INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

A NOTE ON THE INVERTIBILITY OF A CLASS IRREDUCIBLE MATRICES

by

POPA Constantin

PREPRINT SERIES IN MATHEMATICS
No.44/1986

BUCURESTI

A NOTE ON THE INVERTIBILITY OF A CLASS IRREDUCIBLE MATRICES

by

POPA Constantin*)

*) The National Institute for Scientifical and Technical Creation, Computer Center, Bd. Pacii 220, 79622 Bucharest, Romania.

<u>A note on the invertibility of a class of irreducible matrices</u>

Popa Constantin)

ABSTRACT. M.Kaykobad present in [3]a useful criterion about the invertibility of positive matrices. We prove in this paper a generalization of this criterion for arbitrary irreducible matrices and apply the result on proving the invertibility of matrices which appear in problems of interpolation using Bernstein-Bezier curves ([2]). We obtain also like a corrolary the classical criterion for the invertibility of real irreducible diagonally dominant matrices (see [1], [4]).

1. Notations and definitions.

Let $A = (a_{ij})_{i,j}$, $B = (b_{ij})_{i,j}$ square matrices of order $n \geqslant 2$ with real elements. We write: $A \geqslant B$ (resp. $A > B$) if $a_{ij} \geqslant b_{ij}$ (resp. $a_{ij} > b_{ij}$) $(\forall)$ $i,j = 1,\ldots,n$; $A \geqslant o$ (resp. $A > o$) if $a_{ij} \geqslant o$ (resp. $a_{ij} > o$), $(\forall)$ $i,j = 1,\ldots,n$. We use the same notations for vectors $a = (a_i)_i$, $b = (b_i)_i \in \mathbb{R}^n$. By $|A|$ we understand the matrix with elements $|a_{ij}|$, $i,j = 1,\ldots,n$. If $a = (a_i)$, $b = (b_i)$ are two vectors in $\mathbb{R}^n$ we note by $\langle a,b\rangle$ the scalar product of a and b, i.e. $\langle a,b\rangle = \sum_{i=1}^{n} a_i b_i$.

Definition. A n x n real or complex matrix A is <u>reducible</u> if there is a permutation matrix P such that:

$$P A P^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ o & A_{22} \end{bmatrix} \qquad (1)$$

where A_{11} and A_{22} are square matrices. A matrix A is <u>irreducible</u> if it is not reducible.

) The National Institute for Scientific and Technical Creation Bd.Păcii,Nr.220,Sector 6,Bucharest 77538,Romania.

It is well known the following caracterization of irreducible matrices (see for ex. [4], [5]).

Proposition 1. A n x n, real or complex matrix $A = (a_{ij})_{ij}$ is irreducible if and only if, for any two distinct indices $i,j \in \{1,\ldots,n\}$, there is a sequence of nonzero elements of A of the form

$$a_{i,i_1},\ a_{i_1,i_2},\ldots,\ a_{i_m,j} \qquad (2)$$

Corrolary 1. A is irreducible if and only if A^T is irreducible (where A^T is the transpose of A).

Definition. A real or complex n x n matrix $A = (a_{ij})_{i,j}$ is diagonally dominant if:

$$|a_{ii}| \geqslant \sum_{j\neq i} |a_{ij}| \ , \quad i = 1,\ldots,n \qquad (3)$$

The matrix is irreducibly diagonally dominant if it is irreducible diagonally dominant and strict inequality holds in (3) for at least one i.

We know the following result

Proposition 2. If A is irreducibly diagonally dominant then A is invertible.

2. The main theorem

In [3] is proved the following

Theorem (partial statement). Suppose that $A = (a_{ij})_{i,j} \geqslant 0$, $a_{ii} > 0$, $i = 1,\ldots,n$, $b = (b_i)_i > 0$. If we have:

$$b_i > \sum_{j\neq i} a_{ij}\frac{b_j}{a_{jj}} \ , \quad i = 1,\ldots,n \qquad (4)$$

then A is invertible.

We'll make a generalization of this theorem for irreducible matrices with arbitrary real elements. The main result of the paper is

Theorem 1. Let $A = (a_{ij})_{i,j}$ be a real n x n irreducible matrix, with $a_{ii} \neq 0$, $i = 1,\ldots,n$. If a vector $b = (b_i)_i \in \mathbb{R}^n$, $b > 0$, exist with the property:

$$b_i \geq \sum_{j\neq i} \frac{|a_{ij}|}{|a_{jj}|} \cdot b_j \quad , (\forall)\ i = 1,\ldots,n \tag{5}$$

and strict inequality holds in (5) for at leas one $i_0 \in \{1,\ldots,n\}$, then A is invertible.

Proof. Let $D = \operatorname{diag}\{a_{11},\ldots,a_{nn}\}$, $B = AD^{-1} - I$, $A_1 = |A|$, $D_1 = \operatorname{diag}\{|a_{11}|,\ldots,|a_{nn}|\}$ and $B_1 = A_1D_1^{-1} - I$ (where I is the unit n x n matrix). We have $B_1 = |B|$. Let $\rho_1 = \rho(B_1)$ the spectral radius of B_1 and $c = (I-B_1)\cdot b \in \mathbb{R}^n$. Accordingly to (5) we have $c \geq 0$. A_1 irreducible implies A_1^T irreducible (corrolary 1), so B_1^T is irreducible. But $B_1^T \geq 0$, then by Peron-Frobenius theorem ([1] , p.195, th.3.5) there exist a vector $d = (d_i)_i > 0$ such that

$$B_1^T d = \rho_1 d$$

Then we have

$$\langle d, c\rangle = \langle d, (I - B_1)b\rangle = (1 - \rho_1)\ \langle d,b\rangle \geq 0$$

which implies $\rho_1 \leq 1$. Let suppose that $\rho_1 = 1$.
Then $B_1^T d = d$, so $(I-B_1)^T d = 0$ and we have

$$0 = \langle (I-B_1)^T d,\ b\rangle = \langle d,(I-B_1)b\rangle =$$

$$= \sum_{i=1}^{n} d_i\Big(b_i - \sum_{j\neq i}\frac{|a_{ij}|}{|a_{jj}|}\cdot b_j\Big) \geq d_{i_0}\Big(b_{i_0} - \sum_{j\neq i_0}\frac{|a_{i_0j}|}{|a_{jj}|}\cdot b_j\Big) > 0$$

Absurd! (where $i_0 \in \{1,\ldots,n\}$ is an index for which strict inequality holds in (5)). Then $\rho(B_1) < 1$. But if $\rho = \rho(B)$ is the spectral radius of the matrix B we know that ([5] , p.109):
$\rho(B) \leq \rho(|B|) = \rho(B_1) < 1$.
Then $\rho < 1$, so I + B is invertible ([5], p.26). But $I + B = AD^{-1}$ by the definition of matrix B, so that A is invertible. ▨
We may obtain the "transpose" statement of the theorem 1.

Corrolary 2. Let $A = (a_{ij})_{i,j}$ be a real n x n irreducible matrix with $a_{ii} \neq o$, $i = 1,\dots,n$. If a vector $b = (b_i)_i \in \mathbb{R}^n$, $b > o$, exist with the property:

$$b_i \geqslant \sum_{j \neq i} \frac{|a_{ji}|}{|a_{jj}|} \cdot b_j \, , i = 1,\dots,n \qquad (6)$$

and strict inequality holds in (6) for at least one $i_o \in \{1,\dots,n\}$, then A is invertible: ▨

We have also

Corrolary 3. If A is a real n x n matrix with $a_{ii} \neq o, i = 1,\dots,n$ such that A or A^T is irreducibly diagonally dominant, thenA is invertible.

Proof. We apply theorem 1 for the matrix $C = \frac{a_{ij}}{a_{jj}}\, {}_{i,j}$ and the vector $b \in \mathbb{R}^n$, $b = (1,1,\dots,1)$. ▨

Remarks.

a) Corrolary 3 is exactly proposition 2 for the case of real matrices.

b) The condition in theorem 1 about the existence of an index $i_o \in \{1,\dots,n\}$ such that (5) holds strictly is realy necessary. Indeed for the matrix.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & o & o \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & o \\ o & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ o & o & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

which is not invertible ($Ax_o = o$, where $x_o = (1,-1,1,-1)$) we'll prove that $(\forall)$ $b = (b_i)_i \in \mathbb{R}^4$, $b > o$ such that (5) holds, then none of these inequalities cannot be strict.

Indeed, if b satisfies (5) it is easy to see that $b_1 = b_2 = b_3 = b_4 = \alpha > o$. Then if we suppose that one of the inequalities from (5) is strictly satisfied for example with $i = i_o$ we'll have:

$$\alpha > \sum_{j\neq i_o} 2\, a_{i_o j}\,\alpha = 2\alpha \sum_{j\neq i_o} a_{i_o j} = 2\,\alpha\, \frac{1}{2} = \alpha \quad \text{Absurd!}$$

3. Application.

Let $n \geqslant 2$ and u_o, $u_1, \ldots, u_n$, $\lambda_1, \ldots, \lambda_n \in (o,1)$ be arbitrary constants. We consider the following matrices of order $n+1$:

$$A = \begin{bmatrix} b_o & c_o & o & o \cdots\cdots & o & o \\ a_1 & b_1 & c_1 & o \cdots\cdots & o & o \\ o & a_2 & b_2 & c_2 \; o \cdots\cdots & o & o \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ o \cdots\cdots o & & & o \cdots a_{n-1} & b_{n-1} & c_{n-1} \\ o \cdots\cdots o & & & o \cdots o & a_n & b_n \end{bmatrix}$$

$$B = \begin{bmatrix} b'_o & c'_o & o & o & \cdots\cdots & o & a'_o \\ a'_1 & b'_1 & c'_1 & o & \cdots\cdots & o & o \\ o & a'_2 & b'_2 & c'_2 & o \cdots\cdots & o & o \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ o & o & o & o & o \cdots a'_{n-1} & b'_{n-1} & c'_{n-1} \\ c'_n & o \cdots & \cdots & \cdots & o \cdots o & a'_n & b'_n \end{bmatrix}$$

where

$$a_o = o \qquad ; \qquad c_o = \lambda_1$$

$$a_n = 1 - \lambda_n \qquad ; \qquad c_n = o$$

$$a_i = (1-\lambda_i)(1-u_i)^2; \qquad c_i = u_i^2 \lambda_{i+1}, \; i = 1, \ldots, n-1$$

$$b_i = 1 - (a_i + c_i), \; i = o, \ldots, n$$

and

$$a'_o = (1-\lambda_{n+1})(1-u_o)^2 \; ; \quad c^1_o = u_o^2 \lambda_1$$
$$a'_i = (1-\lambda_i)(1-u_i)^2 \; ; \; c'_i = u_i^2 \lambda_{i+1} \, , \quad i = 1,\dots,n$$
$$b_i = 1 - (a'_i + c'_i) \quad , \quad i = o,\dots,n$$

These matrices appear in problems concerning the interpolation with Bernstein-Bezier curves (see for ex. [2]). Using theorem 1 we'll prove that A and B are invertible so that the interpolatio problem has unique solution.

For example for matrix A we'll put

$$\beta_i = \gamma_i \cdot b_i \; , \; i = o,\dots,n \qquad (7)$$

where γ_i are positive numbers which satisfies:

$$\frac{\gamma_o}{\gamma_1} = \frac{\lambda_1}{1-\lambda_1} \; ; \; \frac{\gamma_1}{\gamma_2} = \frac{\lambda_2}{1-\lambda_2} \; ; \dots ; \; \frac{\gamma_{n-1}}{\gamma_n} = \frac{\lambda_n}{1-\lambda_n} \qquad (8)$$

We then have $\beta_i > o$, $i = o,\dots,n$ and:

$$\beta_1 \frac{\lambda_1}{b_1} = \gamma_1 \cdot b_1 \cdot \frac{\lambda_1}{b_1} = \gamma_1 \lambda_1 = \gamma_o \cdot (1-\lambda_1) = \gamma_o \, b_o = \beta_o$$

$$\beta_i \frac{(1-\lambda_{i+1})(1- \quad {}_{i+1})^2}{b_i} + \beta_{i+2} \frac{u^2_{i+1} \lambda_{i+2}}{b_{i+2}} =$$

$$= \gamma_i (1-\lambda_{i+1})(1-u_{i+1})^2 + \gamma_{i+2} \lambda_{i+2} u^2_{i+1} =$$

$$= \gamma_{i+1} \lambda_{i+1} (1-u_{i+1})^2 + \gamma_{i+1} (1-\lambda_{i+2}) u^2_{i+1} =$$

$$= \gamma_{i+1} \left[1-(1-\lambda_{i+1})(1-u_{i+1})^2 - \lambda_{i+2} u^2_{i+1} + (1-u_{i+1})^2 + u^2_{i+1} - \right.$$

$$= \beta_{i+1} - 2 \gamma_{i+1} u_{i+1} (1-u_{i+1}) < \beta_{i+1} \; , \quad i = o,\dots,n-2$$

and

$$\beta_{n-1} \cdot \frac{1-\lambda_n}{b_{n-1}} = \gamma_{n-1}(1-\lambda_n) = \gamma_n \lambda_n = \gamma_n b_n = \beta_n$$

So the matrix A (which is irreducible, [5] pag.1o5) and the vector $\beta = (\beta_o,\dots,\beta_n)$ defined in (7)-(8) verifies conditions of theorem 1. Then A is invertible. For the matrix we observe first that the associated graph is strongly connected so B is irreducible (see [5], p.1o5, 6.2.4). Then using the same vector $\beta = (\beta_0, \ldots, \beta_n)$ like for the matrix A we obtain, using theorem 1, that also B is invertible.

References

1. Arminjon P. - Analyse numérique matricielle (Les presses de l'Univ. de Montreal, 1978)

2. Jebelean P, Anton M. - A interpolation method of Bernstein-Bezier type for plotting curves (Communication held at the Symposium:"The 75-th birthday of A.Myller seminar",Univ.of Iaşi,24-26 oct.1985)(in roumanian)

3. Kaykobad M. - Positive solution of positive linear systems (Linear alg. and appl., 64(1985), 133-14o)

4. Marinescu G. - Numerical analysis (in roumanian) (Ed.Acad. R.S.R., 1974)

5. Ortega I. - Numerical analysis (a second course) (Acad.Press, New York - London, 1972).