B. Indeed, there are mutually disjoint idempotent functions $e_1, \ldots, e_p \in B$ such that

$$b_i = \sum_{j=1}^{p} \alpha_{ij} e_j, \quad 1 \leq i \leq m.$$

Then

$$\sum_{i=1}^{m} b_i J(x_i) = \sum_{j=1}^{p} e_j y_j$$

where

$$y_j = \sum_{i=1}^{m} \alpha_{ij} J(x_i).$$

As F is an A-principal subspace, $\sum_{i=1}^{m} \alpha_{ij} x_i \in F$ and consequently, $y_j \in Pr_B(B(X))$. The relation

$$(e - e_j) e_j y_j = e_j e_k y_k = 0, \qquad j \neq k$$

shows that the $e_j y_j$'s are mutually B-disjoint; therefore, propositions 2.2 and 2.6 imply that $\sum_{j=1}^{p} e_j y_j \in Pr_B(B(X))$.

COROLLARY 2.1. Every maximal A-principal subspace is a closed Riesz subspace and an A-submodule.

PROOF. A maximal A-principal subspace is closed by proposition 2.7 and it is an A-submodule by theorem 2.3; hence, it is also a Riesz subspace by proposition 2.4 iii).

COROLLARY 2.2. Every A-principal subspace is contained in a maximal A-principal Riesz subspace.

PROOF. Apply Zorn's lemma and corollary 2.1.

COROLLARY 2.3. The closed A-submodule generated by an A-principal subspace is an A-principal Riesz subspace.

3. The strong modulus of an operator

Throughout this and the next section, E will be a Riesz space and F will be a separated topological Riesz space.

We say that a subset M of F admits y as a strong supremum and we write y = ssup M if y is an upper bound for M and belongs to the topological closure of $\left\{ \bigvee_{i=1}^{n} y_i \mid n \geq 1,\ y_i \in M \right\}$. Every strong supremum is a supremum in the usual sense.

For every linear operator $U: E \longrightarrow F$ and every $x \in E_+$ we denote by $M_U(x)$ the set

$$\left\{ \sum_{i=1}^{n} |U(x_i)| \mid (x_1,\ldots,x_n) \in D(x) \right\} .$$

As the map $(x_1,\ldots,x_n) \longmapsto \sum_{i=1}^{n} |U(x_i)|$ from D(x) into F is increasing, it follows that $M_U(x)$ is upwards directed.

PROPOSITION 3.1. For any $x \in E_+$, the existence of ssup $M_U(x)$ is equivalent to that of ssup $U([-x,x])$; whenever they exist, they are equal.

PROOF. Our assertion will be a consequence of the following facts:

i) Every element in $M_U(x)$ is a finite supremum of elements in $U([-x,x])$. Indeed,

$$\sum_{i=1}^{n} |U(x_i)| = \sup \left\{ U\left(\sum_{i=1}^{n} \varepsilon_i x_i\right) \mid (\varepsilon_1,\ldots,\varepsilon_n) \in \{-1,1\}^n \right\}$$

and $\sum_{i=1}^{n} \varepsilon_i x_i \in [-x,x]$ whenever $\varepsilon_i = \pm 1$ and $(x_1,\ldots,x_n) \in D(x)$.

ii) For every $y \in U([-x,x])$ there is $z \in M_U(x)$ such that $y \leq z$.

Indeed, if $u \in [-x,x]$ then

$$U(u) = U(u_+) - U(u_-) \leq |U(u_+)| + |U(u_-)| + |U(x - |u|)|$$

and the rightmost element belongs to $M_U(x)$.

It follows from the Riesz decomposition property that

$$\text{ssup } M_U(x_1 + x_2) = \text{ssup } M_U(x_1) + \text{ssup } M_U(x_2)$$

whenever ssup $M_U(x_1)$ and ssup $M_U(x_2)$ exist .

We say that a linear operator $U:E \longrightarrow F$ has a strong modulus if ssup $M_U(x)$ (or, equivalently, ssup $U([-x,x])$) exists for any $x \in E_+$. By the above remark, in such a situation the map $x \longmapsto$ ssup $M_U(x)$ can be uniquely extended to a positive linear operator from E into F; we shall call it the strong modulus of U and denote it by $|U|$. We also consider the operators

$$U_+ = 2^{-1}(|U| + U), \; U_- = 2^{-1}(|U| - U).$$

The notations so introduced are justified by the following obvious proposition :

PROPOSITION 3.2. If U has a strong modulus then $U \in L_r(E,F)$ and

$$|U| = U \vee (-U), \; U_+ = U \vee 0, \; U_- = (-U) \vee 0$$

for the order relation on $L_r(E,F)$.

The set of all linear operators $U:E \longrightarrow F$ having strong moduli will be denoted by $SM(E,F)$. Clearly it is closed under multiplication by scalars and contains $L_r(E,F)_+ \cup (-L_r(E,F)_+)$; we shall see in § 4 that $SM(E,F)$ is generally not closed under addition.

PROPOSITION 3.3. Let G be another topological Riesz space and let $J:F \rightarrow G$ be a continuous Riesz homomorphism. Then for any $U \in SM(E,F)$ we have $JU \in SM(E,G)$ and $|JU| = J|U|$. If F is a closed Riesz subspace of G and J is the inclusion map, then $U \in SM(E,F)$ whenever $U:E \longrightarrow F$ is such that $JU \in SM(E,G)$.

For the proof of the next two propositions we shall need the following lemma :

LEMMA 3.1. Let $U,V \in L_r(E,F)$ be such that $U \in [-V,V]$ and let $x_1,\dots,x_n$, $y_1,\dots,y_n \in E$ be such that $0 \leq x_i \leq y_i$ for $1 \leq i \leq n$. Then

$$0 \leq V(\sum_{i=1}^{n} x_i) - \sum_{i=1}^{n} |U(x_i)| \leq V(\sum_{i=1}^{n} y_i) - \sum_{i=1}^{n} |U(y_i)| .$$

PROOF.

$$\sum_{i=1}^{n} |U(y_i)| - \sum_{i=1}^{n} |U(x_i)| \leq \sum_{i=1}^{n} |U(y_i-x_i)| \leq \sum_{i=1}^{n} V(y_i-x_i) =$$
$$= V(\sum_{i=1}^{n} y_i) - V(\sum_{i=1}^{n} x_i) .$$

PROPOSITION 3.4. Let G be a majorizing Riesz subspace of E and let $U,V \in L_r(E,F)$ be such that $U \in [-V,V]$ and $V|G$ is the strong modulus of $U|G$. Then V is the strong modulus of U.

PROOF. Let $x \in E_+$ be given. As $U \in [-V,V]$, $V(x)$ is an upper bound for $M_U(x)$. As G is majorizing, there is $y \in G$ such that $x \leq y$. Let W be a solid neighborhood of 0 in F. $V|G$ being the strong modulus of $U|G$, there is $(y_1,\ldots,y_n) \in D_G(y)$ such that $V(y) - \sum_{i=1}^{n} |U(y_i)| \in W$. The decomposition property gives an $(x_1,\ldots,x_n) \in D_E(x)$ verifying $x_i \leq y_i$ for $1 \leq i \leq n$. By lemma 3.1,

$$V(x) - \sum_{i=1}^{n} |U(x_i)| \leq V(y) - \sum_{i=1}^{n} |U(y_i)| ,$$

which implies that the leftmost member belongs to W; hence $V(x) = \operatorname{ssup} M_U(x)$.

PROPOSITION 3.5. Suppose that E is a topological Riesz space and let G be a dense Riesz subspace of E. Let $U,V: E \longrightarrow F$ be continuous linear operators. Then the following are equivalent:

i) $U \in SM(E,F)$ and $|U| = V$.

ii) $U|G \in SM(G,F)$ and $|U|G| = V|G$.

PROOF.

i) $\Longrightarrow$ ii) Follows from lemma 1.1.

ii) $\Longrightarrow$ i) Let $x \in E_+$ be given. As $U|G \in [-V|G, V|G]$ it follows by continuity that $U \in [-V,V]$; hence $V(x)$ is an upper bound for $M_U(x)$. Let W be a neighborhood of 0 in F and let W' be a solid neighborhood of 0 such that $W' + W' \subset W$. As G is dense in E there is $y \in G_+$ such that $2V(|x - y|) \in W'$. As $V|G$ is the strong modulus of $U|G$, there is $(y_1,\ldots,y_n) \in D_G(y)$ such that $V(y) - \sum_{i=1}^{n} |U(y_i)| \in W'$. The inequality

$$x \leq \sum_{i=1}^{n} y_i + |x - y|$$

and the decomposition property imply the existence of $(x_1,\ldots,x_{n+1}) \in D_E(x)$ verifying $x_i \leq y_i$ for $1 \leq i \leq n$ and $x_{n+1} \leq |x - y|$. By lemma 3.1,

$$0 \leq V(\sum_{i=1}^{n} x_i) - \sum_{i=1}^{n} |U(x_i)| \leq V(y) - \sum_{i=1}^{n} U(y_i) \quad ;$$

consequently, $V(x) - \sum_{i=1}^{n+1} |U(x_i)| \in W$ which shows that $V(x) = \text{ssup}\, M_U(x)$.

COROLLARY 3.1. Suppose F topologically complete. Then for any $U \in L_r(E,F)$ the following are equivalent :

i) $U \in SM(E,F)$.

ii) $U_x \in SM(\overline{E}_x, \hat{F})$ for any $x \in E_+$.

Moreover, if V is the strong modulus of U, then V_x is the strong modulus of U_x .

PROOF. If V is the strong modulus of U, then $V_x | E_x/N_x$ is clearly the strong modulus of $U_x | E_x/N_x$. As E_x/N_x is a dense Riesz subspace of $\overline{E}_x$ and U_x, V_x are continuous, proposition 3.5 implies that V_x is the strong modulus of U_x .

Conversely, if $U_x \in SM(\overline{E}_x, \hat{F})$ then $U_x | E_x/N_x \in SM(E_x/N_x, F)$ by propositions 3.3 and 3.5, which implies that $U | E_x \in SM(E_x, F)$; as x is arbitrary, $U \in SM(E,F)$.

COROLLARY 3.2. Let E,F be Banach lattices. Then for every $U \in SM(E,F)$ we have $\check{U} \in SM(\check{E}, \check{F})$ and $\check{\check{U}} \in SM(\check{\check{E}}, \check{\check{F}})$. The strong moduli are given by $|\check{U}| = \check{V}$, $|\check{\check{U}}| = \check{\check{V}}$ where $V = |U|$.

PROOF. Let $V = |U|$. As $U'' \in [-V'', V'']$ and E is a majorizing Riesz subspace of $\check{E}$, it follows by proposition 3.4 that $\check{U} \in SM(\check{E}, \check{F})$ and $\check{V} = |\check{U}|$. As $\check{E}$ is dense in $\check{\check{E}}$ and U'', V'' are continuous, it follows by proposition 3.5 that $\check{\check{U}} \in$ $\in SM(\check{\check{E}}, \check{\check{F}})$.

The next result gives a criterion for the existence of the strong modulus in the locally convex case. Before stating it, let us make the following remark. For any $U \in L_r(E,F)$ we can define in a natural way a transpose map $U' \in$ $\in L_r(F', E^{\sim})$; as $E^{\sim}$ is order complete, the usual modulus $|U'|$ of U' always exists.

PROPOSITION 3.6. If $U \in SM(E,F)$ then $|U'| = |U|'$. Conversely, if F is locally convex, $U \in L_r(E,F)$ and $|U'|$ is continuous for $\sigma(F',F)$ and $\sigma(E^{\sim},E)$, then $U \in SM(E,F)$.

PROOF. Suppose that $U \in SM(E,F)$. As $U \in [-|U|, |U|]$ it follows that $|U'| \leq$ $\leq |U|'$. To prove the reverse inequality, let $f \in F'_+$ and $x \in E_+$. As $|U|(x) \in$ $\in \overline{M_U(x)}$ we have

$$\langle |U|'(f), x \rangle = \langle f, |U|(x) \rangle = \sup f(M_U(x)) \ . \qquad (1)$$

By the Hahn-Banach theorem, for any $(x_1, \ldots, x_n) \in D(x)$ there are $f_1, \ldots, f_n \in F'$ such that $|f_i| \leq f$ and $\langle f, |U(x_i)| \rangle = \langle f_i, U(x_i) \rangle$. Consequently,

$$\langle f, \sum_{i=1}^{n} |U(x_i)| \rangle = \sum_{i=1}^{n} \langle U'(f_i), x_i \rangle \leq \sum_{i=1}^{n} \langle |U'|(|f_i|), x_i \rangle \leq$$

$$\leq \sum_{i=1}^{n} \langle |U'|(f), x_i \rangle = \langle |U'|(f), x \rangle .$$

Combining this with (1) we obtain

$$\langle |U|'(f), x \rangle \leq \langle |U'|(f), x \rangle$$

which completes the first part of the proof.

Conversely, suppose that F is locally convex, $U \in SM(E,F)$ and $|U'|$ is continuous for the weak topologies. Then there is $V : E \longrightarrow F$ such that $|U'| = V'$; this implies in particular that $U \in [-V, V]$. Let $x \in E_+$ be given. As $V(x)$ is an upper bound for $M_U(x)$ it remains to prove that $V(x)$ belongs to the topological closure of the solid hull of the convex set $M_U(x)$; to this purpose, it suffices to show that $\langle f, V(x) \rangle \leq 1$ whenever $f \in F'_+$ and $\langle f, y \rangle \leq 1$ for any $y \in M_U(x)$. So consider $f \in F'_+$ with the mentioned property. We have

$$\langle f, V(x) \rangle = \langle V'(f), x \rangle = \langle |U'|(f), x \rangle .$$

But

$$\left\{ \bigvee_{i=1}^{n} U'(g_i) \mid n \geq 1,\ g_i \in [-f, f] \right\} \uparrow |U'|(f) ;$$

as $h \mapsto \langle h, x \rangle$ is an order continuous map on $E^{\sim}$, it follows that

$$\left\{ \langle \bigvee_{i=1}^{n} U'(g_i), x \rangle \mid n \geq 1,\ g_i \in [-f, f] \right\} \uparrow \langle |U'|(f), x \rangle .$$

We have

$$\langle \bigvee_{i=1}^{n} U'(g_i), x \rangle = \sup \left\{ \sum_{i=1}^{n} \langle U'(g_i), x_i \rangle \mid (x_1, \ldots, x_n) \in D(x) \right\}$$

and

$$\sum_{i=1}^{n} \langle U'(g_i), x_i \rangle = \sum_{i=1}^{n} \langle g_i, U(x_i) \rangle \leq \sum_{i=1}^{n} \langle |g_i|, |U(x_i)| \rangle \leq$$

$$\leq \langle f, \sum_{i=1}^{n} |U(x_i)| \rangle \leq 1 .$$

Consequently, $\langle f, V(x) \rangle \leq 1$ and the proof is complete.

An important notion related to strong moduli is that of strong disjointness. We say that $U, V \in SM(E,F)$ are strongly disjoint if for every $x \in E_+$ and every neighborhood W of 0 in F there is $(x_1, \ldots, x_n) \in D(x)$ such that $\sum_{i=1}^{n} |U|(x_i) \wedge |V|(x_i) \in W$.

It is worthwile to note that for any $x \in E_+$ and any positive operators $S,T:E \to F$, the map $(x_1,\dots,x_n) \mapsto \sum_{i=1}^{n} S(x_i) \wedge T(x_i)$ is a decreasing map from $D(x)$ into F.

PROPOSITION 3.7. For every $U \in SM(E,F)$, U_+ and U_- are strongly disjoint. Conversely, if $U,V \in SM(E,F)$ are strongly disjoint, then $U + V \in SM(E,F)$ and $|U + V| = |U| + |V|$.

PROOF. Suppose that $U \in SM(E,F)$. Then using the identity

$$u \wedge v = 2^{-1}(u + v - |u - v|)$$

we obtain that

$$\sum_{i=1}^{n} U_+(x_i) \wedge U_-(x_i) = 2^{-1}(|U|(x) - \sum_{i=1}^{n} |U(x_i)|)$$

for any $x \in E_+$ and $(x_1,\dots,x_n) \in D(x)$; from this relation it follows that U_+ and U_- are strongly disjoint.

Conversely, let $U,V \in SM(E,F)$ be strongly disjoint and let $x \in E_+$ be given. Clearly, $|U|(x) + |V|(x)$ is an upper bound for $M_{U+V}(x)$; to prove that it is a strong supremum, we begin by remarking that in any Riesz space we have

$$|u| + |v| - |u + v| \leq 2(|u| \wedge |v|) \quad . \qquad (1)$$

Indeed, the first formula in theorem 11.9 of [10] gives

$$|u| + |v| = |u + v| \vee |u - v| ;$$

consequently,

$$|u| + |v| - |u + v| = (|u - v| - |u + v|)_+ \leq ||u - v| - |u + v|| .$$

But the rightmost member is equal, by the second formula in the mentioned theorem, to $2(|u| \wedge |v|)$.

Now let W be a neighborhood of 0 in F and let W' be a solid neighborhood of 0 such that $W' + W' + 2W' \subset W$. By the definitions of strong moduli and of strong disjointness and recalling the fact that $D(x)$ is upwards directed, we may find $(x_1,\dots,x_n) \in D(x)$ such that

$$|U|(x) - \sum_{i=1}^{n} |U(x_i)| \in W' ,$$

$$|V|(x) - \sum_{i=1}^{n} |V(x_i)| \in W',$$

$$\sum_{i=1}^{n} |U|(x_i) \wedge |V|(x_i) \in W'.$$

Using (1), we obtain

$$|U|(x) + |V|(x) - \sum_{i=1}^{n} |U(x_i) + V(x_i)| =$$

$$= |U|(x) - \sum_{i=1}^{n} |U(x_i)| + |V|(x) - \sum_{i=1}^{n} |V(x_i)| +$$

$$+ \sum_{i=1}^{n} (|U(x_i)| + |V(x_i)| - |U(x_i) + V(x_i)|) \leq$$

$$\leq |U|(x) - \sum_{i=1}^{n} |U(x_i)| + |V|(x) - \sum_{i=1}^{n} |V(x_i)| + 2\sum_{i=1}^{n} |U(x_i)| \wedge |V(x_i)| .$$

As $|U(x_i)| \leq |U|(x_i)$ and $|V(x_i)| \leq |V|(x_i)$, it follows that the leftmost member of the above inequality belongs to W and the proof is complete.

The connection between strong moduli and the theory developed in § 2 is given by the following result:

THEOREM 3.1. Let G be a principal A-module and let H be an order complete separated principal B-module. Then

$$SM(G,H) \cap L_r''(G,H) = Pr_{A\overline{\otimes}B}(L_r''(G,H)),$$

the topology on $L_r''(G,H)$ being the solid strong topology.

PROOF. Let $U \in SM(G,H) \cap L_r''(G,H)$. By proposition 3.7, U_+ and U_- are strongly disjoint; as they are also continuous on order bounded sets, theorem 4 in [24] implies that they are $A\overline{\otimes}B$-disjoint (observe that the proof of the refered theorem is unaffected by the fact that we require continuity on order bounded sets instead of usual continuity); hence U is $A\overline{\otimes}B$-principal.

Conversely, suppose that U is $A\overline{\otimes}B$-principal, so U_+ and U_- are $A\overline{\otimes}B$-disjoint. The definition of $A\overline{\otimes}B$-disjointness together with lemma 1.2 imply that for any $x \in E_+$ and any neighborhood W of 0 in F there are $a_1,\ldots,a_n \in A$ and $b_1,\ldots,b_n \in B$ such that the following hold (e_1 denotes the unit of A and e_2, the unit of B) :

i) $(a_1,\ldots,a_n) \in D(e_1)$.

ii) $b_i \in [0,e_2]$, $1 \leq i \leq n$.

iii) If $c = \sum_{i=1}^{n} a_i \otimes b_i$ then $|(e_1 \otimes e_2 - c)U_+|(x) \in W'$ and $|cU_-|(x) \in W'$

where W' is a solid neighborhood of 0 such that $W' + W' \subset W$.

But $c \geq 0$ and $e_1 \otimes e_2 - c = \sum_{i=1}^{n} a_i \otimes (e_2 - b_i) \geq 0$; hence $|(e_1 \otimes e_2 - c)U_+| = (e_1 \otimes e_2 - c)U_+$ and $|cU_-| = cU_-$. Consequently,

$$\sum_{i=1}^{n} U_+(a_i x) \wedge U_-(a_i x) \leq \sum_{i=1}^{n} ((e_2 - b_i)U_+(a_i x) + b_i U_-(a_i x)) =$$

$$= ((e_1 \otimes e_2 - c)U_+)(x) + (cU_-)(x)$$

which shows that the leftmost member belongs to W. As W is arbitrary and $(a_1 x, \ldots, a_n x) \in D(x)$, it follows that U_+ and U_- are strongly disjoint; thus, $U \in SM(E,F)$ by proposition 3.7.

COROLLARY 3.3. Suppose F topologically complete. Then for any $U \in L_r(E,F)$, the following are equivalent:

i) $U \in SM(E,F)$.

ii) $U_x \in Pr_{Z(\overline{E}_x) \bar{\otimes} Z(\hat{F})}(L_r(\overline{E}_x, \hat{F}))$ for every $x \in E_+$.

PROOF. By corollary 3.1, $U \in SM(E,F)$ iff $U_x \in SM(\overline{E}_x, \hat{F})$ for every $x \in E_+$; but by theorem 3.1, the latter set equals $Pr_{Z(\overline{E}_x) \bar{\otimes} Z(\hat{F})}(L_r(\overline{E}_x, \hat{F}))$ as $L_r(\overline{E}_x, \hat{F}) = L_r''(\overline{E}_x, \hat{F})$, $\overline{E}_x$ is a principal $Z(\overline{E}_x)$-module and $\hat{F}$ is a principal $Z(\hat{F})$-module.

As SM(E,F) is generally not closed under addition, it makes sense to introduce the following notions:

A strongly modular class of operators from E to F is a vector subspace of $L_r(E,F)$ contained in SM(E,F).

A strongly latticial class of operators from E to F is a strongly modular class which contains the strong modulus of each of its members.

Each strongly latticial class of operators from E to F is a Riesz space for the order induced from $L_r(E,F)$.

PROPOSITION 3.8. Suppose that the topology of F is locally convex. Then a vector subspace $L \subset L_r(E,F)$ is a strongly latticial class iff $\{U' \mid U \in L\}$ is a Riesz subspace of $L_r(F', E^{\sim})$.

PROOF. Follows from proposition 3.6.

PROPOSITION 3.9. Let L_1, L_2 be strongly modular (latticial) classes such that each $U_1 \in L_1$ is strongly disjoint from each $U_2 \in L_2$. Then $L_1 + L_2$ is a strongly modular (latticial) class.

PROOF. Follows from proposition 3.7.

PROPOSITION 3.10. Order disjoint elements in a strongly latticial class are strongly disjoint.

PROOF. Let L be a strongly latticial class and let $U_1, U_2 \in L$ be order disjoint. Put $V = |U_1| - |U_2|$. Then $V \in L$, $V_+ = |U_1|$ and $V_- = |U_2|$; the result follows then from proposition 3.7.

We give now the promised version of theorem 0.1 which holds even in the situation when the topology of F is not order continuous.

THEOREM 3.2. Let G be a principal A-module, H a separated principal B-module and let $L \subset L_r''(G,H)$ be a strongly latticial class and a submodule of the $A \otimes B$-module $L_r(G,H)$. Then L is a principal $A \bar{\otimes} B$-module for the solid strong topology.

PROOF. By proposition 3.10, order disjoint elements in L are strongly disjoint; hence, by theorem 4 in [24], they are $A \bar{\otimes} B$-disjoint. Consequently, L is principal by theorem 2.1.

COROLLARY 3.4. Let G, H and L be as in theorem 3.2 and let $\mathcal{M}$ be a collection of subsets of E_+ with the following property: for every neighborhood W of 0 in F, every $M \in \mathcal{M}$ and every $U \in L_+$ there is $y \in E_+$ such that $U((x-y)_+) \in W$ whenever $x \in M$. Then L is a principal $A \bar{\otimes} B$-module for the solid $\mathcal{M}$-topology. In particular, if $L \subset L_r'(E,F)$ then L is a principal $A \bar{\otimes} B$-module for the solid order precompact topology.

PROOF. Observe that U(M) is topologically bounded for every $U \in L$ and $M \in \mathcal{M}$, hence it makes sense to consider the solid $\mathcal{M}$-topology on L. Our assertion will follow from theorem 3.2 if we show that the solid strong topology is is stronger on each order interval of L than the solid $\mathcal{M}$-topology. Indeed, let $(V_\delta) \subset [-U, U]$ and $V \in [-U, U]$ be such that $V_\delta \longrightarrow V$ for the solid strong topology, let $M \in \mathcal{M}$ and let W be a neighborhood of 0 in F. There is a solid neighborhood W' of 0 such that $2W' + W' \subset W$. By hypothesis there is $y \in E_+$ such that $U((x-y)_+) \in W'$ whenever $x \in M$. Let $U_\delta = V_\delta - V$; as $U_\delta \longrightarrow 0$ for the solid strong topology, there is δ_0 such that $|U_\delta|(y) \in W'$ whenever $\delta \geq \delta_0$. It follows then from the inequality

$$|U_\delta|(x) \leq |U_\delta|((x-y)_+) + |U_\delta|(y) \leq 2U((x-y)_+) + |U_\delta|(y)$$

that $|U_\delta|(M)\subset W$ whenever $\delta\geq\delta_0$; hence $V_\delta\to V$ for the solid $\mathcal{M}$-topology.

We present now a monotone approximation theorem which applies to strongly latticial classes. First, some notations. For any Riesz space G and any subset $M\subset G$ we let $M^{\uparrow}$ be the subset of those $x\in G$ for which there is $(x_n)\subset M$ such that $x_n\uparrow x$; $M^{\downarrow}$ is similarly defined replacing $\uparrow$ by $\downarrow$. In the same way we define $M^{\Uparrow}$ and $M^{\Downarrow}$ replacing sequences by nets. We let M^{o} be the subset of those $x\in G$ for which there are $(x_n)\subset M$ and $(y_n)\subset G$ such that $|x-x_n|\leq y_n$ and $y_n\downarrow 0$; M^{ω} is similarly defined replacing sequences by nets.

The following monotone approximation theorem for principal modules was proved in [24]:

THEOREM 3.3 Let L_0 be an order complete Riesz space endowed with a separated Fatou topology (see [7]) and let L be a Riesz subspace of L_0 which is also a principal A-module (for the induced topology). Then we have

$$[0,x]\subset([0,e]x)^{\uparrow\downarrow\uparrow}\cap([0,e]x)^{\downarrow\uparrow\downarrow},$$
$$[-x,x]\subset([-e,e]x)^{\uparrow\downarrow\uparrow}\cap([-e,e]x)^{\downarrow\uparrow\downarrow}$$

for any $x\in L_+$ and

$$[-|x|,|x|]\subset([-e,e]x)^{o\omega\omega}$$

for any $x\in L$; here e denotes the unit of A, the order intervals from the left are considered in L while the order closures from the right are considered in L_0.

Using the above result one can state

THEOREM 3.4. Let G be a principal A-module, K an order complete Riesz space endowed with a separated Fatou topology, H a Riesz subspace of K which is also a principal B-module (for the induced topology). Consider a strongly latticial class $L\subset L_r^{n}(G,H)$ which is also a submodule of the $A\otimes B$-module $L_r(G,H)$ (and hence, an $A\tilde{\otimes}B$-module). Then we have

$$[0,U]\subset([0,e_1\otimes e_2]U)^{\uparrow\downarrow\uparrow}\cap([0,e_1\otimes e_2]U)^{\downarrow\uparrow\downarrow},$$
$$[-U,U]\subset([-e_1\otimes e_2,e_1\otimes e_2]U)^{\uparrow\downarrow\uparrow}\cap([-e_1\otimes e_2,e_1\otimes e_2]U)^{\downarrow\uparrow\downarrow}$$

for any $U\in L_+$ and

$$[-|U|,|U|]\subset([-e_1\otimes e_2,e_1\otimes e_2]U)^{o\omega\omega}$$

for any $U \in L$; here e_1 (respectively e_2) denotes the unit of A (respectively B), the order intervals from the left are considered in L, the order intervals from the right are considered in $A\tilde{\otimes}B$ and the order closures from the right are considered in $L_r(G,K)$.

PROOF. By theorem 3.2, L is a principal $A\tilde{\otimes}B$-module for the solid strong topology; hence we may apply theorem 3.3 to L and $L_0 = L_r(G,K)$, observing that the solid strong topology on $L_r(G,K)$ is Fatou.

The above result is a generalization in the framework of strongly latticial classes and principal modules of a theorem in [2], which is dealing with the monotone approximation of the components of a positive operator $U:E \longrightarrow F$ by its principal components. Our theorem is more general as, for instance, the theorem in [2] is proved under the assumption that the order continuous dual of the order complete Riesz space F separates it; however, there are order complete Riesz spaces endowed with a separated Fatou topology but admitting no nontrivial order continuous topology (for instance, the Dedekind extension of $C([0,1])$).

PROPOSITION 3.11. Consider the solid strong topology on $L_r(E,F)$. Then the following are true:

i) The map $U \longmapsto |U|$ is uniformly continuous on $SM(E,F)$.

ii) The set of all couples $(U,V) \in SM(E,F) \times SM(E,F)$ such that U and V are strongly disjoint is closed in $SM(E,F) \times SM(E,F)$.

iii) $SM(E,F)$ is closed in $L_r(E,F)$ whenever F is topologically complete.

PROOF.

i) The assertion is a consequence of the following inequality: for any $U,V,P \in SM(E,F)$ such that $U - V \in [-P,P]$ we have

$$|U| - |V| \in [-P,P] \qquad (1)$$

Indeed, it follows from

$$-|V| - P \leq V - P \leq U \leq V + P \leq |V| + P$$

that $|U| \leq |V| + P$; interchanging U and V we also obtain $|V| \leq |U| + P$.

ii) Let (S,T) belong to the closure of the set under consideration, let $x \in E_+$ and let W be a neighborhood of 0 in F. There is a solid neighborhood W' of 0 such that $W' + W' + W' \subset W$. By hypothesis we can find a couple (U,V) of strongly disjoint operators and two positive operators P,Q such that $S - U \in [-P,P]$, $T - V \in [-Q,Q]$, $P(x) \in W'$ and $Q(x) \in W'$. By (1) above, we also have $|S| - |U| \in [-P,P]$ and $|T| - |V| \in [-Q,Q]$. As U,V are strongly disjoint, there is $(x_1,\ldots,x_n) \in D(x)$ such that $\sum_{i=1}^{n} |U|(x_i) \wedge |V|(x_i) \in W'$. It follows then from the inequality

$$|u \wedge v - u' \wedge v'| \leq |u - u'| + |v - v'|$$

that

$$\sum_{i=1}^{n} |S|(x_i) \wedge |T|(x_i) \leq \sum_{i=1}^{n} (|U|(x_i) \wedge |V|(x_i) + ||S|(x_i) - |U|(x_i)| + $$
$$+ ||T|(x_i) - |V|(x_i)|) \leq \sum_{i=1}^{n} |U|(x_i) \wedge |V|(x_i) + P(x) + Q(x) .$$

Consequently, the leftmost member belongs to W and our assertion is proved.

iii) Let $U \in \overline{SM(E,F)}$; to show that $U \in SM(E,F)$ it suffices to prove that for any $x \in E_+$, the net $(x_1,\ldots,x_n) \longmapsto \sum_{i=1}^{n} |U(x_i)|$ $((x_1,\ldots,x_n) \in D(x))$ is Cauchy. We shall make use of the following remark: if a net $(x_\delta)_{\delta \in \Delta}$ has the property that for every neighborhood W of 0 there is a Cauchy net $(y_\delta)_{\delta \in \Delta}$ such that $x_\delta - y_\delta \in W$ for any $\delta \in \Delta$, then $(x_\delta)_{\delta \in \Delta}$ is itself a Cauchy net. So let W be any solid neighborhood of 0 in F. By hypothesis, there is $V \in SM(E,F)$ and a positive operator P such that $U - V \in [-P,P]$ and $P(x) \in W$. We have

$$\left| \sum_{i=1}^{n} |U(x_i)| - \sum_{i=1}^{n} |V(x_i)| \right| \leq \sum_{i=1}^{n} |U(x_i) - V(x_i)| \leq P(x)$$

for any $(x_1,\ldots,x_n) \in D(x)$; consequently, the leftmost member belongs to W. As $V \in SM(E,F)$, the net $(x_1,\ldots,x_n) \longmapsto \sum_{i=1}^{n} |V(x_i)|$ $((x_1,\ldots,x_n) \in D(x))$ is Cauchy; by the above remark, the corresponding net for U is also Cauchy.

From now on to the end of the section, F will be assumed topologically complete.

COROLLARY 3.5. The closure for the solid strong topology of any strongly modular (latticial) class of operators from E to F is a strongly modular (latticial) class.

THEOREM 3.5. Suppose E Archimedean and let $L \subset L_r(E,F)$ be a strongly modular class. Then

$$\left\{ \sum_{i=1}^{n} B_i U_i A_i \;\middle|\; n \geq 1,\ A_i \in Z(E),\ B_i \in Z(F),\ U_i \in L \right\}$$

is again a strongly modular class.

PROOF. We have to show that any V of the form

$$V = \sum_{i=1}^{n} B_i U_i A_i, \quad A_i \in Z(E),\ B_i \in Z(F),\ U_i \in L$$

belongs to SM(E,F). To this purpose, let $x \in E_+$ be given and let $C_i \in Z(\overline{E}_x)$ be the extension to $\overline{E}_x$ of the restriction of A_i to E_x, $D_i \in Z(\hat{F})$ be the extension of B_i to $\hat{F}$. By corollary 3.3, $\{U_x \mid U \in L\}$ is a $Z(\overline{E}_x) \overline{\otimes} Z(\hat{F})$-principal subspace of $L_r(\overline{E}_x, \hat{F})$; therefore, theorem 2.3 shows that V_x, which equals $\sum_{i=1}^{n} D_i (U_i)_x C_i$, is a $Z(\overline{E}_x) \otimes Z(\hat{F})$-principal element of $L_r(\overline{E}_x, \hat{F})$. As x is arbitrary, corollary 3.3 implies that $V \in SM(E,F)$.

THEOREM 3.6. Any maximal strongly modular class $L \subset L_r(E,F)$ is a strongly latticial class closed for the solid strong topology and provided E is Archimedean, a submodule of the $Z(E) \otimes Z(F)$-module $L_r(E,F)$.

PROOF. By corollary 3.5, L is closed; by theorem 3.5, it is a submodule. To see that it is a strongly latticial class, let $U \in L$ and let $V = |U|$. Consider an $x \in E_+$. By corollary 3.1, V_x is the strong modulus of U_x; by corollary 3.3, $\{T_x \mid T \in L\}$ is a $Z(\overline{E}_x) \overline{\otimes} Z(\hat{F})$-principal subspace of $L_r(\overline{E}_x, \hat{F})$. Corollary 2.3 shows that this subspace is contained in a principal Riesz subspace of $L_r(\overline{E}_x, \hat{F})$; it follows in particular that $(\alpha V + T)_x = \alpha V_x + T_x$ is $Z(\overline{E}_x) \overline{\otimes} Z(\hat{F})$-principal for any $\alpha \in \mathbb{R}$ and $T \in L$. As x is arbitrary, corollary 3.3 implies that $\{\alpha V + T \mid \alpha \in \mathbb{R}, T \in L\}$ is a strongly modular class containing L; but L is maximal, hence we must have $V \in L$.

COROLLARY 3.6. Every strongly modular class $L \subset L_r(E,F)$ (in particular, every operator having a strong modulus) is contained in a maximal strongly latticial class.

PROOF. By Zorn's lemma, L is contained in a maximal strongly modular class which, by theorem 3.6, must be latticial.

THEOREM 3.7. Suppose in addition that E is a Riesz A-module, F is a Riesz B-module and consider $L_r(E,F)$ as an $A \otimes B$-module. Then the closure for the solid strong topology of the $A \otimes B$-submodule generated by any strongly latticial class $L \subset L_r(E,F)$ is again a strongly latticial class.

PROOF. By theorem 3.6 and corollary 3.6, L is contained in a closed strongly latticial class L_0 which is also an $A\otimes B$-submodule of $L_r(E,F)$, hence an $A\bar{\otimes}B$-module. The closure of the $A\otimes B$-submodule generated by L is precisely the closure in L_0 of the $A\bar{\otimes}B$-submodule generated by L (use lemma 1.2); consequently, the proof will be concluded by applying the following result:

PROPOSITION 3.12. In any topological Riesz A-module, the closure of the submodule generated by a Riesz subspace is a Riesz subspace.

PROOF. Let G be a topological Riesz A-module, let L be a Riesz subspace of G and let L_0 be the closure of the submodule generated by L. All we need to do is to show that $|\sum_{i=1}^{n} a_i x_i| \in L_0$ whenever $x_1,\ldots,x_n \in L$ and $a_1,\ldots,a_n \in A$. To this purpose, let $x = \sum_{i=1}^{n} |x_i|$. $L \cap G_x$ is a Riesz subspace of G_x containing x and $L_0 \cap G_x$ is a submodule of G_x closed for $\| \ \|_x$; hence, the proof will be finished if we show that in every Riesz A-module with a strong order unit $x \geq 0$, the closure for $\| \ \|_x$ of the submodule generated by a Riesz subspace containigg x is a Riesz subspace. We may of course assume that the Riesz A-module under consideration is complete for $\| \ \|_x$, hence isomorphic to a space C(X), x being identified to the function identically one e. We shall make use of the well known result according to which a closed vector subspace of C(X) containing e is a Riesz subspace iff it is a subalgebra. Let H be a Riesz subspace of the A-module C(X) containing e; replacing, if necessary, H by $\bar{H}$ we may assume that H is closed, hence it is a subalgebra. The formula (1) in the proof of theorem 2.3 shows that the submodule generated by H is also a subalgebra. Consequently, its closure is a subalgebra, hence a Riesz subspace.

4. Examples of strongly modular and strongly latticial classes

a) <u>The class</u> Orth(F).

Recall that $U \in L_r(F,F)$ is called an orthomorphism if $|U(x)| \wedge |y| = 0$ whenever $x,y \in F$ and $|x| \wedge |y| = 0$. The class of all such operators is denoted by Orth(F). It is a strongly latticial class: indeed, it is well known (see [3]) that the modulus of any $U \in$ Orth(F) is given by

$$|U|(x) = |U(x)| \quad , \; x \in F_+$$

which shows in particular that $|U|(x) = \mathrm{ssup}\, U([-x,x])$ for any $x \in F_+$.

b) <u>The class</u> $L_{oru}(E,F)$

A subset of M is called relatively uniformly totally bounded if it is contained in a principal order ideal F_x and it is totally bounded for $\| \|_x$.

A linear operator $U:E \longrightarrow F$ is called oru-compact if it maps order bounded subsets onto relatively uniformly totally bounded subsets. The class of all such operators is denoted by $L_{oru}(E,F)$.

Whenever F is relatively uniformly complete, then any relatively uniformly totally bounded subset M of F admits a strong supremum and, moreover, the set $\{ \mathrm{ssup}\, N \mid N \subset M \}$ is also relatively uniformly totally bounded; consequently, $L_{oru}(E,F)$ is a strongly latticial class in this case. This class was considered in detail in [26].

c) <u>The class</u> $\mathcal{F}(E,F)$

It is the vector subspace of $L_r(E,F)$ generated by all operators $f \otimes y$ with $f \in E^\sim$ and $y \in F$, where $f \otimes y$ denotes the operator $x \longmapsto f(x)y$. As $\mathcal{F}(E,F) \subset L_{oru}(E,F)$, it follows that $\mathcal{F}(E,F)$ is a strongly modular class whenever F is relatively uniformly complete. In general, it is not a strongly latticial class.

d) <u>The class</u> $\overline{\mathcal{F}_G(E,F)}$.

For any order ideal $G \subset E^\sim$ we let $\mathcal{F}_G(E,F)$ be the vector subspace of $L_r(E,F)$ generated by the operators $f \otimes y$ with $f \in G$ and $y \in F$; then we let $\overline{\mathcal{F}_G(E,F)}$ be the closure of $\mathcal{F}_G(E,F)$ for the solid strong topology.

$\overline{\mathcal{F}_G(E,F)}$ is a strongly latticial class whenever F is topologically complete. Indeed, as $\overline{\mathcal{F}_G(E,F)}$ is strongly modular, it suffices by proposition 3.11 and corollary 3.5 to show that $|U| \in \overline{\mathcal{F}_G(E,F)}$ whenever $U \in \mathcal{F}_G(E,F)$. We have $U = \sum_{i=1}^{n} f_i \otimes y_i$ with $f_i \in G$ and $y_i \in F$, hence $|U| \in L_{oru}(E,F)$ and $|U| \leq f \otimes y$ where $f = \sum_{i=1}^{n} |f_i| \in G$ and $y = \sum_{i=1}^{n} |y_i|$; consequently, theorem 3.1 in [26] implies $|U| \in \overline{\mathcal{F}_G(E,F)}$.

When the topology of F is metrizable and complete, $\overline{\mathcal{F}(E,F)}$ coincides with $L_{oru}(E,F)$ (see [26]).

We have the following strong disjointness theorem for $\overline{\mathcal{F}_G(E,F)}$ in case a Riesz space is an element disjoint from every atomic element, the latter being those x such that $[0,|x|]$ contains scalar multiples of $|x|$ only.

THEOREM 4.1. Suppose F topologically complete. Then every Riesz homomorphism $J:E \to F$ is strongly disjoint from every $U \in \overline{\mathcal{F}_G(E,F)}$, G being the band of all diffuse elements in $E^{\sim}$.

PROOF. By proposition 3.11, it suffices to show that J is strongly disjoint from any $U \in \mathcal{F}_G(E,F)$. Clearly, it suffices to consider the case $U = f \otimes y$ with $f \in G_+$ and $y \in F_+$. Let $x \in E_+$ and let W be a solid neighborhood of 0 in F. There is a compact space X and an order isomorphism $T:\overline{E}_x \to C(X)$ such that $T(x) = e$ (the function identically one on X). Let $g = (T')^{-1}(f_x)$ and let μ be the Radon measure on X associated with g. We must have $\mu(\{s\}) = 0$ for any $s \in X$; otherwise, we would find a nonzero Riesz homomorphism $h:C(X) \to \mathbb{R}$ such that $h \leq g$. If $P:E_x \to \overline{E}_x$ denotes the canonical map and if $k = (TP)'(h)$ then k is a Riesz homomorphism on E_x such that $k \leq f|E_x$. We extend k to E by defining

$$k(u) = \sup_{n \geq 1} k(u \wedge nx) \quad , \quad u \in E_+ ;$$

the definition is correct because $k(u \wedge nx) \leq f(u \wedge nx) \leq f(u)$. k is a nonzero Riesz homomorphism such that $k \leq f$; in fact, k is atomic, because if $l \in [0,k]$ then

$$|l(u)| \leq l(|u|) \leq k(|u|) = |k(u)| = 0$$

for any $u \in E$ such that $k(u) = 0$, which implies that $l = \alpha k$ for some $\alpha \in \mathbb{R}$. But f being diffuse, we must have $k \wedge f = 0$, leading thus to a contradiction.

As μ is diffuse, there is a partition $(M_i)_{1 \leq i \leq n}$ into Borel subsets of X such that $\mu(M_i) \leq \varepsilon/3$, $\varepsilon > 0$ being choosen so that $\varepsilon y \in W$. Let N_i be a closed subset of M_i such that $\mu(M_i \setminus N_i) \leq \varepsilon/2n$ and let $(\varphi_i)_{1 \leq i \leq n} \subset T(E_x/N_x)$ be a sequence of positive pairwise disjoint functins such that $g(\varphi_i) \leq \varepsilon/2$ and $\varphi_i(s) = 1$ for $s \in N_i$ (its existence is a consequence of Step 2 in the proof of theorem 2.4 in [23]). By construction we have $g(e - \sum_{i=1}^{n} \varphi_i) \leq \varepsilon/2$. P being a Riesz homomorphism, we may find a sequence $(x_i)_{1 \leq i \leq n} \subset E_x$ of pairwise disjoint elements such that $TP(x_i) = \varphi_i$ and $x_i \in [0,x]$. Put $x_{n+1} = x - \sum_{i=1}^{n} x_i$. Then $(x_1,\dots,x_{n+1}) \in D(x)$ and

$$\sum_{i=1}^{n+1} J(x_i) \wedge f(x_i)y \leq \sum_{i=1}^{n} J(x_i) \wedge f(x_i)y + f(x_{n+1})y =$$

$$= \bigvee_{i=1}^{n} J(x_i) \wedge f(x_i)y + f(x_{n+1})y \leq$$

$$\leq \bigvee_{i=1}^{n} f(x_i)y + f(x_{n+1})y =$$

$$= \bigvee_{i=1}^{n} g(\varphi_i)y + g(e - \sum_{i=1}^{n} \varphi_i)y \leq \varepsilon y$$

as the $J(x_i)$'s are pairwise disjoint; consequently, the leftmost member belongs to W and the proof is complete.

As a consequence of the preceding theorem and proposition 3.9, we see for instance that $\mathrm{Orth}(F) + \overline{\mathcal{F}_G(F,F)}$ is a strongly latticial class whenever F is topologically complete and G is the band of all diffuse elements in $F^{\sim}$.

e) $\mathcal{R}$-modules.

The notion of an $\mathcal{R}$-module was introduced by H.U. Schwarz in [16]. An $\mathcal{R}$-module is a class $\mathcal{B}$ of continuous linear operators between Banach lattices so that, if one denotes by $\mathcal{B}(E,F)$ the set of those U in $\mathcal{B}$ which act between the Banach lattices E and F, the following are supposed to hold:

i) $\mathcal{F}(E,F) \subset \mathcal{B}(E,F)$.

ii) $\mathcal{B}(E,F)$ is a vector space of operators.

iii) $RST \in \mathcal{B}(E_0,F_0)$ whenever $S \in L_r(E_0,E)$, $T \in \mathcal{B}(E,F)$ and $R \in L_r(F,F_0)$.

Normed $\mathcal{R}$-modules are defined as $\mathcal{R}$-modules $\mathcal{B}$ for which the spaces $\mathcal{B}(E,F)$ are endowed with a norm satisfying certain requirements ([16], ch. III, §2). The closure of $\mathcal{F}(E,F)$ with respect to the norm of $\mathcal{B}(E,F)$ is denoted by $\mathcal{B}^0(E,F)$. It is proved ([16], ch. III, §3) that, under certain assumptions, $\mathcal{B}^0(E,F)$ is a Banach lattice of operators. Actually, it follows that $\mathcal{B}^0(E,F)$ is contained in the closure of $\mathcal{F}(E,F)$ for the regular norm (see the proof of Satz 8 in [16], ch. III, §2), hence it is a strongly latticial class.

f) The class M(E,F).

For any Banach lattices E,F, let M(E,F) denote the class of those operators U for which $U(B_E)$ is a relatively uniformly totally bounded subset of F; a norm is introduced on M(E,F) by

$$\|U\|_M = \|\sup U(B_E)\| .$$

Obseving that $M(E,F) \subset L_{\ldots}(E,F)$, one can easily show that M(E,F) is a strongly latticial class.

g) <u>The class</u> $\Lambda(E,F)$.

For any Banach lattices E,F, let $\Lambda(E,F)$ denote the class of those operators $U:E \longrightarrow F$ representable as U_2U_1 where U_1 is a positive operator from E into an AL-space G and U_2 is a compact operator from G into F.

The classes M(E,F) and $\Lambda(E,F)$ were introduced and studied by J. Chaney in [4] in connection with the operatorial representation of M-tensor products.

To prove that $\Lambda(E,F)$ is a strongly latticial class we need the following result:

PROPOSITION 4.1.

i) If $U \in M(E,F)$ then $U' \in SM(F',E')$.

ii) If $U \in L(E,F)$ and $U' \in M(F',E')$ then $U \in SM(E,F)$.

PROOF.

i) By theorem 2.1 in [4], $\mathcal{F}(E,F)$ is dense in M(E,F) for the norm $\| \ \|_M$; as $\|U\|_r \leq \|U\|_M$ for $U \in M(E,F)$, it follows that M(E,F) is contained in the closure of $\mathcal{F}(E,F)$ with respect to $\| \ \|_r$. The result follows now from proposition 3.11 taking into account the continuity of the map $U \longmapsto U'$ with respect to regular norms.

ii) By proposition 3.6, we have to show that $|U'|$ is continuous for $\sigma(F',F)$ and $\sigma(E',E)$; this is equivalent to $|U'|'(E) \subset F$. By the same proposition, $|U'|' = |U''|$. Now $U'' \in SM(E'',F'')$ by i) above and $U''(E'') \subset F$ as U is compact; consequently, $|U''|(E) \subset |U''|(E'') \subset F$.

Returning to the proof of our assertion, it was observed in [4] that $U \in \Lambda(E,F)$ iff $U' \in M(F',E')$; therefore, proposition 4.1 implies that $\Lambda(E,F)$ is strongly modular. Now if $U \in \Lambda(E,F)$, proposition 3.6 gives $|U|' = |U'| \in M(F',E')$; hence $|U| \in \Lambda(E,F)$ which proves that $\Lambda(E,F)$ is strongly latticial.

h) <u>Classes of operators on spaces of continuous functions</u> .

Let X,Y be compact spaces and let $U:C(X) \longrightarrow C(Y)$ be a positive operator. For any $f \in C(X \times Y)$ define the operator $f \otimes U$ as follows: $(f \otimes U)(g)$ is the function $t \longmapsto U(f_t g)(t)$ where f_t is the function $s \longmapsto f(s,t)$.

The set $L = \{ f \otimes U \mid f \in C(X \times Y) \}$ is a strongly latticial class and the map $f \longmapsto f \otimes U$ is a Riesz homomorphism. To see this, let L_1 be a maximal strongly latticial class containing U. L_0 is a $C(X)\otimes C(Y)$-submodule of $L_r(C(X),C(Y))$, hence a $C(X)\bar{\otimes}C(Y)$-module. Let $h:C(X)\bar{\otimes}C(Y) \longrightarrow C(X\times Y)$ be the canonical Riesz homomorphism. We clearly have $cU = h(c)\otimes U$ for any $c\in C(X)\otimes C(Y)$, hence

$$cU = h(c)\otimes U \ , \quad c\in C(X)\bar{\otimes}C(Y)$$

by the density of $C(X)\otimes C(Y)$ in $C(X)\bar{\otimes}C(Y)$. Consequently,

$$|h(c)\otimes U| = |cU| = |c|U = h(|c|)\otimes U = |h(c)|\otimes U \ .$$

As the map $f\longmapsto f\otimes U$ is continuous for the norm of $C(X\times Y)$ and the regular norm and L_0 is closed for the regular norm, we obtain that $f\otimes U\in L_0$ and $|f\otimes U| = |f|\otimes U$ for any $f\in C(X\times Y)$. The proof is complete.

i) <u>When is $L_r(E,F)$ a strongly latticial class</u>?

When E is arbitrary and F is fixed, an answer is provided by the following result:

THEOREM 4.2. If F is order complete and its topology is order continuous, then $L_r(E,F)$ is a strongly latticial class for every Riesz space E. Conversely, if F is topologically complete and $L_r(L_\infty([0,1]),F)$ is a strongly latticial class, then the topology of F is order continuous.

PROOF. If F is order complete and its topology is order continuous, then every order bounded subset of F has a strong supremum; consequently, $L_r(E,F)$ is a strongly latticial class.

Conversely, ley F be topologically complete and let $L_r(E,F)$ be a strongly latticial class for $E = L_\infty([0,1])$. Suppose that the topology of F is not order continuous. Then there are by proposition 24H in [7] $x\in F_+$ and a disjoint sequence $(x_n)\subset[0,x]$ not convergent to 0. Define $U,V:E\longrightarrow F$ by

$$U(f) = \sum_{n=0}^{\infty} \left(\int_0^1 fr_n dt\right)x_n \ ,$$

$$V(f) = \left(\int_0^1 f dt\right)x$$

where r_n is the n-th Rademacher function ($r_n(t) = \text{sign}\sin 2^n t$). The fact that $\int_0^1 fr_n dt \longrightarrow 0$ as $n\longrightarrow\infty$ for $f\in E$ and the topological completeness of F ensure that U is well defined.

Let $L_0\subset L_r(E,\hat{F})$ be a maximal strongly latticial class containing $L_r(E,F)$.

By theorems 3.2 and 3.6, L_0 is a principal $Z(E)\hat{\otimes} Z(\hat{F})$-module for the solid strong topology. As $U\in[-V,V]$ it follows that U belongs to the closure for the solid strong topology of the submodule generated by V (take into account lemma 1.2); as $V([-e,e])$ is totally bounded (e being the function identically one on $[0,1]$) we obtain that $U([-e,e])$ is also totally bounded. But $x_n = U(r_n)\in U([-e,e])$; as any totally bounded disjoint sequence must converge to 0, we have arrived at a contradiction. The proof is complete.

From the above theorem we see why SM(E,F) is generally not closed under addition: otherwise, we would have that $L_r(E,F) = L_r(E,F)_+ - L_r(E,F)_+$ is always strongly latticial.

The next theorem provides an answer in the case when E is a fixed Banach lattice and F is arbitrary.

THEOREM 4.3. Let E be a discrete Banach lattice with order continuous norm. Then $L_r(E,F)$ equals $L_{oru}(E,F)$ for any relatively uniformly complete topological Riesz space F, hence it is a strongly latticial class.

Conversely, if E is a Banach lattice such that $L_r(E,l_\infty)$ is a strongly latticial class, then E is discrete and has order continuous norm.

PROOF. By theorem 2.3 in [4], $1_E\in L_{oru}(E,E)$ whenever E is a discrete Banach lattice with order continuous norm; consequently, $L_r(E,F) = L_{oru}(E,F)$ for every relatively uniformly complete topological Riesz space F.

For the proof of the converse we shall need a lemma.

LEMMA 4.1. Let E be a Banach space and let M be a subset of E with the following property: for every $\varepsilon > 0$ and every sequences $(x_n)\subset M$, $(f_n)\subset B_{E'}$ there is a compact convex set $K\subset E$ such that $\inf_{x\in K} f_m(x_n - x)\leq \varepsilon$ for any m,n. Then M is relatively compact.

PROOF. Suppose that M is not relatively compact. Then there is $\eta > 0$ and a sequence $(x_n)\subset M$ such that $\|x_m - x_n\|\geq\eta$ for $m\neq n$. Choose a double sequence $(f_{mn})_{m,n\geq 1,\, m\neq n}\subset B_{E'}$ such that $f_{mn}(x_m - x_n)\geq\eta$. Define the seminorm p on E by

$$p(x) = \sup_{\substack{m,n\geq 1\\ m\neq n}} {}^{?}|f_{mn}(x)|$$

and let $F = E/p^{-1}(\{0\})$ be the normed space associated with p. Each f_{mn} gives rise to $g_{mn}\in B_{F'}$; denoting by $\|\ \|_1$ the norm on F we have

$$\|y\|_1 = \sup_{\substack{m,n\geq 1\\ m\neq n}} |g_{mn}(y)| \quad , \quad y\in F \, . \tag{1}$$

Let $J:E \longrightarrow F$ be the canonical map and let $y_n = J(x_n)$. By the choice of the f_{mn}'s we have $\|y_m - y_n\|_1 \geq \eta$ for $m \neq n$. We shall derive a contradiction by showing that (y_n) is a totally bounded sequence in F. This will be done by proving that for every $\varepsilon > 0$ there is a compact subset $L\subset F$ such that the distance from each y_n to L is less than ε. So let $\varepsilon > 0$ be given. The relation (1) shows that $co\{\pm g_{mn} \mid m,n \geq 1,\ m\neq n\}$ is $\sigma(F',F)$-dense in $B_{F'}$; consequently, there is a sequence (h_n) which is $\sigma(F',F)$-dense in $B_{F'}$. By hypothesis there is a compact convex set $K\subset E$ such that $\inf_{x\in K} \langle J'(h_m), x_n - x\rangle < \varepsilon/3$ for any m,n. Put $L= J(K)$ and assume that for some n we have $\inf_{y\in L} \|y_n - y\|_1 > \varepsilon$. Then $y_n \notin L + \varepsilon B_F$; as the latter is a closed convex set there is $h\in F'$ such that $\|h\| = 1$ and $\sup h(L+\varepsilon B_F) \leq$ $\leq h(y_n)$. But

$$\sup h(L + \varepsilon B_F) = \sup h(L) + \varepsilon \sup h(B_F) = \sup h(L) + \varepsilon \quad ;$$

consequently, $\inf_{y\in L} h(y_n - y) \geq \varepsilon$. As L is compact and (h_n) is $\sigma(F',F)$-dense in $B_{F'}$, there is m such that $\sup_{y\in L} |h(y) - h_m(y)| \leq \varepsilon/3$ and $|h(y_n) - h_m(y_n)| \leq \varepsilon/3$. It follows that

$$\varepsilon/3 \leq \inf_{y\in L} h_m(y_n - y) = \inf_{x\in K} \langle J'(h_m), x_n - x\rangle < \varepsilon/3 \, .$$

The contradiction so obtained shows that $\inf_{y\in L} \|y_n - y\|_1 \leq \varepsilon$ and the proof is complete.

We return to the proof of the theorem. Let E be a Banach lattice such that $L_r(E,l_\infty)$ is strongly latticial. We shall prove that each order interval in E is compact; a theorem of B. Walsh [28] will then lead to the conclusion.

We remark first that if $L_r(E,l_\infty)$ is strongly latticial, then so is $L_r(G,l_\infty)$ for any closed order ideal G in E (as each $U\in L_r(G,l_\infty)$ is the restriction of some $V\in L_r(E,l_\infty)$). Thus, we may assume without loosing generality that E has a quasi interior element; this guarantees that E is a principal $Z(E)$-module.

Consider an order interval $[0,x]$ in E; as $[0,1_E]x$ is dense in $[0,x]$, all we need to show is that $[0,1_E]x$ is relatively compact. We shall use lemma 4.1; so let $\varepsilon > 0$, $(A_n)\subset [0,1_E]$ and $(f_n)\subset B_{E'}$ be given. Denote by F the Banach lattice of all bounded double sequences of scalars (the norm being the sup norm) and consider the operators $U,V\in L_r(E,F)$ defined by

$$U(y) = (|f_m|(A_n(y)))_{m,n\geq 1}$$

$$V(y) = (|f_m|(y))_{m,n\geq 1} \quad .$$

F is an f-algebra for pointwise multiplication, hence a principal module over itself. As F is topologically and order isomorphic to l_∞, $L_r(E,F)$ is strongly latticial; therefore, theorem 3.2 implies that $L_r(E,F)$ is a principal $Z(E)\widetilde{\otimes} F$-module for the solid strong topology. In particular, as $0\leq U\leq V$, it follows that there is $c\in[0,1_E\otimes e]$ (e being the unit of F) such that

$$\| |U - cV|(x)\| \leq \varepsilon \; ; \tag{1}$$

by lemma 1.2, c can be taken of the form $\sum_{i=1}^{k} B_i\otimes z^i$ with $B_i\in[0,1_E]$ and $(z^1,\dots,z^k)\in D(e)$.

Now we remark that if $S\in L_r(E,F)$ is given by

$$S(y) = (g_{mn}(y))_{m,n\geq 1}$$

with $(g_{mn})\in E'$, then $|S|$ is given by

$$|S|(y) = (|g_{mn}|(y))_{m,n\geq 1} \; .$$

Taking this into account and the fact that $|C'f| = |C|'|f|$ for $C\in Z(E)$ and $f\in E'$, we derive from (1)

$$\sup_{m,n\geq 1} |f_m|(|A_n(x) - \sum_{i=1}^{k} \alpha^i_{mn} B_i(x)|) \leq \varepsilon \; ,$$

the scalars α^i_{mn} being determined by $(\alpha^i_{mn})_{m,n\geq 1} = z^i$. Consequently, if $K = \operatorname{co}\{B_i(x) \mid 1\leq i\leq k\}$ then

$$\inf_{y\in K} f_m(A_n(x) - y) \leq \inf_{y\in K} |f_m|(|A_n(x) - y|) \leq \varepsilon$$

and the proof is complete.

COROLLARY 4.1. Let F be topologically complete and σ-order complete, let E be a Banach lattice and suppose that $L_r(E,F)$ is strongly latticial. Then at least one of the following conditions holds:

i) The topology of F is order continuous.

ii) E is discrete and has order continuous norm.

PROOF. If the topology of F is not order continuous then F contains a Riesz subspace order and topologically isomorphic to l_∞; apply theorem 4.3.

The reader is asked to compare the above result with theorem 10.2 in [14] which characterizes discrete Banach lattices with order continuous norm as being those Banach lattices E with the property that $L_r(E,F)$ is a Riesz space for any Banach lattice F: the distinction is given by the fact that, whenever F is order complete, $L_r(E,F)$ is always a Riesz space while not always a strongly latticial class.

5. Applications to operators on Banach lattices

Throughout the section, E and F will be Banach lattices.

We begin with a principality theorem for strongly latticial classes of operators between Banach lattices.

Call an operator from a Banach space G into F order compact if it maps B_G onto an order precompact subset of F.

PROPOSITION 5.1. Let G be a Banach space. For every $U \in L(E,G)$ the following are equivalent:

i) U' is order compact.

ii) For every $\varepsilon > 0$ there is $f \in E'_+$ such that $\|U(x)\| \leq f(|x|) + \varepsilon\|x\|$ for $x \in E$.

PROOF.

i)$\Longrightarrow$ ii) As $U'(B_{G'})$ is order precompact there is $f \in E'_+$ such that $\|(|U'(g)| - f)_+\| \leq \varepsilon$ whenever $g \in B_{G'}$. Consequently,

$$g(U(x)) = \langle U'(g), x \rangle \leq \langle |U'(g)|, |x| \rangle \leq$$
$$\leq \langle (|U'(g)| - f)_+, |x| \rangle + f(|x|) \leq \varepsilon\|x\| + f(|x|)$$

for any $g \in B_{G'}$, hence

$$\|U(x)\| = \sup_{g \in B_{G'}} g(U(x)) \leq \varepsilon\|x\| + f(|x|) .$$

ii)$\Longrightarrow$i) Let $f \in E'_+$ be such that $\|U(x)\| \leq f(|x|) + 2^{-1}\varepsilon\|x\|$ for $x \in E$. Then

$$\langle (U'(g) - f)_+, x \rangle = \sup_{0 \leq y \leq x} \langle U'(g) - f, y \rangle \leq \sup_{0 \leq y \leq x} (\|U(y)\| - f(y)) \leq \varepsilon/2$$

for any $g \in B_G$, and $x \in B_E \cap E_+$; thus, $\|(U'(g) - f)_+\| \leq \varepsilon/2$. The same is true replacing g by -g; hence

$$\|(|U'(g)| - f)_+\| = \|(U'(g) - f)_+ \vee (U'(-g) - f)_+\| \leq \varepsilon .$$

THEOREM 5.1. Let L be a strongly latticial class of operators from E to F. Suppose that E is a principal A-module, F is a principal B-module and L is a submodule of the $A \otimes B$-module $L_r(E,F)$. Then the following are true:

i) L is a principal $A \tilde{\otimes} B$-module for the solid order precompact topology.

ii) Suppose that E' has order continuous norm and that U' is order compact for every $U \in L$. Then L is a principal $A \tilde{\otimes} B$-module for the regular norm.

PROOF.

i) is a consequence of corollary 3.4 observing that $L_r(E,F) = L_r'(E,F)$.

ii) We shall verify that for every $\varepsilon > 0$ and every $U \in L_+$ there is $y \in E_+$ such that $\|U((x - y)_+)\| \leq \varepsilon$ whenever $x \in B_E$; an application of corollary 3.4 will then conclude the proof.

Indeed, by proposition 5.1 there is $f \in E'_+$ such that $\|U(x)\| \leq f(|x|) + 2^{-1}\varepsilon\|x\|$ for $x \in E$; by theorem 1.2 there is $y \in E_+$ such that $f((x - y)_+) \leq \varepsilon/2$ for $x \in B_E$. Consequently,

$$\|U((x - y)_+)\| \leq f((x - y)_+) + 2^{-1}\varepsilon\|(x - y)_+\| \leq \varepsilon$$

for $x \in B_E$.

The first application is an extension of theorem 4.3 in [22] on ideal properties of those operators which can be approximated with finite rank operators on every compact set.

The compact topology on $L_r(E,F)$ is given by the seminorms $U \longmapsto \sup_{x \in K} \|U(x)\|$ for every compact subset K of E. The notation CA(E,F) (respectively $CA_+(E,F)$) was used in [22] for the closure of $\mathcal{F}(E,F)$ (respectively $\mathcal{P}(E,F)$) with respect to the compact topology ($\mathcal{P}(E,F)$ denotes the cone in $L_r(E,F)$ generated by the operators $f \otimes y$ with $f \in E'_+$ and $y \in F_+$).

THEOREM 5.2. For every strongly latticial class L of operators from E to F, $CA(E,F) \cap L$ and $\{U \mid U \in L, |U| \in CA_+(E,F)\}$ are order ideals in L.

PROOF. Clearly $U \in CA(E,F)$ (respectively $U \in CA_+(E,F)$) iff its restriction tively $CA_+(G_x,F)$); thus, we may assume that E and F have quasi interior elements.

Consider $U,V \in L$ such that $|U| \leq |V|$ and $V \in CA(E,F)$ (respectively $|V| \in CA_+(E,F)$). Let L_0 be a maximal strongly latticial class containing L. By theorem 5.1 i) applied to L_0 and lemma 1.2, there are nets (c_δ), $(d_\delta) \subset Z(E) \otimes Z(F)$ such that $c_\delta V \longrightarrow U$, $d_\delta |V| \longrightarrow |U|$ for the solid order precompact topology and each d_δ is an element of the form $\sum_{i=1}^{n} A_i \otimes B_i$ with $A_i \in Z(E)_+$, $B_i \in Z(F)_+$. It follows that $c_\delta V \longrightarrow U$ and $d_\delta |V| \longrightarrow |U|$ for the compact topology; thus, $U \in CA(E,F)$ (respectively $|U| \in CA_+(E,F)$).

The next application concerns the relation between the order ideal and the closed operator ideal generated by a regular operator; it arises as a generalisation of theorem 3.7 in [22]. We shall need the concept of an operator ideal as defined by A. Pietsch in [13]:

A class $\mathfrak{U}$ of continuous linear operators between Banach spaces is called an operator ideal if the components $\mathfrak{U}(G,H) = \mathfrak{U} \cap L(G,H)$ satisfy the following conditions:

i) $\mathfrak{U}(G,H)$ is a vector subspace of $L(G,H)$ for any Banach spaces G,H.

ii) If $T \in L(G_0,G)$, $S \in \mathfrak{U}(G,H)$ and $R \in L(H,H_0)$ then $RST \in \mathfrak{U}(G_0,H_0)$.

An operator ideal is called closed if all its components are closed for the operator norm.

For every $S \in L(G,H)$ there is a smallest closed operator ideal $\mathfrak{U}_S$ containing S; its component $\mathfrak{U}_S(G_0,H_0)$ is the closed vector subspace of $L(G_0,H_0)$ generated by the operators RST with $T \in L(G_0,G)$ and $R \in L(H,H_0)$.

The notation J_F will be used for the canonical embedding of F into $\check{F}$. The notation $\check{F}'$ will refer to the dual of $\check{F}$ considered as a normed subspace of F".

THEOREM 5.3. Let E_0,F_0 be Banach spaces and let E,F be Banach lattices which are principal modules over their centers. Then the following are true:

i) Let $U,V: E \longrightarrow F$ be order bounded operators such that $|J_F U| \leq |J_F V|$ and let $T \in L(E_0,E)$, $R \in L(F,F_0)$ be such that T and R' are order compact. Then $RUT \in \mathfrak{U}_V$.

ii) Let L be a strongly latticial class of operators from E to F, let $U,V \in L$ be such that $|U| \leq |V|$ and let $T \in L(E_0,E)$ be order compact. Then $UT \in \mathfrak{U}_V$.

PROOF.

i) By theorem 3.2 applied to $F' = L_r'(F,\mathbb{R})$, it follows that F' is a principal $Z(F)$-module for $|\sigma|(F',F)$; applying once again this theorem to $\check{F} = L_r'(F',\mathbb{R})$ (the topology on F' being $|\sigma|(F',F)$), we find that $\check{F}$ is a principal $Z(F)$-module for $|\sigma|(\check{F},F')$. As $|\sigma|(\check{F},F')$ is order continuous, corollary 3.4 applied to $L_r'(E,\check{F})$ shows that there is a net $(c_\delta) \subset Z(E)\bar{\otimes}Z(F)$ such that $|c_\delta| \leq 1_E \otimes 1_F$ and $c_\delta J_F V \longrightarrow J_F U$ for the solid order precompact topology on $L_r'(E,\check{F})$ deduced from $|\sigma|(\check{F},F')$. By lemma 1.2, we may assume that $c_\delta \in Z(E)\otimes Z(F)$ and therefore has the form

$$c = \sum_{i=1}^{n_\delta} A_{i\delta} \otimes B_{i\delta} .$$

Let

$$V_\delta = \sum_{i=1}^{n_\delta} B_{i\delta} V A_{i\delta} ;$$

clearly $RV_\delta T \in \mathcal{U}_V$ and $J_F V_\delta = c_\delta J_F V$. Put $U_\delta = U - V_\delta$; the proof will be concluded if we show that $RU_\delta T \longrightarrow 0$ for the operator norm. As $T(B_{E_0})$ is order precompact, it will suffice to show that $\sup_{x\in K} \| RU_\delta(x)\| \longrightarrow 0$ for any order precompact set K. Let $\varepsilon > 0$ be given. By proposition 5.1 we find $f \in F_+'$ such that

$$\| R(y)\| \leq f(|y|) + (2M)^{-1}\varepsilon \|y\| , \quad y \in F \qquad (1)$$

where $M = \sup_{x\in K} \| \, |J_F V|(|x|)\|$. Observe that

$$|J_F U_\delta| \leq |J_F U| + |J_F V_\delta| = |J_F U| + |c_\delta|\,|J_F V| \leq 2|J_F V| ;$$

thus,

$$\|U_\delta(x)\| = \|J_F U_\delta(x)\| \leq \| \, |J_F U_\delta|(|x|)\| \leq 2 \| \, |J_F V|(|x|)\|$$

for any $x \in E$. Therefore, (1) implies

$$\sup_{x\in K} \|RU_\delta(x)\| \leq \sup_{x\in K} f(|U_\delta(x)|) + \varepsilon .$$

But $f(|U_\delta(x)|) = \langle |J_F U_\delta(x)|, f\rangle$; as $J_F U_\delta \longrightarrow 0$ for the solid order precompact topology, it follows that $\sup_{x\in K} f(|U_\delta(x)|) \longrightarrow 0$. Consequently, $\lim\sup_\delta \sup_{x\in K} \|RU_\delta(x)\| \leq \varepsilon$ and the proof is complete.

ii) Let L_0 be a maximal strongly latticial class containing L. By corollary 3.4 applied to L_0 and lemma 1.2, there is a net $(c_\delta) \subset Z(E)\otimes Z(F)$ such that

$c_\delta V \longrightarrow U$ for the solid order precompact topology. Then $c_\delta VT \to UT$ for the operator norm and $c_\delta VT \in \mathcal{U}_V$; hence $UT \in \mathcal{U}_V$.

We present now a version of theorem 5.3 obtained by replacing $\mathcal{U}_V$ by the smallest regularly closed $\mathcal{R}$-module $\mathcal{R}_V$ containing a regular operator $V:E \longrightarrow F$: this is by definition the $\mathcal{R}$-module whose component $\mathcal{R}_V(E_0,F_0)$ (E_0 and F_0 being arbitrary Banach lattices) is the closure for the regular norm of the subspace of $L_r(E_0,F_0)$ generated by the operators RVT with $T \in L_r(E_0,E)$ and $R \in L_r(F,F_0)$. We need before a lemma:

LEMMA 5.1. B_F is dense in $B_{\check{F}}$ for $|\sigma|(\check{F},F')$.

PROOF. We prove first the following: for every $x \in B_{\check{F}}$, every $f \in B_{F'} \cap F'_+$ and every $\varepsilon > 0$ there is $y \in F$ such that $\langle |x-y|, f\rangle \leq \varepsilon$ and $f(|y|) \leq 1$. Indeed, the canonical map $T:F \longrightarrow (F,f)$ has norm ≤ 1 and (F,f), being an AL-space, is a band in $(F,f)''$; consequently, $T''(x) \in B_{(F,f)}$. As $T(F)$ is dense in (F,f), there is $y \in F$ such that $f(|y|) \leq 1$ and $\|T''(x) - y\| \leq \varepsilon$. As T' preserves order intervals, T'' is a Riesz homomorphism; hence we have for every $z \in F''$

$$\|T''(z)\| = \langle |T''(z)|, g\rangle = \langle T''(|z|), g\rangle = \langle |z|, T'(g)\rangle = \langle |z|, f\rangle$$

where $g \in (F,f)'$ is defined by $T'(g) = f$. Thus,

$$\langle |x-y|, f\rangle = \|T''(x-y)\| = \|T''(x) - y\| \leq \varepsilon .$$

Let $Q:F' \longrightarrow \check{F}'$ denote the map $f \longmapsto \check{f}$. As Q is positive and $J_F' Q = 1_{F'}$, it follows that Q is a Riesz homomorphism and $Q(F')$ is a band in $\check{F}'$, the projection on this band being equal to QJ_F'.

Now suppose that B_F is not dense in $B_{\check{F}}$; then there is $x \in B_{\check{F}}$ and $g:\check{F} \longrightarrow \mathbb{R}$ linear and continuous for $|\sigma|(\check{F},F')$ such that $\sup g(B_F) \leq 1 < g(x)$. Let $f = |J_F'(g)|$. As $f \in B_{F'} \cap F'_+$, it follows by the first part of the proof that for every $\varepsilon > 0$ there is $y \in F$ such that $\langle |x-y|, f\rangle \leq \varepsilon$ and $f(|y|) \leq 1$. The continuity of g implies the existence of $h \in F'_+$ such that $|g| \leq \check{h}$; consequently, $g = QJ_F'(g)$ and $|g| = Q(|J_F'(g)|) = \check{f}$. Hence

$$1 < g(x) \leq |g|(|x|) \leq |g|(|y|) + |g|(|x-y|) =$$
$$= f(|y|) + \langle |x-y|, f\rangle \leq 1 + \varepsilon .$$

As ε is arbitrary, we have arrived at a contradiction; thus, B_F is dense in $B_{\check{F}}$.

THEOREM 5.4. Let E_0, F_0 be Banach lattices and let E, F be Banach lattices which are principal modules over their centers. Then the following are true:

i) Let $U, V \in L_r(E,F)$ be such that $|J_F U| \leq |J_F V|$ and let $T_0 \in L_r(E_0, E)_+$, $R_0 \in L_r(F, F_0)_+$ be such that T_0 and R_0' are order compact. Then $RUT \in \mathcal{R}_V$ for any $R \in [-R_0, R_0]$ and $T \in [-T_0, T_0]$.

ii) Let L be a strongly latticial class of operators from E to F, let $U, V \in L$ be such that $|U| \leq |V|$ and let $T_0 \in L_r(E_0, E)_+$ be order compact. Then $UT \in \mathcal{R}_V$ for any $T \in [-T_0, T_0]$.

iii) Let L, U and V be as in ii) and suppose that E' has order continuous norm and $|V|'$ is order compact. Then $U \in \mathcal{R}_V$.

PROOF.

i) Repeat the proof of i) in theorem 5.3; all we must do in addition is to show that $RU_\delta T \longrightarrow 0$ for the regular norm. We have

$$|J_{F_0} RU_\delta T| = |\check{R} J_F U_\delta T| \leq \check{R}_0 |J_F U_\delta| T_0 . \qquad (1)$$

By proposition 5.1, the restriction of R_0 to B_F is continuous for $|\sigma|(F,F')$ and the norm topology on F_0; hence, by lemma 5.1, we obtain the inclusion $\check{R}(\check{F}) \subset F_0$ and the fact that for every $\varepsilon > 0$ there is $f \in F'_+$ verifying

$$\| \check{R}_0(z) \| \leq \langle |z|, f \rangle + \varepsilon \|z\| , \quad z \in \check{F} .$$

Consequently, by the same arguments as in the proof of i) in theorem 5.3 we find that $\check{R}_0 |J_F U_\delta| T_0 \longrightarrow 0$ for the operator norm; as $\check{R}_0 |J_F U_\delta| T_0$ takes its values in F_0, we obtain from (1) that $RU_\delta T \longrightarrow 0$ for the regular norm.

ii) Repeat the proof of ii) in theorem 5.3 and observe that $\pm (c_\delta VT - UT) \leq |c_\delta V - U| T_0$.

iii) Let L_0 be a maximal strongly latticial class containing L and let L_1 be the order ideal in L_0 formed by those U for which $|U|'$ is order compact. Then the assertion follows from theorem 5.1 ii) applied to L_1.

Applying the above theorems to the case when V is compact we find that the well known results of Dodds - Fremlin [5] and Aliprantis - Burkinshaw [1] which have been proved for the case when F had order continuous norm remain true provided if we replace $L_r(E,F)$ by a strongly latticial class:

COROLLARY 5.1. Let L be a strongly latticial class of operators from E to F and let $U,V \in L$ be such that V is compact and $|U| \leq |V|$. Then the following are true:

i) UT is compact for any order compact operator T from a Banach space into E.

ii) If E' has order continuous norm and $|V|'$ is order compact then U is compact.

PROOF. Observe that U (respectively UT) is compact iff its restriction to the closure of every principal order ideal in E (respectively to the preimage by T of the closure of every principal order ideal in E) is compact; thus, we may assume that E and F have quasi interior elements and the proof follows from theorems 5.3 and 5.4.

As concerns i) in theorem 5.4, it is an improvement of theorem 4.10 in [12] (as, for instance, we do not require the operators U,V to be positive).

We continue our series of applications by considering ideal properties of Dunford - Pettis operators. In [8], N.J. Kalton and P. Saab have proved the following theorems:

Suppose that F has order continuous norm and let $S,T:E \longrightarrow F$ be such that T is a Dunford - Pettis operator and $0 \leq S \leq T$. Then S is a Dunford - Pettis operator.

Suppose that E is an AL-space and that F is weakly sequentially complete. Then the Dunford - Pettis operators form and order ideal in L(E,F) ($= L_r(E,F)$ in this case).

We shall see that the above results remain true for an arbitrary Banach lattice F provided if we ask S and T to belong to the same strongly latticial class. For the proof of our theorem we shall need the following lemma; it is the same as lemma 4.2 in [8] but the proof we present is simpler.

LEMMA 5.2. Let $M \subset E$ be the solid hull of a relatively weakly compact set K. If (x_n) is a disjoint sequence in $M \cap E_+$ then $x_n \longrightarrow 0$ weakly.

PROOF. Let $f \in E'_+$ be given and let $T:E \longrightarrow (E,f)$ be the canonical map. As T(K) is relatively weakly compact, theorem 1.1 gives, for every $\varepsilon > 0$, an $y \in (E,f)$ such that $\|(|T(x)| - y)^+\| \leq \varepsilon$ for any $x \in K$. As T is a Riesz homomorphism, the same relation holds for $x \in M$. Therefore, if $(x_n) \subset M \cap E_+$ is disjoint, the relation

$$f(x_n) = \|T(x_n)\| = \|(T(x_n) - y)_+\| + \|T(x_n) \wedge y\|$$

shows that

$$\limsup_{n \to \infty} f(x_n) \leq \varepsilon + \lim_{n \to \infty} \|T(x_n) \wedge y\| = \varepsilon .$$

As ε is arbitrary, $f(x_n) \to 0$.

THEOREM 5.5. For any strongly latticial class of operators from E to F the following are true:

i) Let $S, T \in L$ be such that T is a Dunford - Pettis operator and $0 \leq S \leq T$. Then S is a Dunford - Pettis operator.

ii) If E is an AL-space, then the set of all Dunford - Pettis operators in L is an order ideal in L.

PROOF. In the beginning we remark that, as in our previous results, we may confine ourselves to the case when E and F have quasi interior elements.

Let $S, T \in L$ be such that T is a Dunford - Pettis operator and $0 \leq S \leq T$ in case i), $|S| \leq |T|$ in case ii). Suppose that $x_n \in E$ and $x_n \to 0$ weakly and let M be the solid hull of $\{x_n \mid n \geq 1\}$. Let L_0 be a maximal strongly latticial class containing L and let L_1 be the order ideal of those $U \in L_0$ with the following property: for every $\varepsilon > 0$ there is $y \in E_+$ such that $\| |U|((x - y)_+)\| \leq \varepsilon$ whenever $x \in M$. By corollary 3.4, L_1 is a principal $Z(E) \overline{\otimes} Z(F)$-module for the solid $\mathcal{M}$ - topology, $\mathcal{M}$ consisting of the single element $M \cap E_+$. We shall see that $T \in L_1$: in case ii), this follows from theorem 1.1; in case i), this follows from theorem 1E in [5] applied to M and to $T'(B_{F'})$ provided if we show that $\lim_{n \to \infty} \sup_{f \in T'(B_{F'})} |f(y_n)| = 0$ for every disjoint sequence (y_n) in $M \cap E_+$. But $\sup_{f \in T'(B_{F'})} |f(y_n)| = \|T(y_n)\|$ and $y_n \to 0$ weakly by lemma 5.2; hence, $\|T(y_n)\| \to 0$ as T is a Dunford - Pettis operator.

It follows that for every $\varepsilon > 0$ there is $c \in Z(E) \overline{\otimes} Z(F)$ such that

$$\sup_{x \in M} \| |S - cT|(x)\| \leq \varepsilon ;$$

by lemma 1.2 we may assume that $c \in Z(E) \otimes Z(F)$. Then cT is a Dunford - Pettis operator and we have

$$\limsup_{n\to\infty} \|S(x_n)\| \leq \sup_{n\geq 1} \| |S - cT|(|x_n|)\| + \lim_{n\to\infty} \| cT(x_n)\| \leq$$

$$\leq \sup_{x\in M} \| |S - cT|(x)\| \leq \varepsilon .$$

As ε is arbitrary, $\|S(x_n)\| \longrightarrow 0$ and the proof is complete.

We conclude this section with a study of ideal and principality properties of the class $M(E,F)$. For any order ideal $G \subset E'$ we shall denote by $M_G(E,F)$ the closure of $\mathcal{F}_G(E,F)$ for $\| \|_M$.

THEOREM 5.6. Let G be an order ideal in E' such that B_E is order precompact for $|\sigma|(E,G)$. Then the following are true:

i) $M_G(E,F) \cap L$ is an order ideal in L for every strongly latticial class L of operators from E to F. In particular, $M_G(E,F)$ is an order ideal in $M(E,F)$.

ii) $M_G(E,F)$ is a principal $A\check{\otimes}B$-module for $\| \|_M$ whenever E is a principal A-module for $|\sigma|(E,G)$ and F is a principal B-module for the norm topology.

PROOF. The proof will be divided into several steps. Before proceeding to it, we remark that the class $M(E,F)$ can be defined for every normed lattice E not necessarily norm complete; the main properties of it remain essentially the same, as one can see by identifying $M(E,F)$ with $M(\overline{E},F)$, where $\overline{E}$ denotes the norm completion of E. This remark will enable us to consider $M(\check{E},F)$.

STEP 1) The solid strong topology and the topology defined by $\| \|_M$ agree on the order interval $[-|U|, |U|]$ of $M(E,F)$ for any $U \in M_G(E,F)$.

PROOF. Let H denote the set of those $U \in M(E,F)$ with the property that the solid strong topology and the topology defined by $\| \|_M$ agree on $[-|U|, |U|]$; by lemma 1.2 in [26], H is an order ideal in $M(E,F)$ closed for $\| \|_M$. Consequently, it will suffice to prove that $f\otimes y \in H$ whenever $f \in G_+$ and $y \in F_+$. To this purpose, it suffices to show that for every $\varepsilon > 0$ there is $x \in E_+$ such that $\|U\|_M \leq \varepsilon + \|U(x)\|$ whenever $U \in [0, f\otimes y]$. As B_E is order precompact for $|\sigma|(E,G)$, there is $x \in E_+$ such that $f((u - x)_+) \leq \|y\|^{-1}\varepsilon$ whenever $u \in B_E$. Hence

$$U(u) \leq U((u - x)_+) + U(x) \leq f((u - x)_+)y + U(x) \leq \|y\|^{-1}\varepsilon\, y + U(x)$$

for any $u \in B_E$; therefore,

$$\|U\|_M = \| \sup U(B_E)\| \leq \| \|y\|^{-1}\varepsilon\, y + U(x)\| \leq \varepsilon + \| U(x)\| \ .$$

STEP 2) Let E be a normed lattice and a principal A-module for $|\sigma|(E,G)$ (G being as in the statement of the theorem), F be a Banach lattice and a principal B-module for the norm topology. Then $M_G(E,F)\cap L$ is an order ideal in L for any strongly latticial class L of operators from E to F, $M_G(E,F)$ is an order ideal in $M(E,F)$ and a principal $A\bar{\otimes}B$-module for $\| \ \|_M$.

PROOF. Let $V\in M_G(E,F)\cap L$ and let $U\in L$ be such that $|U|\leq |V|$. Consider a maximal strongly latticial class L_0 containing L and let L_1 be the order ideal in L_0 generated by V. As the restriction of $|V|$ to order bounded sets is continuous for $|\sigma|(E,G)$ and the norm topology of F, theorem 3.2 implies that L_1 is a principal $A\bar{\otimes}B$-module for the solid strong topology; hence, taking into account lemma 1.2, there is a net$(c_\delta)\subset A\otimes B$ such that $|c_\delta V|\leq |V|$ and $c_\delta V \longrightarrow U$ for the solid strong topology. By step 1) it follows that $(c_\delta V)$ is a Cauchy net for $\| \ \|_M$. As $M(E,F)$ is complete for $\| \ \|_M$ (theorem 2.1 in [4]), we obtain that $U\in M(E,F)$ and $c_\delta V \longrightarrow U$ for $\| \ \|_M$; thus, $U\in M_G(E,F)$ as $c_\delta V\in M_G(E,F)$.

In particular, we have obtained that $M_G(E,F)$ is an order ideal in $M(E,F)$; the fact that it is a principal $A\bar{\otimes}B$-module follows from theorem 3.2 applied to $M_G(E,F)$ and from step 1).

STEP 3) Let $\check{G} = \{\check{g} \mid g\in G\}$. Then $\check{G}$ is an order ideal in $\check{E}'$ such that $B_{\check{E}}$ is order precompact for $|\sigma|(\check{E},\check{G})$.

PROOF. If $Q:E' \longrightarrow \check{E}'$ denotes the map $f\longmapsto \check{f}$ then Q is a one-to-one Riesz homomorphism and $Q(E')$ is a band in $\check{E}'$ (see the proof of lemma 5.1); hence $\check{G} = Q(G)$ is an order ideal in $\check{E}'$. As every $\check{g}$ is continuous for $|\sigma|(\check{E},E')$, the assertion follows from lemma 5.1.

STEP 4) The proof of 1).

Let $V\in M_G(E,F)\cap L$ and $U\in L$ be such that $|U|\leq |V|$. By corollary 3.2, $\check{L} = \{\check{S} \mid S\in L\}$ is a strongly latticial class of operators from E to F'' and the map $S\longmapsto \check{S}$ $(S\in L)$ is a Riesz homomorphism. The map $S\longmapsto \check{S}$ $(S\in M(E,F))$ is an isometry of $M(E,F)$ into $M(\check{E},F)$; hence, it takes $M_G(E,F)$ into $M_{\check{G}}(\check{E},F)$. As $\check{E}$ and F'', being order complete, are principal modules over their centers, steps 2) and 3) show that $M_{\check{G}}(\check{E},F'')\cap\check{L}$ is an order ideal in $\check{L}$. As $|\check{U}|\leq|\check{V}|$ we obtain that $\check{U}\in M_{\check{G}}(\check{E},F'')$; in particular, $U\in M(E,F)$. It remains to show that actually, $U\in M_G(E,F)$. To this purpose, remark that $|V|(B_E)$ is contained in a principal order ideal of F; we may thus assume that F has a quasi interior element. Let $L_0 = \{\check{S} \mid S \in L,\ \check{S}(\check{E}) \subset F\}$; L_0 is a Riesz subspace of $\check{L}$, hence a strongly latticial class of operators from $\check{E}$ to F. As $\check{V} \in M_{\check{G}}(\check{E},F) \cap L_0$, $\check{U} \in L_0$ and $|\check{U}| \leq |\check{V}|$, steps 2) and 3) imply that $\check{U} \in M_{\check{G}}(\check{E},F)$. The restriction map from $M(\check{E},F)$ to $M(E,F)$ being continuous, it follows that $U \in M_G(E,F)$.

Two particular cases of theorem 5.6 are especially important. The first is provided by the situation when E' has order continuous norm:

COROLLARY 5.2. Suppose that E' has order continuous norm. Then $M(E,F) \cap L$ is an order ideal in L for every strongly latticial class L of operators from E to F. If moreover E is a principal A-module and F is a principal B-module, then $M(E,F)$ is a principal $A \overline{\otimes} B$-module for $\| \ \|_M$.

PROOF. By theorem 1.2 we may take $G = E'$ in theorem 5.6 and observe that $M_G(E,F) = M(E,F)$ by theorem 2.1 in [4].

The second assertion in the above corollary was announced (with only a sketch of proof) in [24].

For the second corollary we use the following notation from [4]: let $M_*(E',F)$ denote the subspace of $M(E',F)$ consisting of those U for which $U'(F) \subset E$; by proposition 4.1, $M_*(E',F)$ is a Riesz subspace of $M(E',F)$, hence a strongly latticial class.

COROLLARY 5.3. Suppose that E has order continuous norm. Then $M_*(E',F) \cap L$ is an order ideal in L for every strongly latticial class L of operators from E' to F. If moreover E is a principal A-module and F is a principal B-module, then $M_*(E',F)$ is a principal $A \overline{\otimes} B$-module for $\| \ \|_M$.

PROOF. As E has order continuous norm, the canonical map takes it onto an order ideal G in E''; moreover, theorem 1.2 says that $B_{E'}$ is order precompact for $|\sigma|(E',G)$. By theorem 3.2, E' is a principal A-module for $|\sigma|(E',G)$. By theorem 1.3 in [4], $M_*(E',F) = M_G(E',F)$; hence we may apply theorem 5.6 to get the result.

It is worthwile to restate a part from the above corollary in the language of tensor products as a permanence theorem for principality. Recall that the M-norm on $E \otimes F$ is defined by

$$\|z\|_M = \inf \{ \| \sum_{i=1}^{n} |x_i||y_i| \| \mid n \geq 1,\ x_i \in E,\ y_i \in F,\ z = \sum_{i=1}^{n} x_i \otimes y_i \}.$$

The completion of $E\otimes F$ for the M-norm is called the M-tensor product of E and F and is denoted by $E\hat{\otimes}_M F$; it is a Banach lattice whose positive cone is the closure of

$$\left\{ \sum_{i=1}^{n} x_i \otimes y_i \;\middle|\; n \geq 1,\ x_i \in E_+,\ y_i \in F_+ \right\}.$$

It is known from [4] that the canonical map from $E\otimes F$ into $L_r(E',F)$ extends to a norm and order isomorphism of $E\hat{\otimes}_M F$ onto $M_*(E',F)$; consequently, corollary 5.3 gives

THEOREM 5.7. Suppose that E has order continuous norm, is a principal A-module and F is a principal B-module. Then $E\hat{\otimes}_M F$ is a principal $A\tilde{\otimes}B$-module (the structure of $A\tilde{\otimes}B$-module is given by $(a\otimes b)(x\otimes y) = ax\otimes by$ and principality is considered with respect to norm topologies).

REFERENCES

1. C.D. ALIPRANTIS and O. BURKINSHAW, Positive compact operators on Banach lattices, Math.Z. 174 (1980), 289 - 298.
2. C.D. ALIPRANTIS and O. BURKINSHAW, The components of a positive operator, Math.Z. 184 (1983), 245 - 257.
3. A. BIGARD, K. KEIMEL et S. WOLFENSTEIN, Groupes et anneaux reticules, LNM 608, Berlin - Heidelberg - New York 1977.
4. J. CHANEY, Banach lattices of compact maps, Math.Z. 129 (1972), 1 - 19.
5. P. DODDS and D.H. FREMLIN, Compact operators in Banach lattices, Israel J. Math. 34, 4 (1979), 287 - 320.
6. D.H. FREMLIN, Tensor products of Archimedean vector lattices, American J. Math. 94 (1972), 777 - 798.
7. D.H. FREMLIN, Topological Riesz spaces and measure theory, Cambridge, The University Press, 1974.
8. N.J. KALTON and P. SAAB, Ideal properties of regular operators between Banach lattices, Illinois J. Math. 29 (1985), 382 - 400.
9. H. LEINFELDER, A remark on a paper of L.D. Pitt, Bayreuth Math. Schr. 11 (1982).
10. W.A.J. LUXEMBURG and A.C. ZAANEN, Riesz Spaces I, North Holland Publ. Comp., Amsterdam - London, 1971.
11. P.A. MEYER, Probability and potentials, Blaisdell Publ. Comp., Waltham - Massachusetts - Toronto - London, 1966.
12. B. DE PAGTER, The components of a positive operator, Indag. Math. 45 (1983), 219 - 241.
13. A. PIETSCH, Operator Ideals, Deutscher Verlag der Wissenschaften, Berlin, 1978.
14. A.C. VAN ROOIJ, When do the regular operators between two Riesz spaces form a Riesz space ?, Catholic Univ. of Toernooiveld, Report 8410, 1984.
15. A.R. SCHEP, Kernel operators, Indag. Math. 41 (1979), 39 - 53.
16. H.U. SCHWARZ, Über einige Klassen beschränkter Operatoren in Banachverbänden, Thesis, Jena, 1978.
17. D. VUZA, Extension theorems for strongly lattice - ordered modules and applications to linear operators, Preprint Series Math. INCREST, Bucuresti, 100 ...

18. D. VUZA, Strongly lattice - ordered modules over function algebras I, An. Univ. Craiova 11 (1983), 52 - 63.

19. D. VUZA, Strongly lattice - ordered modules over function algebras II, An. Univ. Craiova 12 (1984), 1 - 9.

20. D. VUZA, The perfect M - tensor product of perfect Banach lattices, Proceedings of the First Romanian - GDR Seminar, LNM 991, Berlin - Heidelberg - New York - Tokyo 1983, 272 - 295.

21. D. VUZA, Module peste inele ordonate comutative, Thesis, Bucureşti, 1983.

22. D. VUZA, Principal modules of linear maps and their applications, Proceedings of the Second International Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics Teubner - Texte Math. 67, Leipzig, 1984, 212 - 219.

23. D. VUZA, Ideals and bands in principal modules, Arch. Math. 45 (1985), 306 - 322.

24. D. VUZA, Sur les treillis vectoriels, en tant que modules principaux sur une f - algebre, C.R. Acad. Sc. Paris 301, n° 17 (1985), 797 - 800.

25. D. VUZA, The theory of principal modules and its applications to linear operators on Riesz spaces, Semesterbericht Funktionalanalysis, Tübingen, Sommersemester 1985, 211 -222.

26. D. VUZA, Oru - compact operators, Preprint Series Math. INCREST, Bucureşti, 67 (1985) (to appear in Operator Theory : Advances and Applications).

27. D. VUZA, Ideal properties of order bounded operators on ordered Banach spaces which are not Banach lattices, Operator Theory : Advances and Applications 17, Birkhäuser Verlag Basel, 1986, 353 - 368.

28. B. WALSH, On characterizing Köthe sequence spaces as vector lattices, Math. Ann. 175 (1968), 253 - 256.