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by

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Ivan SINGER*)

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*) Department of Mathematics, The National Institute for Scientifical and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

A GENERAL DUALITY APPROACH TO MIN-MAX RESULTS IN COMBINATORIAL OPTIMIZATION, VIA COUPLING FUNCTIONS. I: COVERINGS AND PACKINGS IN INCIDENCE TRIPLES

Ivan Singer

Department of Mathematics, INCREST, Bd. Păcii 220, 79622 Bucharest and Institute of Mathematics, str. Academiei 14, 70109 Bucharest, Romania

ABSTRACT

We give a new duality approach (based on the general theory of [11]) to combinatorial min-max relations, using coupling functions on the cartesian product of the primal and dual constraint sets. We give some applications to "all-cardinality" min-max theorems of "covering-packing" type, for which we introduce the unified framework of "incidence triples", "A-covers" and "B-packings", the main coupling function being "the number of incidences".

0. INTRODUCTION

The min-max relations in combinatorial optimization assert that the minimum of a function on a collection of subsets of a finite set is equal to the maximum of another function on a collection of subsets of another finite set.

In the present paper we shall give, instead of the known polyhedral methods (see e.g. [8], [9], [7], [10]), a different duality approach to combinatorial min-max relations, which leaves both collections of sets unchanged, and uses a "coupling" of them, with the aid of a "coupling function" defined on their Cartesian product. This approach is based on [11], where a general theory of dual optimization problems has been developed, which encompasses, theoretically, both the continuous and the discrete case. We think that this method might ultimately lead to an answer e.g. to the following remark, made by Lovász ([6], p.147), in connection with some combinatorial min-max relations: "It is my impression that the common nature of these results is not yet completely understood, especially in terms of discrete programming".

Thus, in the sequel, for each combinatorial min-max equality, we shall regard the minimum side as the optimal value of a primal (or, of a dual) minimization problem, and the maximum side as the optimal value of a dual (respectively, of a primal) maximization problem; then, the min-max equality asserts that, for this primal--dual pair of optimization problems, weak (and hence, by finiteness of the constraint sets, also strong) duality holds, in the general sense of [11]. We shall define and study some coupling functions on the cartesian product of the constraint sets of a primal-dual pair of optimization problems, especially in connection with Lagrangian duality, in the general sense of [11], and we shall give applications to various known "all-cardinality" min-max equalities of "covering-packing" type (i.e., in which both the primal and the dual objective functions are the cardinality function, and the elements of the constraint sets are "coverings" and "packings", in some sense). In order to give a unified framework for such applications, we shall introduce the concepts of "incidence triple" (A,B,ρ), "A-cover" and "B-packing" (which may also have some interest for other applications) and we shall use hypergraphs; in view of further developments (in subsequent papers), we shall define and study coupling functions on the cartesian product of larger collections than necessary here, namely, coupling functions $\phi: 2^A \times 2^B \to \bar{R} = (-\infty, +\infty)$ (where 2^M denotes the collection of all subsets of the set M).

In §1, from the many Lagrangian dual problems (in the sense of [11]), associated to a primal optimization problem, a dual objective set and a coupling function satisfying the "bounding inequalities", we shall choose (uniquely) a convenient "simple Lagrangian dual problem". We shall show that, for any primal-dual pair of combinatorial optimization problems, there exists a coupling function on the cartesian product of their constraint sets, such that the dual problem coincides with the (simple) Lagrangian dual problem (whence the min-max equality coincides with the Lagrangian duality equality). Furthermore, we shall show that, if the min-max equality holds, then, for any coupling function satisfying the bounding and "dual bounding" inequalities, the Lagrangian duality equality also holds; however, the converse is not true. Also, we shall give some other related results.

In §2, we shall define the "incidence triples" (A,B,ρ), mentioned above, and three main coupling functions $\phi_i:2^A \times 2^B \to R$ $(i=1,2,3)$ for them. Also, we shall define, for incidence triples (A,B,ρ), the "A-covers" and "B-packings" mentioned above, and we shall give some duality results for them, where the primal and dual objective functions are the cardinality function, and where the coupling functions are ϕ_1, ϕ_2 and ϕ_3. These results will show the usefulness of ϕ_3 (defined as "the number of incidences"), on which we shall concentrate in the sequel; we shall be especially interested in the incidences between minimum cardinality A-covers and maximum cardinality B-packings . Finally, we shall give some examples of incidence triples, using hypergraphs, and some results on the coupling function ϕ_3 for them.

In §§3-10, we shall apply this approach to various known min-max equalities (König's matching and covering theorems for bipartite graphs, Dilworth's theorem and its polar for finite posets, Menger's theorem and Fulkerson's polar to it, Menger's theorem-vertex form, and the Lucchesi-Younger theorem, for directed graphs).

Some further combinatorial min-max relations, which have other (primal or dual or both) objective functions (involving weights, capacities, etc.), will be studied, introducing other coupling functions, in a subsequent paper. Also, it remains an aim for further study, to obtain suitable general duality theorems (in the sense of [11]), which yield various known combinatorial min-max relations as particular cases and which also lead to new combinatorial min-max relations.

We wish to thank J.-E. Martínez-Legaz for his stimulating interest in our duality approach to combinatorial min-max results and for valuable remarks, and to I. Tomescu for remark 3.1 to our corollary 3.1 b).

1. SOME RELATIONS BETWEEN COMBINATORIAL MIN-MAX EQUALITIES AND LAGRANGIAN DUALITY EQUALITIES

For simplicity, we shall assume, without any special mention, that the sets G and W, occurring in the sequel, are non-empty.

Definition 1.1. Given a finite set G and a function $h:G\to R$, by the (combinatorial) minimization problem associated to the pair (G,h) we shall mean the problem of finding

$$(P) \qquad \min h(G), \tag{1.1}$$

and we shall call optimal element of (P), any $g_o \in G$ such that

$$h(g_o)=\min h(G); \tag{1.2}$$

the (combinatorial) maximization problem associated to (G,h), and its optimal elements $g_o \in G$ are defined similarly, replacing min by max, in (1.1) and (1.2). Any (combinatorial) minimization or maximization problem is called (combinatorial) optimization problem.

Remark 1.1. In concrete problems, instead of $h:G\to R$, we often have $h:G\to M$, where $M\subset R$ (e.g., $M=R_+$, Z, Z_+, $\{0,1\}$, etc.). Also, often G and h have additional structures (e.g., partial order, monotony, modularity, etc.). We shall not explicit these facts here.

Definition 1.2. Given two finite sets G and W and two functions $h:G\to R$ and $\mu:W\to R$, satisfying the "duality inequalities"

$$h(g) \geq \mu(w) \qquad (g \in G,\ w \in W), \qquad (1.3)$$

or, equivalently,

$$\min h(g) \geq \max \mu(W), \qquad (1.4)$$

by the dual problem to (P) of (1.1), associated to the quadruple (G,h,W,μ), we shall mean the maximization problem associated to the pair (W,μ), that is, the problem of finding

$$(Q) \qquad \max \mu(W); \qquad (1.5)$$

in this context, we shall call (P) the primal problem.

The dual problem to a (combinatorial) maximization problem is defined similarly, mutatis mutandis; namely, the dual problem to the primal problem

$$(P') \qquad \max h(G), \qquad (1.6)$$

associated to the quadruple (G,h,W,μ), satisfying (instead of (1.3)) the "duality inequalities"

$$h(g) \geq \mu(w) \qquad (g \in G,\ w \in W), \qquad (1.7)$$

is, by definition, the minimization problem

$$(Q') \qquad \min \mu(W). \qquad (1.8)$$

Remark 1.2. a) In particular, the dual to the dual (1.5), associated to the quadruple (W,μ,G,h) and again to the duality inequalities (1.3) (obtained by applying (1.7) to this case), is nothing else than the primal problem (P) of (1.1).

b) If we consider that W and μ "depend on" G and h, via (1.3), (1.4), then (1.5) is a "dual problem" to (1.1), in the sense of [11] (see [11], formula (1.4)).

Definition 1.3. Under the assumptions of definition 1.2, the (combinatorial) min-max equality corresponding to the quadruple (G,h,W,μ) is, by definition, the equality

$$\min h(G) = \max \mu(W); \qquad (1.9)$$

the (combinatorial) max-min equality corresponding to the quadruple (G,h,W,μ) is, by definition, the equality

$$\max h(G) = \min \mu(W). \qquad (1.10)$$

Remark 1.3. a) Formula (1.9) may be regarded as the min-max equality corresponding to (G,h,W,μ), or as the max-min equality corresponding to (W,μ,G,h).

b) For many known min-max equalities, the duality inequality (1.4) is easy to prove (the "hard part" is the opposite inequality).

Let us recall now one of the concepts of Lagrangian dual problems, in a slightly more general form than needed here (in view of possible applications in subsequent developments).

Definition 1.4. Given two (not necessarily finite) sets F, X, a mapping $u:F\to X$, a set $\Omega\subset X$ with $u(F)\cap\Omega\neq\emptyset$, a set W, a "coupling function" $\phi:X\times W\to R$, and a function $h:F\to R$, ______ the Lagrangian dual problem to the infimization problem

$$(P) \qquad \inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y), \tag{1.11}$$

associated to $(F,u^{-1}(\Omega),h,X,W,\phi)$ is, by definition [11], the problem of finding

$$(Q_L) \qquad \sup \lambda(W), \tag{1.12}$$

where $\lambda:W\to R$ is the function defined by

$$\lambda(w)=\inf_{y\in F}\{h(y)-\phi(u(y),w)\}+\inf_{x\in\Omega}\phi(x,w) \qquad (w\in W); \tag{1.13}$$

the Lagrangian dual problem to a supremization problem is defined similarly, mutatis mutandis.

Remark 1.4. a) We allow also infinite sets in definition 1.4, at the price of replacing min and max by inf and sup, in order to include e.g. linear programming duality. Actually, in [11], the function h and ϕ may have values in $\overline{R}=R\cup\{-\infty,+\infty\}$, using extended additions $\dot{+}$ and $\underset{\cdot}{+}$ on $\overline{R}$, but we shall not need here that generality.

b) As has been observed in [11], we have the "duality inequalities" (1.3) with $G=u^{-1}(\Omega)$, or, equivalently,

$$\inf h(u^{-1}(\Omega))\geq\sup\lambda(W). \tag{1.14}$$

In the sequel we shall be concerned with the case when

$$X=F,\ u=I_F \text{ (the identity operator)},\ \Omega=G; \tag{1.15}$$

in this case, (1.11), (1.13) and (1.14) become, respectively,

$$(P) \qquad \inf h(G), \tag{1.16}$$

$$\lambda(w)=\inf_{y\in F}\{h(y)-\phi(y,w)\}+\inf_{g\in G}\phi(g,w) \qquad (w\in W), \qquad (1.17)$$

$$\inf h(G)\geq \sup \lambda(W). \qquad (1.18)$$

Remark 1.5. a) Since the elements $w\in W$ enter in (1.17) only via the coupling function ϕ, the relations

$$w_1,w_2\in W,\ \phi(y,w_1)=\phi(y,w_2) \qquad (y\in F) \qquad (1.19)$$

imply $\lambda(w_1)=\lambda(w_2)$ (even when $w_1\neq w_2$, which may occur, e.g. for $\phi=\phi_3$ of (2.62) below). Hence, as has been observed in [11], for the study of Lagrangian duals (1.17), it is no restriction of the generality to consider only sets W of functions $w:F\to R$, and to replace $\phi(y,w)$ by $w(y)$, in (1.17). However, in this paper we shall also study the dual problems (1.5), for which, a priori, the relations (1.19) need not imply $\mu(w_1)=\mu(w_2)$. Therefore, in the sequel, we shall continue to work with coupling functions ϕ.

b) Formula (1.17) presents the asymmetry that G is embedded in some set F, while W is coupled with F, with the aid of ϕ. Also, there exist many Lagrangian dual problems (1.12), (1.17) to problem (1.11), according to the choice of the set F containing G, of the function $h|_{F\setminus G}$, and of the coupling function $\phi:F\times W\to R$.

Now, assuming the inequalities (1.20) below, we shall choose (uniquely) a convenient Lagrangian dual problem, associated to (G,h,W,ϕ), where $\phi:G\times W\to R$, as follows.

Definition 1.5. Given two finite sets G,W, a function $h:G\to R$, and a coupling function $\phi:G\times W\to R$, satisfying the "bounding inequalities"

$$\phi(g,w)\leq h(g) \qquad (g\in G,\ w\in W), \qquad (1.20)$$

or, equivalently,

$$\max_{w\in W}\phi(g,w)\leq h(g) \qquad (g\in G), \qquad (1.21)$$

by the simple Lagrangian dual problem to (P) of (1.1), associated to (G,h,W,ϕ), we shall mean the maximization problem

$$(Q_{SL}) \qquad \max \lambda(W), \qquad (1.22)$$

where

$$\lambda(w)=\min_{g\in G}\phi(g,w) \qquad (w\in W), \qquad (1.23)$$

that is, briefly, the problem

$$(Q_{SL}) \qquad \max_{w\in W} \min_{g\in G} \phi(g,w). \qquad (1.24)$$

The simple Lagrangian dual problem to a maximization problem is defined similarly, mutatis mutandis; namely, the simple Lagrangian dual problem to the primal problem

$$(P') \qquad \max h(G) \quad . \qquad (1.25)$$

where $\phi:G\times W\to R$ satisfies (instead of (1.20)) the "bounding inequalities"

$$h(g)\leq\phi(g,w) \qquad (g\in G,\ w\in W), \qquad (1.26)$$

is, by definition, the minimization problem

$$(Q'_{SL}) \qquad \min_{w\in W} \max_{g\in G} \phi(g,w). \qquad (1.27)$$

Remark 1.6. a) (Q_{SL}) is indeed a Lagrangian dual problem in the sense of formula (1.17). For, taking any "abstract element" $y_o\notin G$, and defining $F=G\cup\{y_o\}$, $h(y_o)=\phi(y_o,w)=0$ $(w\in W)$, we obtain $G\subset F$, $h:F\to R$, $\phi:F\times W\to R$ and, by (1.20),

$$\min_{y\in F}\{h(y)-\phi(y,w)\}=0 \qquad (w\in W), \qquad (1.28)$$

whence (1.17) reduces to (1.23). Moreover, taking any set $F\supset G$ and defining $h(y)=\phi(y,w)=0$ $(y\in F\setminus G,\ w\in W)$, we obtain again $h:F\to R$, $\phi:$ $:F\times W\to R$ and (1.28), whence (1.17) reduces to (1.23).

b) For the sequel (see e.g. theorem 1.1) it will be useful that in the simple Lagrangian dual problem (1.23), (1.24), the F of the Lagrangian dual problem (1.17) no longer occurs, and h occurs only in (1.20). However, it may be convenient to use also the general Lagrangian dual problem (1.13), in possible further theoretical developments.

c) One can also define directly the dual problem (1.22)-(1.24), assuming (1.20) (see [11], remark 2.9 f)), but then it need not be a Lagrangian dual problem (1.17), when (1.28) does not hold.

d) The simple Lagrangian dual problem (1.22), (1.23) is a particular case of a dual problem (1.5) (since by (1.20), we have (1.3) for $\mu=\lambda$), with the essential feature that λ of (1.23) is defined with the aid of G and $\phi:G\times W\to R$ (and of $h:G\to R$, via (1.20));

for the dependence of W and μ on G and h, in problem (1.5), see remark 1.2 b).

e) For $\phi:G\times W\to R$, one can make similar observations to those made in remark 1.1 on $h:G\to R$.

Definition 1.6. We shall say that simple Lagrangian duality for (P) of (1.1) is a) symmetric, if the objective function of the simple Lagrangian dual problem to (1.22), (1.23), associated to (W,λ,G,ϕ^-), where

$$\phi^-(w,g)=\phi(g,w) \qquad (w\in W,\ g\in G) \tag{1.29}$$

(whence the condition corresponding to (1.26) becomes $\lambda(w)= = \min_{g'\in G} \phi(g',w)\leq\phi(g,w)$ for all $w\in W$, $g\in G$, so it is satisfied), coincides with the objective function h of the primal problem (P) of (1.1), that is, if

$$h(g)=\max_{w\in W} \phi(g,w) \qquad (g\in G); \tag{1.30}$$

b) weakly symmetric, if we have a), with "objective function" replaced by "value", i.e., if

$$\min h(G)=\min_{g\in G} \max_{w\in W} \phi(g,w). \tag{1.31}$$

The symmetry and weak symmetry of simple Lagrangian duality for (P') of (1.25) are defined similarly, mutatis mutandis.

Remark 1.7. a) For some "natural" coupling functions, not satisfying condition (1.30) (see examples 3.3, 5.1, 6.1, 7.1 and 9.1), simple Lagrangian duality, for (P) of (1.1), will be weakly symmetric (e.g., by (1.32) and remark 1.8).

b) One can also interpret formula (1.30) as the coincidence of the dual problem to (Q) of (1.5), associated to (W,λ,G,h), and the simple Lagrangian dual problem to the same (Q), associated to (W,λ,G,ϕ^-), where ϕ^- is defined by (1.29).

In the sequel, instead of "the simple Lagrangian dual problem (1.24)" we shall say, for brevity, "the Lagrangian dual problem (1.24)"; this will lead to no confusion.

Definition 1.7. Under the assumptions of definition 1.5, the (simple) Lagrangian duality equality associated to (P) of (1.1) and to (G,h,W,ϕ) satisfying (1.20) is, by definition, the equality

$$\min h(G)=\max_{w\in W}\min_{g\in G}\phi(g,w). \qquad (1.32)$$

The Lagrangian duality equality associated to (P') of (1.25) and to (G,h,W,ϕ) satisfying (1.26) is, by definition,

$$\max h(G)=\min_{w\in W}\max_{g\in G}\phi(g,w). \qquad (1.33)$$

Remark 1.8. By (1.21), we have

$$\min h(G)\geq\min_{g\in G}\max_{w\in W}\phi(g,w)\geq\max_{w\in W}\min_{g\in G}\phi(g,w). \qquad (1.34)$$

Hence, the Lagrangian duality equality (1.32) holds if and only if we have (1.31) and

$$\min_{g\in G}\max_{w\in W}\phi(g,w)=\max_{w\in W}\min_{g\in G}\phi(g,w). \qquad (1.35)$$

Thus, in this case, Lagrangian duality, for (P) of (1.1), is weakly symmetric (see remark 1.7 a)).

Theorem 1.1. For any quadruple (G,h,W,μ) as in definition 1.2, there exists a coupling function $\phi:G\times W\to R$ satisfying (1.20), such that

$$\mu(w)=\min_{g\in G}\phi(g,w) \qquad (w\in W), \qquad (1.36)$$

i.e., such that the dual problem (1.5) coincides with the simple Lagrangian dual problem (1.24) (whence the min-max equality (1.9) coincides with the Lagrangian duality equality (1.32)).

Proof. Define the coupling function

$$\phi(g,w)=\mu(w) \qquad (g\in G,\ w\in W). \qquad (1.37)$$

Then, by (1.3) and (1.37), we have (1.20). Also, clearly, by (1.37), we have (1.36).

Remark 1.9. a) One can also interpret formula (1.36) as the symmetry of Lagrangian duality for problem (Q) of (1.5) (see definition 1.6 a)).

b) If (1.36) holds, then any proof of the Lagrangian duality equality (1.32) yields also a proof of the min-max equality (1.9) (and, conversely, any proof of (1.9) yields also a proof of (1.32)).

c) Similarly, for the coupling function ϕ defined by

$$\phi(g,w)=h(g) \qquad (g\in G,\ w\in W), \qquad (1.38)$$

we have (1.30).

In the sequel, we shall see that some coupling functions $\phi:G\times W\to R$, introduced in a natural way, turn out to coincide with (1.37), so theorem 1.1 applies, while some other coupling functions, different from (1.37), will also satisfy (1.36), and thus, for them, we shall still have the properties mentioned after (1.36) and in remark 1.9 b). However, some coupling functions, for which (1.9), (1.32) hold, will not satisfy (1.36) (see examples 4.1 and 8.1).

For general coupling functions (without assuming that they satisfy (1.36)), let us prove

Theorem 1.2. Let (G,h,W,μ) be a quadruple as in definition 1.2, for which the min-max equality (1.9) holds, and let $\phi:G\times W\to R$ be a coupling function, satisfying (1.20) and

$$\phi(g,w)\geqq\mu(w) \qquad (g\in G,\ w\in W): \qquad (1.39)$$

Then

a) The Lagrangian duality equality (1.32) holds, whence also

$$\max \mu(W)=\max_{w\in W}\min_{g\in G}\phi(g,w). \qquad (1.40)$$

b) For any optimal element $g_o\in G$ of the primal problem (1.1), we have

$$h(g_o)=\max_{w\in W}\phi(g_o,w). \qquad (1.41)$$

c) For any $w_o\in W$, the following statements are equivalent:

1°. w_o is an optimal element of the dual problem (1.5).

2°. w_o is an optimal element of the Lagrangian dual problem (1.24), and

$$\mu(w_o)=\min_{g\in G}\phi(g,w_o). \qquad (1.42)$$

d) For any $(g_o,w_o)\in G\times W$, the following statements are equivalent:

1°. g_o is an optimal element of (1.1) and w_o is an optimal element of (1.5).

2°. g_o is an optimal element of (1.1), w_o is an optimal element of the Lagrangian dual problem (1.24), and

$$\mu(w_o)=\phi(g_o,w_o). \tag{1.43}$$

3°. We have

$$h(g_o)=\mu(w_o). \tag{1.44}$$

Proof. a) By (1.20), (1.39) and (1.9), we have

$$\min h(G)\geq\max_{w\in W}\min_{g\in G}\phi(g,w)\geq\max\mu(W)=\min h(G), \tag{1.45}$$

whence the equalities (1.32) and (1.40).

b) By (1.2), (1.32) and (1.20), we have

$$h(g_o)=\min h(G)=\max_{w\in W}\min_{g\in G}\phi(g,w)\leq\max_{w\in W}\phi(g_o,w)\leq h(g_o),$$

whence (1.41).

c) By (1.40) and (1.39), for any $w_o\in W$ we have

$$\max\mu(W)=\max_{w\in W}\min_{g\in G}\phi(g,w)\geq\min_{g\in G}\phi(g,w_o)\geq\mu(w_o),$$

whence the equivalence 1°$\Leftrightarrow$2° follows.

d) By (1.9), for any $(g_o,w_o)\in G\times W$ we have

$$h(g_o)\geq\min h(G)=\max\mu(W)\geq\mu(w_o),$$

whence the equivalence 1°$\Leftrightarrow$3°; moreover, by (1.4), we see that (1.44) even implies the min-max equality (1.9).

3°$\Rightarrow$2°. Assume 3°. Then, by 3°$\Rightarrow$1° and c) above, w_o is an optimal element of (1.24). Also, by (1.20), (1.39) and (1.44), we have

$$h(g_o)\geq\phi(g_o,w_o)\geq\mu(w_o)=h(g_o),$$

whence (1.43).

2°$\Rightarrow$1°. If 2° holds, then, by (1.43), our assumption on w_o, and (1.40), we have

$$\max\mu(W)\geq\mu(w_o)=\phi(g_o,w_o)\geq\min_{g\in G}\phi(g,w_o)=\max_{w\in W}\min_{g\in G}\phi(g,w)=\max\mu(W),$$

whence

$$\mu(w_o)=\max\mu(W). \tag{1.46}$$

Remark 1.10. a) By (1.23), condition (1.39) is equivalent to

$$\lambda(w)=\min_{g\in G}\phi(g,w)\geq\mu(w)\qquad(w\in W). \tag{1.47}$$

Note also that the inequalities (1.39) are nothing else than the "bounding inequalities" (in the sense (1.26)) for the maximization problem (1.5) and the coupling function $\phi^{-}: W \times G \to R$ of (1.29).

b) By the bounding inequalities (1.20), condition (1.39) means that we can insert $\phi(g,w)$ between the terms of the duality inequalities (1.3), i.e.,

$$h(g) \geqq \phi(g,w) \geqq \mu(w) \qquad (g \in G,\ w \in W); \qquad (1.48)$$

or, alternatively, we see that (1.20) and (1.39) imply the duality inequalities (1.3).

c) By (1.39), we have

$$\phi(g,w) \geqq \min_{g' \in G} \phi(g',w) \geqq \mu(w) \qquad ((g,w) \in G \times W); \qquad (1.49)$$

hence, (1.43) <u>holds if and only if we have</u> (1.42) <u>and</u>

$$\phi(g_o,w_o) = \min_{g \in G} \phi(g,w_o). \qquad (1.50)$$

d) The inequality (1.40) shows that, under the assumptions of theorem 1.2, the dual problem (1.5) is "equivalent", in the sense of [11], to the Lagrangian dual problem (1.24), and that, again (as in the case of theorem 1.1), the min-max equality (1.9) coincides with the Lagrangian duality equality (1.32).

e) The converse of theorem 1.2 a) is not valid, i.e., the Lagrangian duality equality (1.32) need not imply the min-max equality (1.9), as shown by example 3.5 below. Further examples, related to the other parts of theorem 1.2, will be given in §§3-10.

f) The above results might suggest new min-max equalities for problems (1.1) for which no min-max equality is known. Indeed, <u>the nature of</u> (G,h) <u>might suggest</u> a choice of a set W, for which one can find a "naturally defined" coupling function $\phi: G \times W \to R$ satisfying (1.20); then, the Lagrangian duality equality (1.32), if true, would be a min-max equality (1.9). Although, even for "natural" ϕ's, (1.32) need not hold (see e.g. the remarks before example 3.5), this method might be useful in some cases (see e.g. remark 10.1).

2. A FRAMEWORK FOR ALL-CARDINALITY COVERING-PACKING DUALITY RESULTS

For simplicity, in the sequel we shall assume, without any special mention, that the sets A and B are non-empty. When this will lead to no confusion, we shall write m instead of {m}.

Definition 2.1. a) Let A and B be two (non-empty) finite sets and let $\rho \subseteq A \times B$ be a binary relation, which we shall call incidence relation. We shall say that $a \in A$ and $b \in B$ are incident, or, that a covers b, and we shall write

$$a\rho b, \tag{2.1}$$

if $(a,b) \in \rho$. We shall denote $(A \times B) \setminus \rho$, i.e., non-incidence, by $\bar{\rho}$.

b) We shall extend ρ to $2^A \times 2^B$, by saying that $y \in 2^A$ and $w \in 2^B$ are incident, in symbols,

$$y\rho w, \tag{2.2}$$

if there exist $a \in y$ and $b \in w$ such that $a\rho b$. If $y \in 2^A$ and $w \in 2^B$ are not incident, we shall write $y\bar{\rho}w$.

c) Any triple (A,B,ρ) as above, will be called an incidence triple.

Definition 2.2. For an incidence triple (A,B,ρ), we define the coupling functions $\phi_i : 2^A \times 2^B \to R$ $(i=1,2,3)$, respectively, by

$$\phi_1(y,w) = |\{a \in y \mid a\rho w\}| \qquad (y \in 2^A,\ w \in 2^B), \tag{2.3}$$

$$\phi_2(y,w) = |\{b \in w \mid y\rho b\}| \qquad (y \in 2^A,\ w \in 2^B). \tag{2.4}$$

$$\phi_3(y,w) = |\{(a,b) \in y \times w \mid a\rho b\}| \qquad (y \in 2^A,\ w \in 2^B), \tag{2.5}$$

where $|M|$ denotes the cardinality of the (finite) set M.

Remark 2.1. a) In other words, $\phi_1(y,w)$ is the number of elements a of y, which are incident with the set w (and, a similar remark holds for $\phi_2(y,w)$), while $\phi_3(y,w)$ is the "number of incidences" between the sets y and w. We use the notation $y \in 2^A$ instead of $g \in 2^A$, in view of possible subsequent applications of (1.17).

b) With the notations

$$\rho(y) = \{b \in B \mid y\rho b\} = \bigcup_{a \in y} \{b \in B \mid a\rho b\} \qquad (y \in 2^A), \tag{2.6}$$

$$\rho^{-1}(w) = \{a \in A \mid a\rho w\} = \bigcup_{b \in w} \{a \in A \mid a\rho b\} \qquad (w \in 2^B), \tag{2.7}$$

we have

$$\phi_1(y,w)=|\rho^{-1}(w)\cap y| \qquad (y\in 2^A,\ w\in 2^B), \quad (2.8)$$

$$\phi_2(y,w)=|\rho(y)\cap w| \qquad (y\in 2^A,\ w\in 2^B); \quad (2.9)$$

in particular, $\phi_1(y,\emptyset)=0$ $(y\in 2^A)$ and $\phi_2(\emptyset,w)=0$ $(w\in 2^B)$.

c) If we define the "incidence function" $c:A\times B\to R$, by

$$c(a,b)=\phi_3(\{a\},\{b\})=\begin{cases}1 & \text{if } a\rho b\\ 0 & \text{if } a\bar{\rho}b,\end{cases} \quad (2.10)$$

then $\phi_3(y,\emptyset)=0$ $(y\in 2^A)$, $\phi_3(\emptyset,w)=0$ $(w\in 2^B)$ and

$$\phi_3(y,w)=\sum_{a\in y}\sum_{b\in w} c(a,b) \qquad (\emptyset\neq y\in 2^A, \emptyset\neq w\in 2^B). \quad (2.11)$$

From these formulas it follows that $\phi_3:2^A\times 2^B\to R$ is "bimodular" (i.e., modular in each variable) and that, embedding $2^A\times 2^B$ into $R^{|A|}\times R^{|B|}$ in the usual way (i.e., identifying each $m\in 2^M$ with its incidence vector in $R^{|M|}$), ϕ_3 can be extended canonically to a bilinear coupling function $R^{|A|}\times R^{|B|}\to R$.

d) We have

$$\phi_1(y,\{b\})=\phi_3(y,\{b\}) \qquad (y\in 2^A,\ b\in B), \quad (2.12)$$

$$\phi_2(\{a\},w)=\phi_3(\{a\},w) \qquad (a\in A,\ w\in 2^B). \quad (2.13)$$

Definition 2.3. Let (A,B,ρ) be an incidence triple.

a) We shall say that $g\in 2^A$ is an A-cover (for B), if

$$|\{a\in g|a\rho b\}|\geq 1 \qquad (b\in B), \quad (2.14)$$

i.e., if $g\rho b$ $(b\in B)$; or, with the notation (2.6), if $\rho(g)=B$. We shall denote

G=the collection of all A-covers g. (2.15)

b) We shall say that $w\in 2^B$ is a B-packing (for A), if

$$|\{b\in w|a\rho b\}|\leq 1 \qquad (a\in A), \quad (2.16)$$

i.e., if for each $a\in A$ there exists at most one $b\in w$ such that $a\rho b$. We shall denote

W=the collection of all B-packings w. (2.17)

Remark 2.2. a) In order to ensure that $G\neq\emptyset$, i.e., the existence of A-covers, it is sufficient (and necessary, since every set containing an A-cover is an A-cover) to make the assumption that A is an A-cover, i.e.,

$$|\rho^{-1}(b)|=|\{a\in A|a\rho b\}|\geq 1 \qquad (b\in B), \quad (2.18)$$

or, in other words, $A\rho b$ $(b\in B)$; or, with the notation (2.6), $\rho(A)=B$.

b) B-packings always exist, i.e., $W \neq \emptyset$, since e.g. the empty set $\emptyset$ and all singletons $\{b\}$, where $b \in B$, are B-packings. However, in order to study the duality equalities (2.25) and (2.34) below, it will be convenient (see remark 2.5 b)) to make the assumption

$$|\rho(a)| = |\{b \in B | a\rho b\}| \geq 1 \qquad (a \in A), \qquad (2.19)$$

i.e., $a\rho B$ $(a \in A)$, or, with the notation (2.7), $\rho^{-1}(B)=A$; in other words, A should have no "isolated" elements (i.e., which are not incident with any $b \in B$).

Note also that (2.19) is symmetric to (2.18), and hence it ensures the existence of B-covers for the "polar" incidence relation $\rho^{\circ} \subseteq B \times A$ defined by

$$(b,a) \in \rho^{\circ} \Longleftrightarrow (a,b) \in \rho. \qquad (2.20)$$

c) One can associate to A,B and ρ as above, a bipartite "incidence graph" $(A \cup B, \rho)$, with the edges defined by

$$(a,b) \in \rho \Longleftrightarrow a\rho b; \qquad (2.21)$$

for the case of example 2.2 below, such incidence graphs are defined e.g. in [1]. However, note that the A-covers (respectively, B-packings) involve only subsets of A (respectively, B), so they are completely different from the vertex covers (respectively, matchings) of the graph $(A \cup B, \rho)$.

Proposition 2.1. We have

$$|g| \geq |w| \qquad (g \in G,\ w \in W). \qquad (2.22)$$

Proof. Let $g \in G$ and $w \in W$. Since g is an A-cover, we can define a mapping $f: w \to g$ by choosing, for each $b \in w$, an $a=f(b) \in g$ such that $a\rho b$. Then, since w is a B-packing, for any $b_1, b_2 \in w$ with $b_1 \neq b_2$ we have $f(b_1) \neq f(b_2)$ (since otherwise $a\rho b_1$, $a\rho b_2$ and $b_1 \neq b_2$ would contradict (2.16)), so f is a one-to-one mapping of w into g, which proves (2.22).

Proposition 2.1 shows that for G,W of (2.15), (2.17) and for

$$h(g)=|g| \qquad (g \in G), \qquad (2.23)$$

$$\mu(w)=|w| \qquad (w \in W), \qquad (2.24)$$

we have the duality inequalities (1.3), (1.4). However, the min--max equality

$$\min_{g\in G}|g|=\max_{w\in W}|w| \tag{2.25}$$

need not hold (see e.g. the remarks before example 3.5).

Let us give now some relations between the coupling functions ϕ_i of (2.3)-(2.5) and the sets G,W of (2.15), (2.17).

Proposition 2.2. We have

$$\phi_1(y,w)=\phi_3(y,w) \qquad (y\in 2^A,\ w\in W), \tag{2.26}$$

$$\phi_2(g,w)=|w| \qquad (g\in G,\ w\in 2^B). \tag{2.27}$$

Proof. By (2.3), (2.17), (2.16) and (2.5), we obtain

$$\phi_1(y,w)=|\{a\in y|a\rho w\}|=|\{a\in y|\exists b\in w,\ a\rho b\}|=$$

$$=|\{(a,b)\in y\times w|a\rho b\}|=\phi_3(y,w) \qquad (y\in 2^A,\ w\in W).$$

Finally, by (2.4), (2.15) and (2.14), we get

$$\phi_2(g,w)=|\{b\in w|g\rho b\}|=|w| \qquad (g\in G,\ w\in 2^B).$$

In the sequel, it will be more convenient to use ϕ_3.

Proposition 2.3. a) $y\in 2^A$ is an A-cover (i.e., $y\in G$) if and only if

$$\phi_3(y,\{b\})\geq 1 \qquad (b\in B). \tag{2.28}$$

b) $w\in 2^B$ is a B-packing (i.e., $w\in W$) if and only if

$$\phi_3(\{a\},w)\leq 1 \qquad (a\in A). \tag{2.29}$$

Proof. a) By (2.12), (2.3) and (2.14), we have $y\in G$ if and only if

$$\phi_3(y,\{b\})=\phi_1(y,\{b\})=|\{a\in y|a\rho b\}|\geqq 1 \qquad (b\in B).$$

b) By (2.13), (2.4) and (2.16), we have $w\in W$ if and only if

$$\phi_3(\{a\},w)=\phi_2(\{a\},w)=|\{b\in w|a\rho b\}|\leqq 1 \qquad (a\in A).$$

Remark 2.3. If $B=\{b_1,\dots,b_m\}$, then, defining $u:2^A\to Z_+^m$ by

$$u(y)=(\phi_3(y,\{b_1\}),\dots,\phi_3(y,\{b_m\})) \qquad (y\in 2^A), \tag{2.30}$$

we can also write (2.28) in the form

$$u(y)\geq \varepsilon_m=(1,\dots,1)\ (\in Z_+^m) \qquad (y\in 2^A). \tag{2.28'}$$

Similarly, if $A=\{a_1,\dots,a_n\}$, then, defining $u':2^B\to Z_+^n$ by

$$u'(w)=(\phi_3(\{a_1\},w),\dots,\phi_3(\{a_n\},w)) \qquad (w\in 2^B), \tag{2.31}$$

we can write (2.29) in the form

$$u'(w)\leq \varepsilon_n=(1,\dots,1)\ (\in Z_+^n) \qquad (w\in 2^B). \tag{2.29'}$$

Then, (2.25) becomes

$$\min_{\substack{y\in 2^A \\ u(y)\geqq\varepsilon_m}} |y| = \max_{\substack{w\in 2^B \\ u'(y)\leqq\varepsilon_n}} |w|. \qquad (2.25')$$

Since the operators u, u' of (2.30), (2.31) are "modular" (so is each component function) and since the functions

$$h(y)=|y| \qquad (y\in 2^A), \qquad (2.23')$$

$$\mu(w)=|w| \qquad (w\in 2^B), \qquad (2.24')$$

are modular (and even $u(\emptyset)=0$, $u'(\emptyset)=0$, $h(\emptyset)=0$, $\mu(\emptyset)=0$), this suggests to construct directly a theory of "modular programming" (instead of the usual linear programming approach to linear objective combinatorial optimization problems).

Corollary 2.1. We have

$$\phi_3(y,w)\leqq|y| \qquad (y=\{a_1,\dots,a_q\}\in 2^A,\ w\in W), \qquad (2.32)$$

$$\phi_3(g,w)\geqq|w| \qquad (g\in G,\ w=\{b_1,\dots,b_p\}\in 2^B). \qquad (2.33)$$

Proof. By (2.11), $\phi_3(y,\emptyset)=0$, (2.29) and (2.28), we obtain

$$\phi_3(y,w)=\sum_{i=1}^{q}\phi_3(\{a_i\},w)\leqq q=|y|,$$

$$\phi_3(g,w)=\sum_{j=1}^{p}\phi_3(g,\{b_j\})\geqq p=|w|.$$

Corollary 2.1 shows that for (G,h,W,μ) and $\phi=\phi_3$ of (2.15), (2.23), (2.17), (2.24) and (2.5), we have the bounding inequalities (1.20) and (1.39). However, the Lagrangian duality equality

$$\min_{g\in G}|g|=\max_{w\in W}\min_{g\in G}\phi_3(g,w) \qquad (2.34)$$

need not hold (see e.g. the remarks before example 3.5).

Theorem 2.1. If the min-max equality (2.25) holds for G and W of (2.15) and (2.17), then so does the Lagrangian duality equality (2.34), and for any minimum cardinality A-cover g_o and any maximum cardinality B-packing w_o, the number of incidences between g_o and w_o is

$$\phi_3(g_o,w_o)=|g_o|=|w_o|. \qquad (2.35)$$

Proof. This follows from theorem 1.2 a), d), applied to the quadruple (G,h,W,μ) of (2.15), (2.23), (2.17), (2.24), and to $\phi=\phi_3$ of (2.5).

Theorem 2.2. If the min-max equality (2.25) holds, then for any pair (g_o, w_o) as in theorem 2.1, we have

$$|\{b \in w_o | a \rho b\}| = 1 \qquad (a \in g_o), \qquad (2.36)$$

$$|\{a \in g_o | a \rho b\}| = 1 \qquad (b \in w_o), \qquad (2.37)$$

i.e., each $a \in g_o$ is incident with exactly one $b \in w_o$ and for each $b \in w_o$ there exists exactly one $a \in w_o$ such that a and b are incident.

Proof. For any optimal $g_o = \{a_1, \ldots, a_q\} \in G$ and any optimal $w_o \in W$ we have, by (2.11) and (2.35),

$$\sum_{i=1}^{q} \phi_3(\{a_i\}, w_o) = \phi_3(g_o, w_o) = |g_o| = q \ ,$$

whence, by (2.29), we obtain

$$\phi_3(\{a_i\}, w_o) = 1 \qquad (i = 1, \ldots, q).$$

Hence, for each $a_i \in g_o$ there exists $b_i \in w_o$ such that $a_i \rho b_i$, and this b_i is unique (by (2.16)), which proves (2.36). Furthermore, by (2.14), for each $b \in w_o$ there exists $a_i \in g_o$ such that $b = b_i$, and thus $f: a_i \to b_i$ maps g_o onto w_o. Hence, since $|g_o| = |w_o|$ (by (2.35)), f is one-to-one, which yields (2.37).

Remark 2.4. a) One can also give the following proof of (2.37) above, which does not use formulas: By (2.25), for any optimal pair $(g_o, w_o) \in G \times W$ we have $|g_o| = |w_o|$. Now, since $g_o \in G$, each $b \in w_o$ is covered by at least one $a = a_b \in g_o$, and since $w_o \in W$, distinct b's are covered by distinct $a = a_b$'s, so we need at least $|w_o|$ distinct elements $a \in g_o$ to cover all $b \in w_o$. But, since $|g_o| = |w_o|$, all elements $a \in g_o$ are used up in this process, and each $a \in g_o$ is used exactly once, which completes the proof.

b) Theorem 2.2 implies again the second part of theorem 2.1. Therefore, in the particular cases which we shall consider in the sequel, we shall first give theorem 2.2 and then mention, as a consequence, formula (2.35).

Corollary 2.2. Assume that the min-max equality (2.25) holds. Then

a) Given any minimum cardinality A-cover $g_o = \{g_1, \ldots, a_q\}$, for each $a_i \in g_o$ one can select $b_i \in B$ with $a_i \rho b_i$, in such a way that $w_o = \{b_1, \ldots, b_q\}$ is a maximum cardinality B-packing.

b) Given any maximum cardinality B-packing $w_o=\{b_1,\ldots,b_q\}$, for each $b_i\in w_o$ one can select $a_i\in A$ with $a_i\rho b_i$, in such a way that $g_o=\{a_1,\ldots,a_q\}$ is a minimum cardinality A-cover.

Proof. a) By finiteness, there exists a maximum cardinality B-packing w_o, and then theorem 2.2 applies.

b) The proof is similar, choosing any minimum cardinality A-cover g_o.

In the sequel, we shall also use

Remark 2.5. a) We can ensure that $G\neq\emptyset$, i.e., that (2.18) holds, replacing the set B, if necessary, by its subset

$$B'=\{b\in B \mid \exists\, a\in A,\ a\rho b\}=\{b\in B \mid A\rho b\}, \tag{2.38}$$

which satisfies (2.18) (with B replaced by B').

b) For the study of the equalities (2.25) and (2.34) it is convenient, and it is no restriction of the generality, to assume (2.19), replacing the set A, if necessary, by its subset

$$A'=\{a\in A \mid \exists\, b\in B,\ a\rho b\}=\{a\in A \mid a\rho B\}, \tag{2.39}$$

which satisfies (2.19) (with A replaced by A'). Indeed, if $g_o\in G$ is an A-cover containing an element $a_o\in A\setminus A'$, then $g_o\setminus a_o$ is an A-cover, so g_o is not a minimum cardinality A-cover (thus, the elements of $A\setminus A'$ cannot belong to any minimum cardinality A-cover); for, if $a_o\in A\setminus A'$, then there is no $b_o\in B$ such that $a_o\rho b_o$, and hence we have $g_o\rho b$ $(b\in B)$ if and only if $(g_o\setminus a_o)\rho b$ $(b\in B)$. Furthermore, $w\in W$ is a B-packing for A if and only if it is a B-packing for A'. Also, clearly, $\phi_3(a_o,b)=0$ $(a_o\in A\setminus A',\ b\in B)$, whence $\phi_3(g,w)=\phi_3(g\setminus a_o,w)$ $(g\in G,\ w\in W,\ a_o\in A\setminus A')$.

Let us give now a simple example of an incidence relation.

Example 2.1. Let M be a finite set. If $A,B\subseteq M$, we define the incidence of any $a\in A$ and $b\in B$ by

$$a\rho b \Leftrightarrow a=b. \tag{2.40}$$

Then, extending ρ to $2^A\times 2^B$ by definition 2.1 b), for any $y\in 2^A$ and $w\in 2^B$ we obtain

$$y\rho w \Leftrightarrow y\cap w\neq\emptyset. \tag{2.41}$$

The incidence relation (2.41) can be used to build the two main examples which we shall use in the sequel, as follows.

Example 2.2. Let M be a finite set and let A=M and $B\subseteq 2^M$. Then, (A,B) is a set system, with "ground set" M, in the sense of [1]; if, in addition,

$$b\neq\emptyset \qquad (b\in B), \qquad (2.42)$$

$$\bigcup_{b\in B} b=A, \qquad (2.43)$$

then (A,B) is called [3] a hypergraph, with vertices $a\in A$ and edges $b\in B$. One can use (2.41) to define the incidence of any $a\in A$ and $b\in B$ by

$$a\rho b \Leftrightarrow a\cap b\neq\emptyset \Leftrightarrow a\in b; \qquad (2.44)$$

this is nothing else than the usual incidence relation (and hence $(\phi_3(a,b))_{a\in A,b\in B}$ is the usual "incidence matrix") for set systems [1] and hypergraphs [3]. Then, by definition 2.1 b), for any $y\in 2^A$ and $w\in 2^B$ we obtain

$$y\rho w \Leftrightarrow |\{(a,b)\in y\times w|a\in b\}|\geq 1; \qquad (2.45)$$

also, conditions (2.18), (2.19) mean nothing else than (2.42) and (2.43). Furthermore, $g\in 2^A$ is an A-cover, in the sense (2.14), if and only if

$$g\cap b\neq\emptyset \qquad (b\in B), \qquad (2.46)$$

i.e., if and only if g is a transversal [3] of the hypergraph (A,B). Also, $w\in 2^B$ is a B-packing, in the sense (2.16), if and only if

$$b_i\cap b_j=\emptyset \qquad (b_i,b_j\in w,\ b_i\neq b_j) \qquad (2.47)$$

i.e., if and only if w is a matching [3] of the hypergraph (A,B); or, in other words, the collection w of subsets of M is a packing into M, in the usual sense (e.g. [2]). In this case, the inequality (2.22) is known (see e.g. [3], p.424, theorem 5).

Remark 2.6. For every incidence triple (A,B,ρ), satisfying (2.18), (2.19), there exist a hypergraph $(A,\tilde{B})$ as in example 2.2, and a mapping $b\to\tilde{b}$ of B into $\tilde{B}$ (not necessarily one-to-one), such that, for any $b,b'\in B$ with $\tilde{b}=\tilde{b'}$, we have

$$a\rho b' \Leftrightarrow a\in\tilde{b}. \qquad (2.48)$$

Indeed, it is enough to take

$$\tilde{b}=\rho^{-1}(b)=\{a\in A|a\rho b\} \qquad (b\in B), \qquad (2.49)$$

$$\tilde{B}=\{\tilde{b}|b\in B\}, \qquad (2.50)$$

since then (2.48) is obvious, and (2.18), (2.19) yield (2.42) and (2.43) (with b, B replaced by $\tilde{b}$ and $\tilde{B}$ respectively).

Theorem 2.3. Under the assumptions of example 2.2, if $A=M\in B$ (in particular, if every superset b' of each set $b\in B$ belongs to B), then

$$|y|=\max_{w\in W}\phi_3(y,w) \qquad (y\in 2^A). \tag{2.51}$$

Proof. By our assumption, $\{A\}\in W$; also, for any $y\in 2^A$ we have, clearly,

$$\phi_3(y,\{A\})=|y|, \tag{2.52}$$

which, together with (2.32), proves (2.51).

Remark 2.7. From remark 2.5 it follows that, under the assumptions of example 2.2, we can achieve (2.42), (2.43), replacing the set system (A,B) by the hypergraph (A',B'), where

$$A'=\bigcup_{b\in B} b,\ B'=B\backslash\emptyset. \tag{2.53}$$

Example 2.3. Let M be a finite set and let $A\subseteq 2^M$ and B=M. Then, (B,A) is a set system, and, if

$$a\neq\emptyset \qquad (a\in A), \tag{2.54}$$

$$\bigcup_{a\in A} a=B, \tag{2.55}$$

then (B,A) is a hypergraph, with edges $a\in A$ and vertices $b\in B$. One can use (2.41) to define the incidence of any $a\in A$ and $b\in B$ by

$$a\rho b \Longleftrightarrow a\cap b\neq\emptyset \Longleftrightarrow b\in a, \tag{2.56}$$

i.e., by the usual incidence of the edge a and the vertex b in the hypergraph (B,A); this ρ is, essentially, the "polar" of (2.44), in the sense (2.20). Conditions (2.18), (2.19) mean now (2.55) and (2.54). Furthermore, $g\in 2^A$ is an A-cover, in the sense (2.14), if and only if

$$\bigcup_{a\in g} a=B, \tag{2.57}$$

i.e., if and only if g is a covering of the hypergraph (B,A), in the sense of [3], p.420 (the term "edge cover" might be more appropriate); or, in other words, the collection g of subsets of M is a covering of M, in the usual sense. Also, $w\in 2^B$ is a B-packing, in the sense (2.16), if and only if

$$|w \cap a| \leq 1 \qquad (a \in A) \qquad (2.58)$$

i.e., if and only if w is a strongly stable set in the hypergraph (B,A), in the sense of [3], p.448 (one might also use the term "independent set", or "coclique"). In this case, the inequality (2.22) is known (see e.g. [3], p.449, theorem 1).

Remark 2.8. In the survey paper [2], the "set packing problem" and the "set covering problem" are formulated in terms of 0-1 matrices, and then it is observed that, since every 0-1 matrix which has no zero rows or zero columns is the incidence matrix of some hypergraph, these problems can be also formulated in terms of hypergraphs. However, our approach, via coupling functions for incidence triples, is different.

Theorem 2.4. Under the assumptions of example 2.3, if for each $b \in B=M$ we have $\{b\} \in A$ (in particular, if (2.54), (2.55) hold and every non-empty subset a' of each set $a \in A$ belongs to A), then

$$|w| = \min_{g \in G} \phi_3(g,w) \qquad (w \in 2^B). \qquad (2.59)$$

Proof. By our assumption, $g_1 = \{\{b\} | b \in B\} \in G$; also, clearly, for any $w \in 2^B$ we have

$$\phi_3(g_1,w) = |w|, \qquad (2.60)$$

which, together with (2.33), proves (2.59).

Remark 2.9. From remark 2.5 it follows that, under the assumptions of example 2.3, we can achieve (2.54), (2.55), replacing the set system (B,A) by the hypergraph (B',A'), where

$$B' = \bigcup_{a \in A} a, \quad A' = A \setminus \emptyset. \qquad (2.61)$$

In the frameworks of examples 2.2 and 2.3, we have

$$\phi_3(y,w) = |\{(a,b) \in y \times w | a \cap b \neq \emptyset\}| \qquad (y \in 2^A,\ w \in 2^B), \qquad (2.62)$$

where $a \cap b \neq \emptyset$ means $a \in b$ for example 2.2 and $b \in a$ for example 2.3, so ϕ_3 is the usual incidence function for the hypegraph (A,B) (respectively, (B,A)).

In both cases, one can also define a new coupling function $\phi_4 : 2^A \times 2^B \to R$, by

$$\phi_4(y,w) = |\{a \cap b | a \in y,\ b \in w,\ a \cap b \neq \emptyset\}| \qquad (y \in 2^A,\ w \in 2^B); \qquad (2.63)$$

in other words, $\phi_4(y,w)$ is the "number of non-empty intersections" between y and w. By (2.62) and (2.63), we have

$$\phi_3(y,w) \geq \phi_4(y,w) \qquad (y \in 2^A,\ w \in 2^B), \quad (2.64)$$

where strict inequality may also occur (since several different intersecting pairs (a,b) may have the same intersection $a \cap b$).

Let us observe now that, in the case of example 2.2, we have

$$\phi_4(y,w) = \phi_3(y,w) \qquad (y \in 2^A,\ w \in W). \quad (2.65)$$

Indeed, if (a_1,b_1), $(a_2,b_2) \in A \times w$ satisfy

$$a_1 \cap b_1 = a_2 \cap b_2 \neq \emptyset,$$

then, in the case of example 2.2, $a_1 = a_2 \in b_1 \cap b_2$, in contradiction with (2.47).

However, in the case of example 2.3, (2.65) need not hold, even for $y \in G$ and $w \in W$ (in example 4.1, with y=E, $w=\{v_1'\}$, we have $\phi_3(y,w)=2>1=\phi_4(y,w)$).

In §§3-10, we shall study min-max equalities of the form (2.25), for various sets G and W, so h and μ will be the functions (2.23), (2.24). We shall consider only certain pairs (G,W), for which we shall be able to choose an incidence triple (A,B,ρ) such that G and W are the sets (2.15) (or "almost" (2.15), e.g. in (8.1)) and (2.17). We shall also study the Lagrangian duality equality (2.34), with $\phi=\phi_3$ of (2.5)=(2.62), and some related problems.

3. MATCHINGS AND VERTEX COVERS IN BIPARTITE GRAPHS

Throughout the sequel, for simplicity, by a "graph" we shall mean, without any special mention, a finite simple graph, i.e. (see e.g. [3], p.5), having no loops and no multiple edges, and we shall consider only graphs without isolated vertices.

Let us recall (see e.g. [9], theorem 2)

"König's matching theorem". In a bipartite graph $\mathfrak{G}=(V=V'\cup V'',E)$, the maximum cardinality of a matching (i.e., of a set of pairwise disjoint edges) is equal to the minimum cardinality of a vertex cover (i.e., of a set of vertices intersecting all edges).

We can write this theorem in the form (2.25), by choosing

G=the collection of all vertex covers g, (3.1)

W=the collection of all matchings w. (3.2)

Now, let

$$A=V(=M),\ B=E\setminus\emptyset\subset 2^V \tag{3.3}$$

(where we identify each edge with the set of its two endpoints), and let ρ be the usual incidence of vertices and edges in graphs, defined, for any $v(=a)\in V$ and $e(=b)\in E$, by (2.44) of example 2.2; note that conditions (2.42) and (2.43) are now satisfied. Then, the A-covers are nothing else than the vertex covers of $\mathcal{G}$, and the B-packings coincide with the matchings of $\mathcal{G}$; hence, the sets G of (2.15) and (3.1) are the same, and so are the sets W of (2.17), (3.2), and the results of §2 apply.

Thus, since the min-max equality (2.25) is now König's matching theorem, from theorems 3.1, 3.2, and corollary 3.1, we obtain the following results, for a bipartite graph $\mathcal{G}$:

Theorem 3.1. We have (2.34) (so König's matching theorem coincides with the Lagrangian duality equality (1.32), with $\phi=\phi_3$), and, for any minimum cardinality vertex cover g_o and any maximum cardinality matching w_o, each vertex $v\in g_o$ belongs to exactly one edge $e\in w_o$, and each edge $e\in w_o$ contains exactly one vertex $v\in g_o$; hence, the number of incidences between g_o and w_o is (2.35).

Corollary 3.1. a) Given any minimum cardinality vertex cover $g_o=\{v_1,\ldots,v_q\}$, for each $v_i\in g_o$ one can select an edge $e_i\in E$, containing v_i, in such a way that $w_o=\{e_1,\ldots,e_q\}$ is a maximum cardinality matching.

b) Given any maximum cardinality matching $w_o=\{e_1,\ldots,e_q\}$, from each edge $e_i\in w_o$ one can select a vertex v_i of e_i, in such a way that $g_o=\{v_1,\ldots,v_q\}$ is a minimum cardinality vertex cover.

Remark 3.1. Corollary 3.1 b) is known; for an algorithmic proof, see e.g. [12], pp.279-280.

Moreover, there holds now

Theorem 3.2. For G and ϕ_3 of (3.1), (2.5), we have

$$|w|=\min_{g\in G}\phi_3(g,w) \qquad (w\in 2^E). \tag{3.4}$$

Proof. For any $w=\{e_1=(v_1',v_1''),\ldots,e_p=(v_p',v_p'')\}\in 2^E$, define

$$g_w=V\setminus\{v_1'',\ldots,v_p''\}; \tag{3.5}$$

note that some of the v'_i's (and some of the v''_j's) may coincide, but we write w in the above form, for the computation of $\phi_3(g_w,w)$. Then, $g_w \cap e_i=\{v'_i\}\neq\emptyset$ $(i=1,\dots,p)$ and, clearly, $g_w \cap e\neq\emptyset$ for all $e\in E\setminus w$, so $g_w\in G$. Furthermore, $v'_i \cap w=\{v'_i\}\neq\emptyset$ $(i=1,\dots,p)$ and $v\cap w=\emptyset$ $(v\in g_w\setminus\{v'_1,\dots,v'_p\}=V\setminus\{v'_1,v''_1,\dots,v'_p,v''_p\})$, whence

$$\phi_3(g_w,w)=p=|w|, \tag{3.6}$$

which, together with (2.33), proves (3.4).

Finally, let us give now some counter-examples related to the preceding results.

Formula (2.26) need not hold for $w\in 2^E\setminus W$, as shown by

Example 3.1. Let $V'=\{v'_1\}$, $V''=\{v''_1,v''_2\}$, $E=\{e_1=(v'_1,v''_1)$, $e_2=(v'_1,v''_2)\}$. Then for $y=g=V'\in G$ and $w=E\in 2^E\setminus W$, we have $\phi_1(y,w)=$ $=|\{v\in y|v\rho w\}|=1<2=|\{(v,e)\in y\times w|v\rho e\}|=\phi_3(y,w)$.

Under the assumptions of theorem 1.2, the converse of the statement in theorem 1.2 b) is not valid, as shown by

Example 3.2. Let $V'=\{v'_1\}$, $V''=\{v''_1\}$, $E=\{e_1=(v'_1,v''_1)\}$, and let $\mathcal{G}=\{V'\cup V'',E\}$. Then, for $g_o=V'\cup V''\in G$ we have $\phi_3(g_o,e_1)=2=|g_o|$, whence (1.41) (since $W=\{\{e_1\},\emptyset\}$), but g_o is not a minimum cardinality vertex cover.

Formula (1.41) need not hold for all $g_o\in G$, as shown by

Example 3.3. For $\mathcal{G}$ of example 3.1, and $g_o=V''\in G$, we have $\phi_3(g_o,e_1)=\phi_3(g_o,e_2)=1<2=|g_o|$, so (1.41) is not satisfied (since $W=\{\{e_1\},\{e_2\},\emptyset\}$).

Formula (1.50) need not hold for all $(g_o,w_o)\in G\times W$ such that g_o is an optimal element of (1.1), as shown by

Example 3.4. Let $V'=\{v'_1,v'_2\}$, $V''=\{v''_1,v''_2\}$, $E=\{e_1=(v'_1,v''_1)$, $e_2=$ $=(v'_1,v''_2)$, $e_3=(v'_2,v''_1)\}$, and let $\mathcal{G}=\{V'\cup V'',E\}$. Then, $g_o=\{v'_1,v''_1\}\in G$ is a minimum cardinality vertex cover, but for $w_o=\{e_1\}\in W$ we have $2=\phi_3(g_o,w_o)>\min_{g\in G}\phi_3(g,w_o)=1$ (attained, e.g., for $g=V'\in G$).

In the sequel, we shall not mention examples corresponding to examples 3.1, 3.2 and 3.4 (with the exception of example 4.2), but we shall give some, which correspond to example 3.3 (see examples 5.1, 6.1, 7.1 and 9.1).

It is well-known that for non-bipartite graphs the min-max equality (2.25) need not hold (e.g., for the triangle K_3 we have $\min_{g\in G}|g|=2>1=\max_{w\in W}|w|$), but one can still consider the coupling function ϕ_3 of (2.5). The Lagrangian duality equality (2.34) need not hold either, for this case, as shown again by the triangle K_3 (since $\max_{w\in W}\min_{g\in G}\phi_3(g,w)=1$). Moreover, even when the Lagrangian duality equality (2.34) holds true, the min-max equality (2.25) need not hold, and hence (3.4) need not hold even for maximum cardinality matchings $w\in W$, as shown by

Example 3.5. Let $\mathcal{G}=K_4$, the complete graph on four vertices. Then for any maximum cardinality matching $w\in W$ we have $|w|=2<3=$ $=\min_{g\in G}\phi_3(g,w)=\min_{g\in G}|g|$.

Remark 3.2. For maximum cardinality matchings in arbitrary (not necessarily bipartite) graphs, the known max-min equalities (see e.g. [5], Ch.6, theorem 7.1 and [9], theorem 11) are not "all-cardinality" equalities (namely, (3.1) and (2.23) do not hold); these max-min equalities suggest to use another coupling function $\phi:G\times W\to R$, which we shall give elsewhere.

4. EDGE COVERS AND COCLIQUES IN BIPARTITE GRAPHS

Let us recall (see e.g. [9], corollary 2 a)

"König's covering theorem". In a bipartite graph $\mathcal{G}=(V=V'\cup V'',E)$, the minimum cardinality of an edge cover (i.e., of a set of edges covering all vertices) is equal to the maximum cardinality of a coclique (i.e., of a set of pairwise non-adjacent vertices).

We can write this theorem in the form (2.25), by choosing

$$G=\text{the collection of all edge covers } g, \quad (4.1)$$

$$W=\text{the collection of all cocliques } w. \quad (4.2)$$

Now, let

$$A=E\setminus\emptyset\subset 2^V, \quad B=V(=M) \quad (4.3)$$

(where we identify each edge with the set of its two endpoints), and let ρ be the incidence defined, for each $e(=a)\in V$ and $v(=b)\in V$, by (2.56) of example 2.3; note that conditions (2.54) and (2.55) are now satisfied. Then, the A-covers are nothing else than the edge covers of $\mathcal{G}$, and the B-packings coincide with the cocliques of $\mathcal{G}$. Thus, since the min-max equality (2.25) is now König's covering theorem, from theorems 2.1, 2.2, and corollary 2.1 we obtain the following results, for a bipartite graph $\mathcal{G}$:

Theorem 4.1. We have (2.34) (so König's covering theorem coincides with the Lagrangian duality theorem (1.32), with $\phi=\phi_3$), and for any minimum cardinality edge cover g_o and any maximum cardinality coclique w_o, each edge $e\in g_o$ contains exactly one vertex $v\in w_o$, and each vertex $v\in w_o$ belongs to exactly one edge $e\in g_o$; hence, the number of incidences between g_o and w_o is (2.35).

Corollary 4.1. a) Given any minimum cardinality edge cover $g_o=\{e_1,\dots,e_q\}$, from each edge $e_i\in g_o$ one can select a vertex v_i of e_i, in such a way that $w_o=\{v_1,\dots,v_q\}$ is a maximum cardinality coclique.

b) Given any maximum cardinality coclique $w_o=\{v_1,\dots,v_q\}$, for each $v_i\in w_o$ one can select an edge $e_i\in E$, containing v_i, in such a way that $g_o=\{e_1,\dots,e_q\}$ is a minimum cardinality edge cover.

In contrast with the situation of §3 (see example 3.3), there holds now

Theorem 4.2. For W and ϕ_3 of (4.2), (2.5), we have

$$|y|=\max_{w\in W}\phi_3(y,w) \qquad (y\in 2^E). \tag{4.4}$$

Proof. For any $y=\{e_1=(v_1',v_1''),\dots,e_q=(v_q',v_q'')\}\in 2^E$, define

$$w_y=\{v_1',\dots,v_q'\}; \tag{4.5}$$

note that some of the v_i''s (and some of the v_j'''s) may coincide, but we write y and w_y in the above form, for the computation of $\phi_3(y,w_y)$. Then, since $w_y\subseteq V'$, we have $w_y\in G$. Furthermore, $y\cap v_i'\ni v_i'\neq\emptyset$ $(i=1,\dots,q)$, whence

$$\phi_3(y,w_y)\geq q=|y|, \tag{4.6}$$

which, together with (2.32), proves (4.4).

A result corresponding to theorem 3.2 need not hold, even for $w\in W$, as shown by

Example 4.1. For $\mathcal{G}$ of example 3.1, G is a singleton, namely, the only edge cover of $\mathcal{G}$ is $g_o=E$. Hence, for the coclique $w=\{v_1'\}\in W$ we have

$$|w|=1<2=\phi_3(g_o,w)=\min_{g\in G}\phi_3(g,w); \tag{4.7}$$

nevertheless, equality holds for the coclique $w_o=\{v_1'',v_2''\}$, and, in fact, for any "optimal pair" (g_o,w_o) (by theorem 4.1).

Example 4.1 and remark 1.10 c) suggest the question, whether formula (1.50) holds for all $(g_o,w_o)\in G\times W$ such that g_o is optimal for problem (1.1), where G and W are those of (4.1) and (4.2), respectively. The answer is negative, as shown by

Example 4.2. Let $V'=\{v_1',v_2'\}$, $V''=\{v_1'',v_2'',v_3''\}$, $E=\{e_1=(v_1',v_1''),$ $e_2=(v_1',v_2'')$, $e_3=(v_1',v_3'')$, $e_4=(v_2',v_1'')$, $e_5=(v_2',v_3'')\}$, and let $\mathcal{G}=(V'\cup V'',E)$. Then, for the minimum cardinality edge covers $g_o=\{e_2,e_4,e_5\}$ and $g_o'=\{e_2,e_3,e_4\}$ and the coclique $w=\{v_2',v_2''\}\in W$ we have $\phi_3(g_o,w)=3>2=\phi_3(g_o',w)$.

For non-bipartite graphs, the equality (4.4) need not hold, even for minimum cardinality edge covers $y\in G$, as shown by

Example 4.3. Let $\mathcal{G}=K_4$ of example 3.5. Then every minimum cardinality edge cover $g\in G$ is a matching, and hence $\max_{w\in W}\phi_3(g,w)=1<2=|g|$ (since $W=\{\{v_1\},\ldots,\{v_4\},\emptyset\}$). Note also that, in this example, we have (1.35), but not (1.31), (2.34), (2.25).

5. CHAIN COVERS AND ANTICHAINS IN POSETS

Let us recall (see e.g. [9], theorem 7)

"Dilworth's theorem". In a finite poset P, the minimum cardinality of a chain cover of P (i.e., of a set of chains covering all elements of P) is equal to the maximum cardinality of an antichain (i.e., of a set of pairwise incomparable elements).

In the usual formulation of this theorem (see e.g. [1], theorem 8.14), the words "chain cover" are replaced by "partition of P into chains" (i.e., disjoint chain cover), but it is easy to see that the two formulations are equivalent.

We can write the above theorem in the form (2.25), by choosing

G=the collection of all chain covers g, (5.1)

W=the collection of all antichains w. (5.2)

Now, let

$$A=L\setminus\emptyset \subset 2^P,\quad B=P(=M), \tag{5.3}$$

where L denotes the collection of all chains ℓ in P, and let ρ be the incidence defined, for each $\ell(=a)\in L$ and $p(=b)\in P$, by (2.56) of example 2.3; note that conditions (2.54) and (2.55) are now satisfied. Then, the A-covers are nothing else than the chain covers of P, and the B-packings are precisely the antichains of P. Thus, since the min-max equality (2.25) is now Dilworth's theorem, from theorems 2.1, 2.2 and corollary 2.1 we obtain the following results, for a finite poset P:

Theorem 5.1. We have (2.34) (so Dilworth's theorem coincides with the Lagrangian duality equality (1.32), with $\phi=\phi_3$), and for any minimum cardinality chain cover g_o and any maximum cardinality antichain w_o, each chain $\ell\in g_o$ contains exactly one $p\in w_o$, and each $p\in w_o$ belongs to exactly one chain $\ell\in g_o$; hence, the number of incidences between g_o and w_o is (2.35).

Corollary 5.1. a) Given any minimum cardinality chain cover $g_o=\{\ell_1,\dots,\ell_q\}$, from each chain $\ell_i\in g_o$ one can select an element $p_i\in\ell_i$, in such a way that $w_o=\{p_1,\dots,p_q\}$ is a maximum cardinality antichain.

b) Given any maximum cardinality antichain $w_o=\{p_1,\dots,p_q\}$, for each $p_i\in w_o$ one can select a chain $\ell_i\in L$, containing p_i, in such a way that $g_o=\{\ell_1,\dots,\ell_q\}$ is a minimum cardinality chain cover.

Furthermore, from theorem 2.4 we obtain

Theorem 5.2. For G and ϕ_3 of (5.1), (2.5), we have

$$|w|=\min_{g\in G}\phi_3(g,w) \qquad (w\in 2^P). \tag{5.4}$$

A result corresponding to theorem 4.2 need not hold, even for $y\in G$, as shown by

Example 5.1. Let $P=\{p_1,p_2\}$, with $p_1>p_2$. Then, for $y=g'=$ $=\{\{p_1\},\{p_2\}\}\in G$ we have $\phi_3(g',\{p_1\})=\phi_3(g',\{p_2\})=1<2=|g'|$, so (1.41) is not satisfied (since $W=\{\{p_1\},\{p_2\},\emptyset\}$).

6. ANTICHAIN COVERS AND CHAINS IN POSETS

Let us recall (see e.g. [9], theorem 8) the following

"Polar" to Dilworth's theorem. In a finite poset P, the maximum cardinality of a chain is equal to the minimum cardinality of an antichain cover (i.e., of a set of antichains covering all elements of P).

This result is "polar", to Dilworth's theorem of §5, in the sense (see [9], p.456) that interchanging "chains" and "antichains" carries one to the other.

We can write the above theorem in the form (2.25), by choosing

$$G = \text{the collection of all antichain covers } g, \tag{6.1}$$

$$W = L (= \text{the collection of all chains } w). \tag{6.2}$$

Now, let

$$A = \Gamma \setminus \emptyset \subset 2^P, \quad B = P (=M), \tag{6.3}$$

where Γ denotes the collection of all antichains γ in P, and let ρ be the incidence defined, for each $\gamma(=a)\in\Gamma$ and $p(=b)\in B$, by (2.56) of example 2.3; note that conditions (2.54) and (2.55) are now satisfied. Then, the A-covers are nothing else than the antichain covers of P, and the B-packings are precisely the chains of P. Thus, since the min-max equality (2.25) is now the above "polar" result to Dilworth's theorem, from theorems 2.1, 2.2 and corollary 2.1 we obtain the following results, for a finite poset P:

Theorem 6.1. We have (2.34) (so the above "polar" to Dilworth's theorem coincides with the Lagrangian duality equality (1.32), with $\phi=\phi_3$), and for any minimum cardinality antichain cover g_o and any maximum cardinality chain w_o, each antichain $\gamma\in g_o$ contains exactly one $p\in w_o$, and each $p\in w_o$ belongs to exactly one antichain $\gamma\in g_o$; hence, the number of incidences between g_o and w_o is (2.35).

Corollary 6.1. a) Given any minimum cardinality antichain cover $g_o=\{\gamma_1,\dots\gamma_q\}$, from each antichain $\gamma_i\in g_o$ one can select an element $p_i \in \gamma_i$, in such a way that $w_o=\{p_1,\dots,p_q\}$ is a maximum cardinality chain.

b) Given any maximum cardinality chain $w_o=\{p_1,\dots,p_q\}$, for each $p_i \in w_o$ one can select an antichain $\gamma_i \in \Gamma$, containing p_i, in such a way that $g_o=\{\gamma_1,\dots,\gamma_q\}$ is a minimum cardinality antichain cover.

Remark 6.1. a) In this paper, "chain" means a set of pairwise comparable elements of P; if one also requires these elements to be arranged in increasing order, then in corollary 6.1 a) one should replace "$w_o=\{p_1,\dots,p_q\}$" by "some permutation w_o of $\{p_1,\dots,p_q\}$".

b) Corollary 6.1 b) is known and it admits a simple direct proof, which is also the usual proof of the above "polar" to Dilworth's theorem (see e.g. [1], proof of proposition 8.15); namely, it is enough to take

$$\gamma_i=\{p \in P \mid d(p)=i\} \qquad (i=1,\dots,q), \qquad (6.4)$$

where d(p) denotes the length of the largest chain in P, with (upper) endpoint p.

Furthermore, from theorem 2.4 we obtain

Theorem 6.2. For G and ϕ_3 of (6.1), (2.5), we have

$$|w|=\min_{g \in G} \phi_3(g,w) \qquad (w \in 2^P). \qquad (6.5)$$

A result corresponding to theorem 4.2 need not hold, even for $y \in G$, as shown by

Example 6.1. Let $P=\{p_1,p_2\}$, with p_1 and p_2 incomparable. Then, for $y=g'=\{\{p_1\},\{p_2\}\} \in G$ we have $\phi_3(g',\{p_1\})=\phi_3(g',\{p_2\})=1<2=|g'|$, so (1.41) is not satisfied (since $W=\{\{p_1\},\{p_2\},\emptyset\}$).

7. ARC-DISJOINT r-s-PATH PACKINGS AND r-s-SEPARATING ARC SETS IN DIRECTED GRAPHS

Let us recall (see e.g. [1], theorem 8.1)

"Menger's theorem". Let $\mathcal{D}=(V,U)$ be a directed graph, and let $r \neq s \in V$. Then the maximum number of pairwise arc-disjoint r-s-paths (i.e., directed paths from r to s) is equal to the minimum cardinality of an r-s-separating arc set (i.e., of a set of arcs such that after its removal from $\mathfrak{D}$, there remains no r-s-path).

In another formulation of this theorem (see [9], theorem 4), the words "r-s-separating arc set" are replaced by "r-s-cut" (i.e., a set $\delta^{-}(V')$, with $V' \subset V$, $r \notin V'$, $s \in V'$, where $\delta^{-}(V')$ denotes the set of all arcs entering V'), but the two formulations are equivalent; indeed, each r-s-cut is r-s-separating, and each r-s-separating arc set contains an r-s-cut (see e.g. [5], p.121).

We can write the above theorem in the form (2.25), by choosing

G=the collection of all r-s-separating arc sets g, (7.1)

W=the collection of all sets w of pairwise arc-disjoint r-s-paths. (7.2)

An r-s-path in $\mathfrak{D}$ is any sequence $\pi=(v_0=r, u_1, v_1, u_2, \dots, v_{n-1}, u_n, v_n=s)$, where $v_i \in V$ $(i=0,\dots,n)$, $u_i=\overrightarrow{v_{i-1}, v_i} \in U$ $(i=1,\dots,n)$, and $n \geq 1$, and we shall identify it with the sequence of arcs $\pi=(u_1,\dots,u_n)$. We recall that an r-s-path $\pi=(u_1,\dots,u_n)$ is said to be simple (see e.g. [3], p.8), if $u_i \neq u_j$ for all $i \neq j$. It is well-known that each r-s-path π contains a simple r-s-path (in fact, any shortest r-s-subpath of π, i.e., having the least number of arcs, is simple). Using this fact, let us observe

Lemma 7.1. In Menger's theorem one can replace r-s-paths by simple r-s-paths.

Proof. Let $w_0=\{\pi_1,\dots,\pi_p\}$ be a maximum cardinality set of pairwise disjoint r-s-paths. Then each π_j contains a simple r-s-path π'_j, whence $w'=\{\pi'_1,\dots,\pi'_p\}$ is a maximum cardinality set of pairwise disjoint simple r-s-paths. On the other hand, an arc set g disconnects all r-s-paths (i.e., $g \in G$) if and only if g disconnects all simple r-s-paths (since each r-s-path contains a simple r-s-path). Thus, replacing r-s-paths by simple r-s-paths does not alter the max and min in Menger's theorem, which completes the proof.

In the sequel, we shall consider only simple r-s-paths. Let

$$A=\bigcup_{\pi \in \Pi(r,s)} \pi\ (=M) \subset U,\quad B=\Pi(r,s) \subset 2^{A}, \tag{7.3}$$

where $\Pi(r,s)$ denotes the set of all simple r-s-paths $\pi=(u_1,\dots,u_n)$ in $\mathfrak{D}$ (these A and B are obtained by applying remark 2.7 to U and

$\Pi(r,s)\subset 2^U$), and let ρ be the incidence defined, for each $u(=a)\in \in\bigcup_{\pi\in\Pi(r,s)}\pi$ and $\pi(=b)\in\Pi(r,s)$, by (2.44) of example 2.2. Then, (2.42) and (2.43) are satisfied, and the A-covers are nothing else than the r-s-separating arc sets in $\mathfrak{D}$, while the B-packings coincide with the sets of pairwise arc-disjoint simple r-s-paths in $\mathfrak{D}$. Thus, since the min-max equality (2.25) is now Menger's theorem, from theorems 2.1, 2.2, and corollary 2.1 we obtain the following results for a directed graph $\mathfrak{D}=(V,U)$ and for $r\neq s\in V$:

Theorem 7.1. We have (2.34) (so Menger's theorem coincides with the Lagrangian duality equality (1.32), with $\phi=\phi_3$), and for any minimum cardinality r-s-separating arc set g_o and any maximum cardinality set w_o of pairwise arc-disjoint simple r-s-paths, each arc $u\in g_o$ belongs to exactly one r-s-path $\pi\in w_o$, and each r-s-path $\pi\in w_o$ contains exactly one arc $u\in g_o$; hence, the number of incidences between g_o and w_o is (2.35).

Corollary 7.1. a) Given any minimum cardinality r-s-separating arc set $g_o=\{g_1,\dots,u_q\}$, for each $u_i\in g_o$ one can select a simple r-s--path $\pi_i\in\Pi(r,s)$, containing u_i, in such a way that $w_o=\{\pi_1,\dots,\pi_q\}$ is a maximum cardinality set of arc-disjoint simple r-s-paths.

b) Given any maximum cardinality set $w_o=\{\pi_1,\dots,\pi_q\}$ of arc--disjoint simple r-s-paths, from each $\pi_i\in w_o$ one can select an arc u_i, in such a way that $g_o=\{u_1,\dots,u_q\}$ is a minimum cardinality r-s--separating arc set.

Moreover, there holds now

Theorem 7.2. For G and ϕ_3 of (7.1), (2.5), we have

$$|w|=\min_{g\in G}\phi_3(g,w)\qquad (w\in 2^{\Pi(r,s)}). \tag{7.4}$$

Proof. For any $w=\{\pi_1=(u_{11},\dots,u_{1,j_1}),\dots,\pi_p=(u_{p1},\dots,u_{p,j_p})\}\in \in 2^{\Pi(r,s)}$, define

$$g_w=U\setminus\{u_{12},\dots,u_{1,j_1},\ u_{22},\dots,u_{2,j_2},\dots,u_{p2},\dots,u_{p,j_p}\}; \tag{7.5}$$

note that some of the arcs u_{ik} may coincide, but we write w in the above form, for the computation of $\phi_3(g_w,w)$. Then, $g_w\cap\pi_i=\{u_{i1}\}\neq\emptyset$ $(i=1,\dots,p)$ and, clearly, $g_w\cap\pi\neq\emptyset$ $(\pi\in\Pi(r,s)\setminus w)$, so $g_w\in G$. Furthermore, $\{u_{i1}\}\cap w=\{u_{i1}\}\neq\emptyset$ $(i=1,\dots,p)$ and $\{u\}\cap w=\emptyset$ $(u\in g_w\setminus\{u_{11},u_{21},\dots$ $\dots,u_{p1}\})=U\setminus\bigcup_{u_{ik}\in\pi_1\cup\dots\cup\pi_p} u_{ik})$, whence

$$\phi_3(g_w,w)=p=|w|, \tag{7.6}$$

which, together with (2.33), proves (7.4).

A result corresponding to theorem 4.2 need not hold, even for $y\in G$, as shown by

Example 7.1. Let $V=\{r,v_1,v_2,v_3,s\}$, $U=\{\overrightarrow{rv_1},\overrightarrow{v_1v_2},\overrightarrow{v_1v_3},\overrightarrow{v_2s},\overrightarrow{v_3s}\}$, and let $\mathfrak{G}=(V,U)$. Then $g_o=\{\overrightarrow{v_2s},\overrightarrow{v_3s}\}\in G$, but for $\pi_1=(\overrightarrow{rv_1},\overrightarrow{v_1v_2},\overrightarrow{v_2s})$, $\pi_2=(\overrightarrow{rv_1},\overrightarrow{v_1v_3},\overrightarrow{v_3s})\in\Pi(r,s)$ we have $\phi_3(g_o,\{\pi_1\})=\phi_3(g_o,\{\pi_2\})=1<2=|g_o|$, so (1.41) is not satisfied (since $W=\{\{\pi_1\},\{\pi_2\},\emptyset\}$).

8. r-s-PATHS AND r-s-SEPARATING ARC SET PACKINGS IN DIRECTED GRAPHS

Let us recall (see [4], p.311)

Fulkerson's "polar" to Menger's theorem. Let $\mathfrak{G}=(V,U)$ be a directed graph and let $r\neq s\in V$. Then the minimum number of arcs in an r-s-path is equal to the maximum cardinality of a set of pairwise disjoint r-s-separating arc sets.

Again, there is another equivalent formulation (see [9], corollary 5 a), in which the words "r-s-separating arc sets" are replaced by r-s-cuts" $\delta^-(V')$ (see §7, the remark after Menger's theorem).

We can write the above theorem in the form (2.25), by choosing

$G=\Pi(r,s)$ (=the collection of all simple r-s-paths g), (8.1)

W=the collection of all sets w of pairwise disjoint r-s-separating arc sets; (8.2)

indeed, an r-s-path with a minimum number of arcs is necessarily simple. Now, let

$$A=U(=M),\ B=\sum(r,s)\subset 2^U, \tag{8.3}$$

where $\sum(r,s)$ denotes the collection of all r-s-separating arc sets σ, and let ρ be the incidence defined, for each $u(=a)\in U$ and $\sigma(=b)\in$ $\in\sum(r,s)$, by (2.44) of example 2.2; note that conditions (2.42) and (2.43) are now satisfied. Then, the B-packings are nothing else —

than the sets of pairwise disjoint r-s-separating arc sets, so the sets W of (2.17) and (8.2) are the same. However, the A-covers are now the sets g' of arcs such that each r-s-separating arc set σ contains at least one arc $u\in g'$, so the set G' of all A-covers does not coincide with the set G of (8.1). Therefore, let us prove

Lemma 8.1. Each r-s-path g is an A-cover, and each A-cover g' contains an r-s-path. Hence, the minimum cardinality A-covers are nothing else than the minimum cardinality simple r-s-paths.

Proof. The first statement is obvious, since, by definition, each r-s-path g intersects each r-s-separating arc set σ.

Assume now that g' is an A-cover. Then, g' contains at least one arc, belonging to some r-s-path, of the r-s-separating set $\delta^+(r)=\{\overrightarrow{rv}_1,\ldots,\overrightarrow{rv}_q\}$ (the set of all arcs leaving r). If $g'\cap\delta^+(r)=\{\overrightarrow{rv}_{i_1},\overrightarrow{rv}_{i_2},\ldots,\overrightarrow{rv}_{i_k}\}$, then g' contains at least one arc, belonging to some r-s-path, of the r-s-separating set

$$\delta^+(\{v_{i_1},\ldots,v_{i_k}\})\cup\{\delta^+(r)\setminus g'\}, \tag{8.4}$$

and hence at least one arc, belonging to some r-s-path, of $\delta^+(\{v_{i_1},\ldots,v_{i_k}\})$; thus, g' contains at least one path from r, consisting of two arcs and contained in some r-s-path. Considering all arcs leaving the "positive" endpoints of all two-arc paths from r contained in g' and in some r-s-path, and continuing in this way, we obtain an r-s-path g, contained in g'.

Finally, if g'_o is a minimum cardinality A-cover, then it contains an r-s-path g, which, in turn, contains a simple r-s-path g_o; then, since g_o is an A-cover, it follows that $g'_o=g_o$=a simple r-s-path.

Since the min-max equality (2.25) is now Fulkerson's "polar" to Menger's theorem, from lemma 8.1, theorems 2.1, 2.2, and corollary 2.1, we obtain the following results, for a directed graph $\mathcal{D}=(V,U)$ and for $r\neq s\in V$:

Theorem 8.1. We have (2.34) (so Fulkerson's polar to Menger's theorem coincides with the Lagrangian duality equality (1.32), with

$\phi=\phi_3$) and for any r-s-path g_o with a minimum number of arcs and any maximum cardinality set w_o of pairwise disjoint r-s-separating arc sets, each arc $u\in g_o$ belongs to exactly one r-s-separating arc set $\sigma\in w_o$, and each $\sigma\in w_o$ contains exactly one $u\in g_o$; hence, the number of incidences between g_o and w_o is (2.35).

Corollary 8.1. a) Given any r-s-path $g_o=(u_1,\dots,u_q)$ with a minimum number of arcs, for each $u_i\in g_o$ one can select an r-s-separating arc set σ_i, containing u_i, in such a way that $w_o=\{\sigma_1,\dots,\sigma_q\}$ is a maximum cardinality set of pairwise disjoint r-s-separating arc sets.

b) Given any maximum cardinality set $w_o=\{\sigma_1,\dots,\sigma_q\}$ of pairwise disjoint r-s-separating arc sets, from each $\sigma_i\in w_o$ one can select an arc u_i, in such a way that some permutation g_o of $\{u_1,\dots,u_q\}$ is an r-s-path with a minimum number of arcs.

Remark 8.1. One can also give the following direct proof of corollary 8.1 a) (and hence of Fulkerson's polar to Menger's theorem): Let $u_i=\overrightarrow{v_{i-1}v_i}$ $(i=1,\dots,q)$, where $v_o=r$, $v_q=s$. Choose $\sigma_1=\delta^+(r)\in\in\sum(r,s)$; then $u_1\in\sigma_1$. Let $\delta^+(r)=\{\overrightarrow{rv_{i_1}},\dots,\overrightarrow{rv_{i_k}}\}$. If $s\in\{v_{i_1},\dots,v_{i_k}\}$, then $\overrightarrow{rs}$ is an r-s-path with a minimum number of arcs, so $|g_o|=q=1$, and we are done (with $w_o=\{\sigma_1\}$). If $s\notin\{v_{i_1},\dots,v_{i_k}\}$, choose $\sigma_2=\delta^+(\{v_{i_1},\dots,v_{i_k}\})\in\sum(r,s)$; then $\sigma_1\cap\sigma_2=\emptyset$ and $u_2\in\sigma_2$. If s belongs to the set of all "positive" endpoints of the arcs of σ_2, then $|g_o|=q=2$, and we are done (with $w_o=\{\sigma_1,\sigma_2\}$). If not, then, continuing in this way, we arrive, finally, at a set $w_o=\{\sigma_1,\dots,\sigma_q\}$ of pairwise disjoint r-s-separating arc sets, with $u_i\in\sigma_i$ $(i=1,\dots,q)$; then, since $|w_o|=|g_o|$, w_o is of maximum cardinality (by theorem 1.2 d), implication $3°\Rightarrow 1°$).

A result corresponding to theorem 7.2 need not hold, even for $w\in W$, and even using r-s-cuts instead of r-s-separating sets (since r-s-cuts are "smaller"), as shown by

Example 8.1. Let $V=\{r,v_1,v_2,s\}$, $U=\{\overrightarrow{rv_1},\overrightarrow{v_1v_2},\overrightarrow{v_2s}\}$, and let $\mathfrak{D}==(V,U)$. Then $\sigma=\{\overrightarrow{rv_1},\overrightarrow{v_2s}\}\in\sum(r,s)$ (and $\sigma=\delta^-(V')$, where $V'=\{v_1,s\}\subset V$, so σ is an r-s-cut), whence $w=\{\sigma\}\in W$, and G is the singleton consisting of the r-s-path $g=(\overrightarrow{rv_1},\overrightarrow{v_1v_2},\overrightarrow{v_2s})$, but $\phi_3(g,w)=2>1=|w|$.

However, since $U\in\sum(r,s)$, from theorem 2.3 we obtain

Theorem 8.2. For W and ϕ_3 of (8.2), (2.5), we have

$$|y|=\max_{w\in W}\phi_3(y,w) \qquad (y\in 2^U). \tag{8.5}$$

If we use r-s-cuts instead of r-s-separating sets, then theorem 8.2 need not remain valid, even for $y\in G$, as shown by

Example 8.2. Let $V=\{r,v,s\}$, $U=\{u_1=\overrightarrow{rs},u_2=\overrightarrow{rv},u_3=\overrightarrow{vs}\}$, and let $\mathcal{D}=(V,U)$. Then $y=g'=(u_2,u_3)\in G$, but for $\sigma_1=\{u_1,u_2\}=\delta^-(\{v,s\})$, $\sigma_2=\{u_1,u_3\}=\delta^-(\{s\})$, which are the only r-s-cuts in $\mathcal{D}$, we have $\sigma_1\cap\sigma_2=\{u_1\}\neq\emptyset$ (hence $\{\sigma_1,\sigma_2\}\notin W$) and

$$\phi_3(g',\{\sigma_1\})=\phi_3(g',\{\sigma_2\})=1<2=|g'|.$$

9. INTERNALLY VERTEX-DISJOINT r-s-PATH PACKINGS AND r-s-SEPARATING VERTEX SETS IN DIRECTED GRAPHS

Let us recall (see e.g. [9], corollary 4 a) or [1], theorem 8.2)

"*Menger's theorem-vertex form*". *Let $\mathcal{D}=(V,U)$ be a directed graph, and let $r\neq s\in V$, $\overrightarrow{rs}\notin U$. Then the maximum number of pairwise internally vertex-disjoint (i.e., having in common only the vertices r and s) r-s-paths is equal to the minimum cardinality of an r-s-separating vertex set (i.e., of a set of vertices in $V\setminus\{r,s\}$, intersecting all r-s-paths).*

We can write this thoerem in the form (2.25), by choosing

G=the collection of all r-s-separating vertex sets g, (9.1)

W=the collection of all sets w of pairwise internally vertex-disjoint r-s-paths. (9.2)

In this section it will be convenient *to identify each r-s-path* $\pi=(v_o=r,u_1,v_1,u_2,\dots,v_{n-1},u_n,v_n=s)$ *with the sequence of vertices* $\pi=(v_1,\dots,v_{n-1})\in V\setminus\{r,s\}$; similarly to §7, we may restrict ourselves *to consider only elementary* r-s-*paths* π, i.e. ([3], p.8), such that $v_i\neq v_j$ for all $v_i,v_j\in\pi$, $i<j$ (including $i=0$ and $j=n$).

Now, let

$$A=\bigcup_{\pi\in\Pi^{\varepsilon}(r,s)} \pi(=M)\subset V\setminus\{r,s\}, B=\Pi^{\varepsilon}(r,s)\subset 2^{A}, \qquad (9.3).$$

where $\Pi^{\varepsilon}(r,s)$ denotes the set of all elementary r-s-paths $\pi=(v_1,\ldots,v_{n-1})$ in $\mathcal{D}$, and let ρ be the incidence defined, for each $v(=a)\in\bigcup_{\pi\in\Pi^{\varepsilon}(r,s)}\pi$ and $\pi(=b)\in\Pi^{\varepsilon}(r,s)$, by (2.44) of example 2.2. Then, (2.42) and (2.43) are satisfied, and the A-covers are nothing else than the r-s-separating vertex sets, while the B-packings coincide with the sets of pairwise internally vertex-disjoint elementary r-s-paths. Thus, since the min-max equality (2.25) is now Menger's theorem-vertex form, from theorems 2.1, 2.2 and corollary 2.1, one obtains vertex forms of theorem 7.1 and corollary 7.1, which we omit. Also, corresponding to theorem 7.2, we have now

$$|w|=\min_{g\in G} \phi_3(g,w) \qquad (w\in 2^{\Pi^{\varepsilon}(r,s)}), \qquad (9.4)$$

with a similar proof (replacing arcs by vertices). Finally, corresponding to example 7.1, we can give now

Example 9.1. Let $\mathcal{D}=(V,U)$ be as in example 7.1. Then $g_o=\{v_2,v_3\}\in G$, but for $\pi_1=(v_1,v_2)$, $\pi_2=(v_1,v_3)\in\Pi^{\varepsilon}(r,s)$ we have $\phi_3(g_o,\{\pi_1\})=\phi_3(g_o,\{\pi_2\})=1<2=|g_o|$, so (1.41) is not satisfied (since $W=\{\{\pi_1\},\{\pi_2\},\emptyset\}$).

We omit the similar treatment of the extension to the case when r, s are replaced by disjoint subsets R and S of V, and the sets of pairwise (internally) vertex-disjoint r-s-paths are replaced by sets of pairwise vertex-disjoint R-S-paths, i.e., paths starting in R and ending in S (related to [1], proposition 8.3 or [9], p.454). Also, we omit the similar treatment of undirected graphs $\mathcal{G}=(V,E)$ (related to [1], theorem 8.4 and propositions 8.5, 8.6).

10. DIRECTED CUT COVERINGS AND PACKINGS IN DIRECTED GRAPHS

Let us recall (see e.g. [9], theorem 18) the

"Lucchesi-Younger theorem". Let $\mathcal{D}=(V,U)$ be a directed graph. Then the minimum cardinality of a directed cut covering is equal to the maximum cardinality of a set of pairwise disjoint directed cuts.

We recall that, by definition, a directed cut is a set of arcs of the form $\delta^-(V')$, where $\emptyset \neq V' \neq V$ and $\delta^+(V') = \emptyset$, and a directed cut covering is a set U' of arcs intersecting all directed cuts.

We can write the above theorem in the form (2.25), by choosing

$$G = \text{the collection of all directed cut coverings } g, \tag{10.1}$$

$$W = \text{the collection of all sets } w \text{ of pairwise disjoint directed cuts.} \tag{10.2}$$

Now, let

$$A = \bigcup_{c \in C} c \,(=M) \subset U, \quad B = C \setminus \emptyset \subset 2^A, \tag{10.3}$$

where C denotes the collection of all directed cuts $c = \delta^-(V')$, and let ρ be the incidence defined, for each $u(=a) \in \bigcup_{c \in C} c$ and $c(=b) \in C$, by (2.44) of example 2.2. Then, (2.42) and (2.43) are satisfied, and the A-covers coincide with the directed cut coverings, while the B-packings are the sets of pairwise disjoint directed cuts. Thus, since the min-max equality (2.25) is now the Lucchesi-Younger theorem, from theorems 2.1, 2.2 and corollary 2.1 we obtain the following results for a directed graph $\mathcal{D} = (V, U)$:

Theorem 10.1. We have (2.34) (so the Lucchesi-Younger theorem coincides with the Lagrangian duality equality (1.32), with $\phi = \phi_3$), and for any minimum cardinality directed cut covering g_o and any maximum cardinality set w_o of pairwise disjoint directed cuts, each arc $u \in g_o$ belongs to exactly one directed cut $c \in w_o$ and each directed cut $c \in w_o$ contains exactly one arc $u \in g_o$; hence, the number of incidences between g_o and w_o is (2.35).

Corollary 10.1. a) Given any minimum cardinality directed cut covering $g_o = \{u_1, \ldots, u_q\}$, for each $u_i \in g_o$ one can select a directed cut $c_i \in C$, containing u_i, in such a way that $w_o = \{c_1, \ldots, c_q\}$ is a maximum cardinality set of pairwise disjoint directed cuts.

b) Given any maximum cardinality set $w_o = \{c_1, \ldots, c_q\}$ of pairwise disjoint directed cuts, from each $c_i \in w_o$ one can select an arc u_i, in such a way that $g_o = \{u_1, \ldots, u_q\}$ is a minimum cardinality directed cut covering.

Moreover, there holds now

Theorem 10.2. For G and ϕ_3 of (10.1), (2.5), we have

$$|w|=\min_{g\in G} \phi_3(g,w) \qquad (w\in 2^C). \qquad (10.4)$$

Proof. The proof is similar to that of theorem 7.2, replacing $\pi_i\in\Pi(r,s)$ by $c_i\in C$.

A result corresponding to theorem 4.2 need not hold, even for $y\in G$, as shown by

Example 10.1. Let $V=\{v_1,v_2,v_3\}$, $U=\{\overrightarrow{v_1v_2},\overrightarrow{v_1v_3},\overrightarrow{v_2v_3}\}$, and let $\mathcal{D}=(V,U)$. Then $g_o=U\in G$, but for $c_1=\{\overrightarrow{v_1v_3},\overrightarrow{v_2v_3}\}=\delta^-(\{3\})\in C$, $c_2=\{\overrightarrow{v_1v_2},\overrightarrow{v_1v_3}\}=\delta^-(\{v_2,v_3\})\in C$ we have $\phi_3(g_o,\{c_1\})=\phi_3(g_o,\{c_2\})=2<3=|g_o|$, so (1.41) is not satisfied (since $W=\{\{c_1\},\{c_2\},\emptyset\}$).

Remark 10.1. It has been conjectured by Edmonds and Giles (see e.g. [9], p.476) that the "polar" min-max equality to the Lucchesi-Younger theorem holds, i.e., that in any directed graph, the minimum cardinality of a directed cut is equal to the maximum number of pairwise disjoint directed cut coverings. Let us note that if the Lagrangian duality equality (2.34) could be shown, then it would be itself a valid min-max equality (1.9) (though, a priori, not an all-cardinality min-max equality (2.25)), and it would lend a support to the above conjecture (2.25) (see theorem 1.2 a)), choosing G and W in the obvious way, A=U, B=the set of all directed cut covers ($\subset 2^U$), and proving an analog of lemma 8.1.

One can treat similarly other equalities (2.25), e.g. Edmonds' disjoint branching theorem (see e.g. [9], theorem 17), Schrijver's theorem on strong connectors (see e.g. [9], theorem 21), etc.

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Addendum. Corollary 3.1 a) is known; for a proof if König's matching theorem, via corollary 3.1 a), see L. Lovász, Three short proofs in graph theory, J. Combinatorial Theory (B) 19(1975), 269--271.