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APPROACH SYSTEMS

by

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INTRODUCTION

In order to approximate a space S, endowed with several structures, by a family $(S_w)_W$ of simpler (possibly discrete) spaces, one should also take into account certain relations between the approximants. Accordingly, in this paper we define the approach systems, which present some similarities with the projective systems. In the case of usual projective systems the relations between approximants are functions (see [1],[6],[8],[9],[10]). Fleming Topsøe in [11] studied systems of spaces connected by correspondences (multivalued functions). The approximants in the approach systems are connected by transition probabilities, which give more information than multivalued functions. The usual projective systems are a special case of the approach systems for which all the transition probabilities are of Dirac type. The limit of an approach system is defined by a universality property (analogous to the limit of projective systems).

An example will show the advantages of our tratament. Let us try to approximate the real line, R, with the sequence of dyadic sets $(S_w)_W$, where $S_n=\{ k/2^n \mid k\in Z \}$. Whatever be the family of functions $(f_{mn})_{m\ n}$ such that $(S_n, f_{mn}/N)$ is a projective system we do not obtain R endowed with the usual topology λ, as the projective limit. Nevertheless we can find an approach system whose limit is (R, λ) (Theorem 2.16.).

The transition probabilities between spaces determine maps between the associate measure spaces. This fact allows one to associate with an approach system the usual projective system of measure spaces. Under certain assumptions

(Theorem 2.9.), the limit of this projective system is exactly the measure space for which the basic space is the limit of the approach system. We show that every local compact, separable, metrizable space is the limit of a discrete approach system, while the space of the measures defined on it is the limit of the projective system attached to this approach system (Theorem 2.20.).

In the first paragraph we study the approach systems of measurable spaces, the general background in which we can define all further concepts. In the second paragraph we study the approach systems of topological spaces. In this context we obtain the existence theorems for the limit of some approach systems. The difficulty consists in the fact that, unlike projective systems, in general there is no canonical limit of an approach system. In Theorem 2.10. we deal with the case where the terms of an approach system are included in a separable complete metric space S. We give conditions on the transition probabilities under which the limit of this approach system is also included in S. Theorem 2.23. uses Choquet's theory to prove that the limit of an approach system formed by compact metric spaces exists.

The discrete approach systems have a special importance. Nevertheless in view of applications in the theory of stochastic processes we need study more general approach systems. This is the basic reason for which we have considered the general case.

1. Approach systems for measurable spaces

Throughout the paper W will denote a directed ordered set, i.e. for every $w, w' \in W$, there is $w'' \in W$ such that $w \leq w''$ and $w' \leq w''$.

DEFINITION 1.1. Let $(E_w, \sigma_w)_W$ be a family of measurable spaces and $(\varphi_{ww'})_{\substack{w \leq w' \\ w, w' \in W}}$ a family of transition probabilities such that, for every $w \leq w' \leq w''$,

(i) $\varphi_{ww'}$ is a transition probability from $(E_{w'}, \sigma_{w'})$ to (E_w, σ_w);

(ii) $\varphi_{ww}(x, \ ) = \mathcal{E}_x$, $x \in E_w$;

(iii) $\varphi_{w'w''}\varphi_{ww'} = \varphi_{ww''}$, where $\varphi_{w'w''}\varphi_{ww''}(x,A) = \int \varphi_{ww'}(y,A)\varphi_{w'w''}(x,dy)$, $x \in E_{w''}$, $A \in \sigma_w$.

Then $(E_w, \sigma_w, \varphi_{ww'}/W)$ is called an approach system of measurable spaces.

DEFINITION 1.2. Let $\mathcal{E}=(E_w, \sigma_w, \varphi_{ww'}/W)$ be an approach system of measurable spaces, (E,σ) be a measurable space and $(\psi_w)_{w \in W}$ be a family of transition probabilities such that, for every $w \leq w'$,

(i) ψ_w is a transition probability from (E, σ) to (E_w, σ_w);

(ii) $\psi_{w'}\varphi_{ww'} = \psi_w$.

Then $(E, \sigma, \psi_w/W)$ is called a measurable space with approximations for $\mathcal{E}$, while ψ_w is called the w-approximation of E.

If, in addition, for any $(E', \sigma', \psi'_w/W)$, a measurable space with approximations for $\mathcal{E}$, there exists ψ, a unique transition probability from (E', σ') to (E, σ) such that $\Psi\Psi_w = \Psi'_w$, $w \in W$, then $(E, \sigma, \Psi_w/W)$ is called a limit of $\mathcal{E}$. In this case Ψ is called the representation of E' in E.

PROPOSITION 1.3. Let $(E, \sigma, \Psi_w/W)$ be a limit of the approach system of measurable spaces $\mathcal{E}=(E_w, \sigma_w, \ \ _{ww'}/W)$. Then σ is the σ-algebra generated by the set $\{\Psi_w(\ ,A) \ / \ w \in W,\ A \in \sigma_w\}$. Also $\{\Psi_w(f) \ / \ w \in W,\ f \text{ measurable, bounded}\}$ is a real vector space which generates σ, and σ separates the points of E.

Proof. Let $\sigma' = \sigma(\Psi_w(\ ,A)/w \in W,\ A \in \sigma_w)$. Clearly, $\sigma \supset \sigma'$. Since $(E, \sigma', \Psi_w/W)$ is a measurable space with approximations for $\mathcal{E}$, there is Ψ, the representation of (E, σ') in (E, σ). Let η be the representation of (E, σ) in itself. We have $\eta(x,\ ) = \varepsilon_x$, $x \in E$. The transition probability $(\eta_{/E\times\sigma'})\Psi$ is the representation of (E, σ) in itself, too. Consequently, $\Psi = \eta$ and $\sigma \subset \sigma'$.

For a contradiction, suppose that there are $x \neq y$ in E with $\Psi_w(x,\ ) = \Psi_w(y,\ )$, for every $w \in W$. Denote $F = E - \{y\}$. Then $(F, \sigma_{/F}, \Psi_{w/F\times\sigma_w}/W)$ is a measurable space with approximation for $\mathcal{E}$. Let $\Psi_1(z,\ ) = \varepsilon_z$,

$\Psi_2(z,\ ) = \chi_{F-\{x\}}(z)\,\varepsilon_z + \chi_{\{x\}}(z)\,\varepsilon_y$, $z \in F$. We have

$\Psi_1\Psi_w = \Psi_{w/F\times\sigma_w} = \Psi_2\Psi_w$, $w \in W$. Consequently,

$(E, \sigma, \Psi_w/W)$ is not a limit of $\mathcal{E}$.

PROPOSITION 1.4. Let $(E, \sigma, \Psi_w/W)$ be a limit of the approach system of measurable spaces $\mathcal{E}=(E_w, \sigma_w, \varphi_{ww'}/W)$.

(i) If $E' \subset E$, $\sigma' = \sigma/_{E'}$ and $\psi'_w = \psi_{w/E' \times \sigma_w}$, $w \in W$, then $(E', \sigma', \psi'_w/W)$ is a measurable space with approximations for $\mathcal{E}$, having the representation $x \mapsto \varepsilon_x$, $x \in E'$.

(ii) If W' is cofinal in W, then $(E, \sigma, \psi_w/W')$ is a limit of $(E_w, \sigma_w, \varphi_{ww'}/W')$. If $(E, \sigma, \psi_w/W')$ is a limit of the approach system of measurable spaces $(E_w, \sigma_w, \varphi_{ww'}/W')$ then $(E, \sigma, \psi_w/W)$ is a limit of $\mathcal{E}$, where $\psi_w = \psi_{w'} \varphi_{ww'}$ for every $w \in W - W'$ and $w' > w$, $w' \in W'$.

(iii) If W' is cofinal in W, W' is countable and σ_w is countably generated, for every $w \in W'$, then σ is countably generated and each point of E is measurable.

Proof. (i) and (ii). Obvious.

(iii) We shall apply Propositions 1.3. and 1.4.(ii). Let $\mathcal{G}_w$ be a countable subset of σ_w such that: $\sigma(\mathcal{G}_w) = \sigma_w$, $\mathcal{G}_w$ is closed under finite intersections and $E_w \in \mathcal{G}_w$.

For every $w \in W'$, we have

$$\{A \in \sigma_w \,/\, \psi_w(\ ,A) \text{ is } \sigma(\psi_w(\ ,B)/B \in \mathcal{G}_w)\text{-measurable}\} = \sigma_w$$

We have

$$\sigma(\psi_w(\ ,B)/w \in W',\ B \in \mathcal{G}_w) = \sigma\Big(\bigcup_{w \in W'} \sigma(\psi_w(\ ,A)/A \in \sigma_w)\Big) = \sigma.$$

Because σ is generated by a countable set of real functions, it follows that σ is countably generated and, in addition, since σ separates the points, the points are measurable.

THEOREM 1.5. If $(E, \sigma, \psi_w/W)$ and $(E', \sigma', \psi'_w/W)$ are two limits of the approach system of measurable spaces

$\mathcal{E} = (E_w, \sigma_{w'}, \varphi_{ww'}/W)$ and each point of E and E' is measurable then there exists a unique isomorphism for measurable spaces $f : E \longrightarrow E'$, such that $x \longmapsto \varepsilon_{f(x)}$ is the representation of E in E'. As a result, $\Psi_w(x, \ ) = \Psi'_w(f(x), \ )$, $x \in E$, $w \in W$.

Proof. Let Ψ be the representation of E' in E and Ψ' the representation of E in E'. We have $\Psi'\Psi(x, \ ) = \varepsilon_x$, $x \in E$, because $\Psi'\Psi$ is the representation of E in itself. Also, $\Psi\Psi'(x', \ ) = \varepsilon_{x'}$, $x' \in E'$.

For every $x \in E$, we have

$$1 = \Psi'\Psi(x, \{x\}) = \int \Psi(x', \{x\}) \, \Psi'(x, dx').$$

We obtain $\Psi'(x, \{x' \in E' / \Psi(x', \{x\}) = 1\}) = 1$. There is a x' in E' such that $\Psi(x', \{x\}) = 1$. For this x' we have

$$1 = \Psi\Psi'(x', \{x'\}) = \int \Psi'(y, \{x'\}) \, \Psi(x', dy) = \Psi'(x, \{x'\}).$$

As a result, for every $x \in E$, there is just one $x' \in E'$, such that $\Psi'(x, \{x'\}) = \Psi(x', \{x\}) = 1$.

Let $f : E \longrightarrow E'$ be defined by $f(x) = x'$, $x \in E$. Then f is a bijection. We have $\Psi(x, \ ) = \varepsilon_{f(x)}$, $x \in E$ and $\Psi'(x', \ ) = \varepsilon_{f^{-1}(x')}$, $x' \in E'$. Obviously, f and f^{-1} are measurable.

For every measurable space (A, α), denote by M(A), or $M(A, \alpha)$ the vector space of real valued measures on (A, α), endowed with the σ-algebra generated by the set of functions $\{\mu \longmapsto \int f d\mu, \ \mu \in M(A) \ / \ f \ \alpha\text{-measurable, bounded}\}$.

If (A, α) and (B, β) are two measurable spaces and φ is a transition probability from (A, α) to (B, β), then we shall denote by $\hat{\varphi} : M(A) \longrightarrow M(B)$ the morphism for

vector spaces defined, for every $\mu\in M(A)$, by
$\hat{\varphi}(\mu)=\mu\varphi=\int\varphi(x,\ )\mu(dx)$. Clearly, $\hat{\varphi}$ is measurable.

LEMMA 1.6. For every $w\leq w'$ in W, let (E_w,σ_w) be a measurable space and $\varphi_{ww'}$ a transition probability from $(E_{w'},\sigma_{w'})$ to (E_w,σ_w). Then $(E_w,\sigma_w,\varphi_{ww'}/W)$ is an approach system of measurable spaces if and only if $(M(E_w),\hat{\varphi}_{ww'}/W)$ is a projective system for vector measurable spaces.

Proof. Obviously, for every $w\leq w'\leq w''$,
$\varphi_{w'w''}\varphi_{ww'}=\varphi_{ww''}$ if and only if $\hat{\varphi}_{ww'}\circ\hat{\varphi}_{w'w''}=\hat{\varphi}_{ww''}$.

THEOREM 1.7. If $(E,\sigma,\Psi_w/W)$ is a limit for the approach system of measurable spaces $\mathcal{E}=(E_w,\sigma_w,\varphi_{ww'}/W)$ and if for every $\mu\in M(E)$, $\neq 0$, there is $w\in W$ such that $\mu\Psi_w\neq 0$, then $(M(E),\Psi_w/W)$ is a limit for the projective sistem $(M(E_w),\hat{\varphi}_{ww'}/W)$.

Proof. Put $M=\{(\mu_w)_W\in\prod_W M(E_w)\ /\ \mu_{w'}\varphi_{ww'}=\mu_w,\ w<w'\}$
and, for every $w\in W$, let π_w be the restriction to M of the w-projection of $\prod_{v\in W}M(E_v)$. We endow M with the σ-algebra generated by $\{\pi_w\ /\ w\in W\}$. The measurable space $(M,\pi_w/W)$ is a limit of the projective system $(M(E_w),\hat{\varphi}_{ww'}/W)$. Let $\pi:M(E)\rightarrow M$ be the function defined by $\pi(\mu)=(\mu\Psi_w)_W$, $\mu\in M(E)$. Then π is the only function such that for every $w\in W$ $\pi\pi_w=\hat{\Psi}_w$. For every $w\in W$ and every σ_w-measurable and bounded function $f:E_w\rightarrow R$, the function $x\mapsto\Psi_w(x,f)$, $x\in E$, is σ-measurable and the function $\mu\mapsto\mu\Psi_w^f$, $\mu\in M(E)$, is measurable, too. Therefore π

is measurable.

Let $P=\{(\mu_w)_W \in M \,/\, \mu_w$ is a probability measure$\}$ and endow P with the σ-algebra of M. We have
$M=\{a\mu-b\nu \,/\, a,b\in R,\ \mu,\nu\in P\}$.

For each $w\in W$ denote by π'_w the transition probability from P to E_w defined by $\pi'_w((\mu_v)_W,\ )=\mu_w$, for every $(\mu_v)_W\in P$. Obviously $(P,\pi'_w/W)$ is a space with approximations for $\mathcal{E}$. Let θ be the representation of P in E. Then $\theta(\mu,\ )$ is a probability measure such that
$\theta(\mu,\ )\varphi_w=\mu_w$, for every $w\in W$ and $\mu=(\mu_w)_W\in P$.
Consequently, π is a surjection and since Ker $\pi=0$ it is an bijection.

The function $\mu\longmapsto\int f(y)\pi^{-1}(\mu)(dy)=\theta(\mu,f)$, $\mu\in M$, is measurable. Therefore π^{-1} is measurable.

2. Topological approach systems

DEFINITION 2.1. Let $(E_w,\tau_w)_W$ be a family of topological spaces and, for every $w\leq w'$, let $\varphi_{ww'}$ be a transition probability from $(E_{w'},\sigma(\tau_{w'}))$ to $(E_w,\sigma(\tau_w))$.

Then $(E_w,\tau_w,\varphi_{ww'}/W)$ is called a topological approach system (TAS) if, for every $w\leq w'\leq w''$ in W

(i) $\varphi_{ww}(x,\ )=\varepsilon_x$, $x\in E_w$;

(ii) $\varphi_{w'w''}\varphi_{ww'}=\varphi_{ww''}$;

(iii) the function $x\longmapsto\varphi_{ww'}(x,\ )$, $x\in E_{w'}$, is weakly continuous (i.e. $\varphi_{ww'}(C_b(E_w))\subset C_b(E_{w'})$).

DEFINITION 2.2. Let $\mathcal{E}=(E_w,\tau_w,\varphi_{ww'}/W)$ be a TAS, let (E,τ) be a topological space and, for every $w\in W$, let

Ψ_w be a transition probability from $(E, \sigma(\tau))$ to $(E_w, \sigma(\tau_w))$. Then $(E, \tau, \Psi_w/W)$ is called a topological space with approximations for $\mathcal{E}$, if

(i) the function $x \longmapsto \Psi_w(x, \ )$, $x \in E$, is weakly continuous

(ii) $\Psi_{w'} \varphi_{ww'} = \Psi_w$, for every $w \leq w'$.

The topological space with approximations $(E, \tau, \Psi_w/W)$ is called the limit of $\mathcal{E}$, if τ is the smallest topology on E such that, for every topological space with approximations for $\mathcal{E}$, $(E', \tau', \Psi'_w/W)$, there exists a unique transition probability Ψ from $(E', \sigma(\tau'))$ to $(E, \sigma(\tau))$ such that

(iii) $x \longmapsto \Psi(x, \ )$, $x \in E'$, is τ'-weakly continuous,

(iv) $\Psi\Psi_w = \Psi'_w$, $w \in W$.

In this case Ψ is called the representation of E' in E.

PROPOSITION 2.3. Let $(E, \tau, \Psi_w/W)$ be a limit of the TAS $\mathcal{E}=(E_w, \tau_w, \varphi_{ww'}/W)$. The set $V=\{\Psi_w(f)/w \in W, f \in C_b(E_w)\}$ is a real vector space, which generates the topology τ. The topology τ is Hausdorff.

Proof. Obviously, V is a vector space and $\tau \supset$ topV. Thus $(E, \text{top}V, \Psi_w/W)$ is a topological space with approximations for $\mathcal{E}$. Let Ψ be the representation of (E, topV) in (E, τ) and let Ψ' be the transition probability from $(E, \sigma(\tau))$ to $(E, \sigma(\text{top}V))$, defined by $\Psi'(x, \ )=\varepsilon_x$, $x \in E$. Then $\Psi'\Psi$ is a representation of (E, τ) in itself. So $\Psi'\Psi(x, \ )=\varepsilon_x$, $x \in E$. We have, for every $x \in E$,

$$\Psi(x, \ )=\int \Psi(y, \ )\Psi'(x, dy)=\varepsilon_x \ .$$ Consequently,

$C_b(E, \tau)=C_b(E, \text{top}V)$ and $\sigma(\text{top}V)=\sigma(\tau)$.

Let $(E', \tau', \Psi'_w/W)$ be a topological space with approximations for $\mathcal{E}$ and let η be the representation of (E', τ') in (E, τ).

Let ζ be a transition probability from $(E', \sigma(\tau'))$ to $(E, \sigma(\tau))$ such that $\zeta(C_b(E, \mathrm{top}V)) \subset C_b(E')$ and $\zeta\Psi_w = \Psi'_w$ for every $w \in W$. Then $\zeta\Psi = \zeta$ is the representation of (E', τ') in (E, τ). We obtain $\zeta = \eta$. Consequently $\tau = \mathrm{top}V$.

If the points x, y are not separated by τ (i.e. $x \in D$ if and only if $y \in D$, for every $D \in \tau$), then x and y are not separated by $\sigma(\tau)$. Consequently $\Psi_w(x, \ ) = \Psi_w(y, \ )$, for every $w \in W$. As in 1.3. we get a contradiction.

REMARK 2.4. If $(E, \tau, \Psi_w/W)$ is a space with approximation for the TAS $\mathcal{E} = (E_w, \tau_w, \varphi_{ww'}/W)$ such that for every space with approximations for $\mathcal{E}$, there exists a unique transition probability wich satisfies 2.2.(iii) and (iv), then $(E, \mathrm{top}V, \Psi_w/W)$ is a limit for $\mathcal{E}$, but it does not follow that $\mathrm{top}V = \tau$, although $\mathrm{top}V$ and τ have the same Borel sets, the same continuous function and define the same weak topology on the space of real valued measures on $(E, \sigma(\tau))$. For an example consider λ and $\mathrm{top}(\lambda, Q)$, where λ is the usual topology on R and Q is the set of rational numbers.

THEOREM 2.5. If $(E, \tau, \Psi_w/W)$ and $(E', \tau', \Psi'_w/W)$ are two limits of the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$, then there is only one homeomorphism $f: E' \to E$ such that $\Psi(x, \ ) = \varepsilon_{f(x)}$, $x \in E'$, is the representation of E' in E. For every $w \in W$ we have $\Psi_w(f(\ ), \ ) = \Psi'_w$.

Proof. Let Ψ be the representation of E' in E and let Ψ' be the representation of E in E'. We have $\Psi\Psi'(x, \ ) = \varepsilon_x$,

$x \in E'$, and $\Psi'\Psi(y, \ ) = \varepsilon_y$, $y \in E$. Because τ and τ' are Hausdorff we obtain that the points are closed, and similarly to 1.5. we find a bijection $f: E' \to E$, such that $\Psi(x, \ ) = \varepsilon_{f(x)}$ and $\Psi_w(f(x), \ ) = \Psi'_w(x, \ )$, $x \in E'$, $w \in W$.

Obviously, $\Psi(C_b(E)) \subset C_b(E')$. Because τ is generated by real functions we obtain that f is continuous. Similarly, f^{-1} is continuous .

PROPOSITION 2.6. Let $\mathcal{E} = (E_w, \tau_w, \varphi_{ww'}/W)$ be a TAS and let W' be cofinal in W.

(i) If $(E, \tau, \Psi_w/W)$ is a limit of $\mathcal{E}$ then $(E, \tau, \Psi_w/W')$ is a limit of $(E_w, \tau_w, \varphi_{ww'}/W')$.

If $(E, \tau, \Psi_w/W')$ is a limit of $(E_w, \tau_w, \varphi_{ww'}/W')$ then $(E, \tau, \Psi_w/W)$ is a limit of $\mathcal{E}$, where $\Psi_w = \Psi_{w'} \varphi_{ww'}$ for every $w \in W - W'$ and $w' \in W'$, $w' \geq w$.

(ii) If $(E, \tau, \Psi_w/W)$ is a limit of $\mathcal{E}$, W' is countable and (E_w, τ_w) is a separable metrizable space for every $w \in W'$, then (E, τ) is a separable metrizable space.

Proof. (i) Obviously.

(ii) For every $w \in W'$ let $\mathcal{F}_w$ be a denumerable set uniformly dense in $C_b(E_w)$. Then $\{\Psi_w(f) / f \in \mathcal{F}_w\}$ is uniformly dense in $\{\Psi_w(f) / f \in C_b(E_w)\}$ and $\mathrm{top}(\Psi_w(f) / f \in \mathcal{F}_w) = \mathrm{top}(\Psi_w(f) / f \in C_b(E_w))$.

We have $\tau = \mathrm{top}(\bigcup_{w \in W'} \mathrm{top}(\Psi_w(f) / f \in \mathcal{F}_w)) = \mathrm{top}(\Psi_w(f) / f \in \mathcal{F}_w, w \in W')$. Consequently, the topology τ is separable and metrizable because it is generated by a countable set of real functions.

If (S, τ) is a topological space denote by $M(S)$, or by

$M(S, \tau)$, the vector space of signed finite measures on $(S, \sigma(\tau))$ endowed with the weak topology.

LEMMA 2.7. Let (S, τ) and (S', τ') be two topological spaces.

(i) If φ is a transition measure from $(S', \sigma(\tau'))$ to $(S, \sigma(\tau))$ such that $\varphi(C_b(S)) \subset C_b(S')$, then the function $\hat{\varphi}: M(S') \to M(S)$ is a continuous morphism for vector spaces, where $\hat{\varphi}$ is defined by $\hat{\varphi}(\mu) = \mu\varphi$ for every $\mu \in M(S)$.

(ii) If (S, τ) and (S', τ') are separable metrizable spaces and $\hat{\varphi}: M(S') \to M(S)$ is a continuous morphism for vector spaces, then there exists a transition measure φ from $(S', \sigma(\tau'))$ to $(S, \sigma(\tau))$ which represents $\hat{\varphi}$ (i.e. $\hat{\varphi}(\mu) = \mu\varphi$, for every $\mu \in M(S')$). In addition, $\varphi(C_b(S)) \subset C_b(S')$. If the morphism $\hat{\varphi}$ is positive (i.e. $\hat{\varphi}(\mu) \geqslant 0$ for every $\mu \geqslant 0$) and unitary (i.e. $\hat{\varphi}(\mu)(S) = \mu(S')$ for every $\mu \geqslant 0$) then φ is a transition probability.

Proof. (i) Obviously.

(ii) Define $\varphi(x, \ ) = \hat{\varphi}(\varepsilon_x)$, $x \in S'$. Because the function $x \mapsto \varepsilon_x$, $x \in S'$, is continuous, it follows that the function $x \mapsto \varphi(x, \ )$ is weakly continuous.

For every $D \in \tau$, there is an increasing sequence $(f_n)_N$ in $C_b(S)$ such that $0 \leq f_n \leq 1$ and $\lim f_n = \chi_D$. It follows that $\tau \subset \{D \in \sigma(\tau) / \varphi(\ , D)$ is $\sigma(\tau')$-measurable$\}$. Therefore, the function $\varphi(\ , A)$ is $\sigma(\tau')$-measurable for every $A \in \sigma(\tau)$.

For every measure $\mu = \sum_{i=1}^{n} a_i \varepsilon_{x_i} \in M(E)$ we have
$$\hat{\varphi}(\mu) = \sum_{i=1}^{n} a_i \varphi(x_i, \ ) = \mu\varphi .$$

For every $\mu \in M(E)$ there is a sequence $(\mu_n)_N$ with $\lim \mu_n = \mu$ and $\mu_n = \sum_{i=1}^{n} a_i^n \varepsilon_{x_i^n}$, $n \in N$.

We have $\hat{\varphi}(\mu) = \lim \hat{\varphi}(\mu_n) = \lim \mu_n \varphi = \mu \varphi$.

If $f \in C_b(E)$ and $(x_n)_N$ converges to x in E, then $\varphi(x,f) = \lim \hat{\varphi}(\varepsilon_{x_n})(f) = \lim \varphi(x_n, f)$. Consequently $\varphi(C_b(E)) \subset C_b(E')$.

REMARK 2.8. If (S,τ), (S',τ') and (S'',τ'') are topological spaces, φ is a transition measure from $(S',\sigma(\tau'))$ to $(S,\sigma(\tau))$ and η is a transition measure from $(S'',\sigma(\tau''))$ to $(S',\sigma(\tau'))$ such that $\varphi(C_b(S)) \subset C_b(S')$ and $\eta(C_b(S')) \subset C_b(S'')$, then $\widehat{\eta\varphi} = \hat{\varphi} \circ \hat{\eta}$.

Conversely, if (S,τ), (S',τ') and (S'',τ'') are separable metrizable spaces, $\hat{\varphi}: M(S') \longrightarrow M(S)$ and $\hat{\eta}: M(S'') \longrightarrow M(S')$ are continuous morphisms of topological vector spaces, then $\hat{\varphi}\,\hat{\eta}$ is represented by $\eta\varphi$.

THEOREM 2.9. (i) If $(E_w, \tau_w, \varphi_{ww'}/W)$ is a TAS, then $(M(E_w), \hat{\varphi}_{ww'}/W)$ is a projective system for topological vector spaces.

(ii) If $(E_w, \tau_w)_W$ is a family of separable metrizable spaces and $(M(E_w), \hat{\varphi}_{ww'}/W)$ is a projective system for topological vector spaces with $\hat{\varphi}_{ww'}$ positive and unitary for every $w < w'$, then $(E_w, \tau_w, \varphi_{ww'}/W)$ is a TAS.

(iii) If (E,τ) and (E_w, τ_w), $w \in W$, are separable metrizabbe spaces, then $(E, \tau, \varphi_w/W)$ is a limit for the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$ if and only if the projective system

$(M(E_w), \hat{\varphi}_{ww'}/W)$ has the limit $(M(E,\tau), \hat{\psi}_w/W)$, where $\hat{\varphi}_{ww'}$ and $\hat{\psi}_w$ are positive and unitary, $w \leq w'$ in W.

Proof. (i) and (ii). Obviously.

(iii) We suppose that $(E, \tau, \psi_w/W)$ is a limit of the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$. Similarly to the proof of Theorem 1.7. we consider the set M and the projections π_w, $w \in W$. We endow M with the topology $top(\pi_w / w \in W)$. Then $(M, \pi_w/W)$ is a limit of the projective system $(M(E_w), \hat{\varphi}_{ww'}/W)$. Define $\pi : M(E) \longrightarrow M$ by $\pi(\mu)=(\mu\psi_w)_{w \in W}$. Clearly, π is a bicontinous isomorphism of topological vector spaces and $\pi_w \pi = \hat{\psi}_w$, $w \in W$.

Conversely, assume $(M(E), \hat{\psi}_w/W)$ is a limit of the projective system of topological vector spaces. $(M(E_w), \hat{\varphi}_{ww'}/W)$.

Let ψ_w be the transition probability which represents $\hat{\psi}_w$, $w \in W$. Obviously, $(E, \tau, \psi_w/W)$ is a space with approximations of the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$.

Let $(E', \tau', \psi'_w/W)$ be a space with approximations of the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$. Because $(M(E), \hat{\psi}_w/W)$ is a limit of $(M(E_w), \hat{\varphi}_{ww'}/W)$ there exists a unique continuous morphism $\hat{\psi}: M(E') \longrightarrow M(E)$ such that $\hat{\psi}_w \hat{\psi} = \hat{\psi}'_w$, $w \in W$. Define $\psi(x, \,) = \hat{\psi}(\varepsilon_x)$, $x \in E'$. Because for every $f \in C_b(E)$ the function $x \longmapsto \psi(x,f)$ is continuous it follows that for every $A \in \sigma(\tau)$ the function $x \longmapsto \psi(x,A)$, $x \in E'$, is measurable. Next, ψ is the unique transition probability from $(E', \sigma(\tau'))$ to $(E, \sigma(\tau))$ such that $\psi(C_b(E)) \subset C_b(E')$ and $\psi' \psi_w = \psi'_w$ for every $w \in W$. Consequently, $(E, \tau, \psi_w/W)$ is a limit of the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$.

The following theorem gives conditions under wich a (closed subset of a) complet separable metric space (S,d) can be the limit of a TAS, whose approximants are closed sets in S.

Theorem 2.10. Let (S,d) be a complete separable metric space and τ_d the topology of the distance d. Let $\mathcal{E}=(E_n, \tau_n, \varphi_{mn}/N)$ be a TAS which satisfies, for every $m\in N$, the following conditions

(i) E_m is a closed subset of S and $\tau_m=\tau_d/E_m$;

(ii) For every convergent sequence $(x_n)_N \in \prod_N E_n$, $(\varphi_{mn}(x_n, \ ))_{n>m}$ is weakly convergent in $M(E_m)$;

(iii) If the sequence $(x_n)_N \in \prod_N E_n$ converges to $x\in E_m$, then $\lim_n \varphi_{mn}(x_n, \ )=\varepsilon_x$;

(iv) There is a convergent series $\sum_N \varepsilon_n$ with positive terms such that, for every $n\in N$,
$\inf_{x\in E_{n+1}} \varphi_{nn+1}(x, B(x, \varepsilon_n)\cap E_n)>1-\varepsilon_n$

Denote

$E=\{x\in S$ / there is a sequence $(x_n)_N \in \prod_N E_n$ convergent to $x\}$

and $\tau=\tau_{d/E}$. Define, for every $m\in N$ and $x\in E$,

$\varphi_m(x, \ )=\lim_n \varphi_{mn}(x_n, \ )$, where the sequence $(x_n)_N\in \prod_N E_n$ converges to x.

Under these assumptions the TAS $\mathcal{E}$ has the limit $(E, \tau, \varphi_n/N)$ and the projective system $(M(E_n), \hat{\varphi}_{mn}/N)$ has the limit $(M(E), \hat{\varphi}_n/N)$.

Proof. If $\mu \in M(E_m)$ denote by $\bar{\mu}$ the extension of μ to $(S, \sigma(\tau_d))$, $\bar{\mu}=\mu(\cdot \cap E_m)$. Similarly, if φ is a transition probability from a measurable space (E', σ') to $(E_m, \sigma(\tau_m))$ denote by $\bar{\varphi}$ the transition probability from (E', σ') to $(S, \sigma(\tau_d))$ defined by $\bar{\varphi}(x, \ )=\varphi(x, \cdot \cap E_m)$, $x \in E'$. Obviously, the sequence $(\mu_n)_N$ of $M(E_m)$ is weakly convergent if and only if $(\bar{\mu}_n)_N$ is weakly convergent in $M(S)$.

Since (S,d) is a complet separable metric space it follows that $(M(S),L)$ is also a complet separable metric space, where L is the Levy's distance (i.e., for every $\mu, \nu \in M(S)$, $L(\mu,\nu)=\max(l(\mu,\nu), l(\nu,\mu))$, where $l(\mu,\nu)=\inf\{\varepsilon / \mu(A) \le \nu(A^{\varepsilon})+\varepsilon$, for any A closed in $S\}$ and $A^{\varepsilon}=\{x \in S / d(x,A) \le \varepsilon\}$. Denote L_m the Levy's distance on $M(E_m)$. Obviously, $L_m(\mu,\nu)=L(\mu,\nu)$, $\mu,\nu \in M(E_m)$.

For every $m<n$ denote $r_{mn}=\sum_{k=m}^{n-1}\varepsilon_k$, $r_m=\sum_{k \ge m}\varepsilon_k$,

$a_{mn}=\prod_{k=m}^{n-1}(1-\varepsilon_k)$ and $a_m=\prod_{k=m}(1-\varepsilon_k)$.

We shall use the following lemmas.

LEMMA 2.11. Let $m<n$. For every $x_n \in E_n$, we have

$\bar{\varphi}_{mn}(x_n, B(x_n, r_{mn})) \ge a_{mn}$.

For every $x \in E$, we have $\bar{\varphi}_m(x, \bar{B}(x, r_m)) \ge a_m$. Consequently, $\sup_{x \in E} d(x,E_m) \le r_m$ and E is closed in S.

LEMMA 2.12. If $m<n$, $\mu \in M(E_m)$, $\nu \in M(E_n)$ such that $\nu\varphi_{mn}=\mu$ and $|\mu| \le a$, $a \in R$, then

$L(\bar{\mu}, \bar{\nu}) \le a \max(r_{mn}, a_{mn}^{-1}-1)$.

LEMMA 2.13. For every bounded uniformly continous function $f: S \longrightarrow R$ the sequence $(\varphi_n(f/E_n))_N$ is uniformly convergent to $f/_E$.

Proof of Lemma 2.11. Induction with respect to n-m. The case n-m=1 result from (ii). We suppose that the lemma is true for n-m-1. Let $x_n \in E_n$. We have $\bar{\varphi}_{mn}(x_n, B(x_n, r_{mn})) \geqslant$

$$\geqslant \int_{B(x_n, r_{m+1n}) \cap E_{m+1}} \bar{\varphi}_{mm+1}(y, B(y, \varepsilon_m))\, \varphi_{m+1n}(x_n, dy) \geqslant$$

$$\geqslant (1-\varepsilon_m)\, \bar{\varphi}_{m+1n}(x_n, B(x_n, r_{m+1n})) \geqslant a_{mn}\,.$$

For every $m \in N$, $\varepsilon > 0$ and for every sequence $(x_n)_N \in \prod_N E_n$ convergent to x, we have $\bar{\varphi}_m(x, B(x, r_m+\varepsilon)) \geqslant$

$$\geqslant \lim \bar{\varphi}_{mn}(x_n, B(x_n, r_{mn})) \geqslant \prod_{k=m} (1-\varepsilon_k) = a_m$$

It follows that $\bar{\varphi}_m(x, \bar{B}(x, r_m)) \geqslant a_m$. Obviously, $\sup\limits_{x \in E} d(x, E_m) \leqslant r_m$

If the sequence $(x_n)_N$ of E converges to $x \in S$, then we find an increasing natural sequence $(n_k)_{k \in N}$ and a sequence $(y_{n_k})_N \in \prod_{k \in N} E_{n_k}$ such that $d(x_k, y_{n_k}) < 2^{-k}$ for every $k \in N$. Obviously, $\lim y_{n_k} = x$. For every $n \in N$ such that $n_{k-1} < n < n_k$, denote y_n an arbitrary element of $B(y_{n_k}, r_{nn_k}) \cap E_n$, wich is not empty. Obviously, $(y_n)_N \in \prod_N E_n$ and $\lim y_n = x$.

Proof of Lemma 2.12, Let $m < n$, $\mu \in M(E_m)$ and $\nu \in M(E_n)$ such that $\nu\varphi_{mn} = \mu$ and $|\mu| \leqslant a$.

Obviously, $l(\bar{\mu}, \bar{\nu}) \leqslant \inf\{\varepsilon / \mu(A) \leqslant \bar{\nu}(A^{\varepsilon}) + \varepsilon\ , A \in \sigma(\tau_m)$

For every $A \in \sigma(\tau_m)$, we have $\mu(A) =$

$$= \int_{A^{r_{mn}} \cap E_n} \varphi_{mn}(y, A)\, \nu(dy) + \int_{C(A^{r_{mn}}) \cap E_n} \varphi_{mn}(y, A)\, \nu(dy) \leqslant$$

$$\leqslant \bar{\nu}(A^{r_{mn}}) + a(1-a_{mn}).$$

Consequently, $l(\bar{\mu}, \bar{\nu}) \leqslant a \max(r_{mn}, 1-a_{mn})$.

For every $B \in \sigma(\tau_n)$, we have

$$\bar{\mu}(B^{r_{mn}}) \geqslant \int_B \bar{\varphi}_{mn}(y, B(y, r_{mn}))\, \nu(dy) \geqslant a_{mn}\, \nu(B).$$

Therefore $\nu(B) \le a_{mn}^{-1}\bar{\mu}(B^{r_{mn}}) \le \bar{\mu}(B^{r_{mn}}) + (a_{mn}^{-1} - 1)a$.

It follows that $l(\bar{\nu}, \bar{\mu}) \le a \max(r_{mn}, a_{mn}^{-1} - 1)$.

Consequently, $L(\bar{\mu}, \bar{\nu}) \le a \max(r_{mn}, a_{mn}^{-1} - 1)$.

Proof of Lemma 2.13. Let $f : S \longrightarrow R$ be a bounded uniformly continuous function. There is a sequence $(\delta_n)_N$ decreasing to 0 such that, for every $n \in N, x, y \in S$, if $d(x,y) \le r_n$, then $|f(x)-f(y)| < \delta_n$.

Let s be a bound of f. For every $x \in E$, we have

$$|f(x)-f_n(x)| \le \int |f(x)-f_{/E_n}(y)|\, \Psi_n(x,dy) \le$$

$$\le \int_{B(x,r_n)\cap E_n} |f(x)-f_{/E_n}(y)|\, \Psi_n(x,dy) + 2s\Psi_n(x, CB(x,r_n)) \le$$

$$\le \delta_n + 2s(1-a_n) .$$

Obviously, $(f_n)_N$ is a sequence uniformly convergent to $f_{/E}$.

We continue the proof of Theorem 2.10. proving that $(E, \mathcal{T}, \Psi_n/N)$ is a topological space with approximations for the TAS $\mathcal{E}$.

Let $m<n$ and $f \in C_b(E_n)$. Let $(x_n)_N \in \prod_N E_n$ be a sequence convergent to x. We have $\Psi_n \varphi_{mn}(x,f) =$

$$= \lim_p \int \varphi_{mn}(y,f)\, \varphi_{np}(x_p, dy) = \lim_p \varphi_{mp}(x_p, f) = \Psi_m(x,f) .$$

Consequently, $\Psi_n \varphi_{mn} = \Psi_m$.

Let $m \in N$. Assuming that $x \longmapsto \Psi_m(x, \ )$, $x \in E$, is not a continuous function. Therefore there are $\varepsilon > 0$ and a sequence $(x_n)_N$ in E, with $\lim x_n = x$ such that

$L_m(\Psi_m(x, \ ), \Psi_m(x_n, \ )) > \varepsilon$. We can find an increasing sequence $(n_k)_N$ of natural integers and $(y_k)_N \in \prod_N E_{n_k}$ such that $\lim d(y_k, x_k) = 0$ and, for every $k \in N$,

$L_m(\Psi_m(x_k, \ ), \varphi_{mn_k}(y_k, \ )) < \varepsilon/2$.

There is a sequence $(x'_n)_N \in \prod_N E_n$ convergent to x such that $x'_{n_k} = x_{n_k}$.

We have $L_n(\Psi_n(x, \ ), \varphi_{mn_k}(x'_{n_k}, \ )) > \varepsilon/2$, wich contradicts (ii). Therefore, the function $x \longmapsto \Psi_n(x, \ )$, $x \in E$, is weakly continuous. It follows that Ψ_n is a transition probability from $(E, \sigma(\tau))$ to $(E_n, \sigma(\tau_m))$. Consequently, $(E, \tau, \Psi_n/N)$ is a space with approximations for $\mathcal{E}$.

We have top $V \subset \tau$, where $V = \{\Psi_n(g) \ / \ n \in N, g \in C_b(E_n)\}$.

Conversly, using Lemma 2.13., it follows that

top $V \supset \tau = \text{top}\{f_{/E} \mid f : S \longrightarrow R$, f bounded, uniformly continuous $\}$.

Let $(E', \tau', \Psi'_n/N)$ be a space with approximations for the TAS $\mathcal{E}$. Using Lemma 2.12. we obtain that, for every $x' \in E'$, the sequence $(\Psi'_n(x', \ ))_N$ is L-Cauchy. Define $\Psi(x', \ ) = (\lim_n \Psi'_n(x', \ ))/\sigma(\tau)$, $x' \in E'$.

Let $n \in N$ and $x' \in E'$. For every $f \in C_b(E_n)$ we have

$$\Psi\Psi_n(x', f) = \lim_m \int \Psi_n(y, f)\, \Psi'_m(x', dy) =$$

$$= \lim_m \int \int_{E_n} \varphi_{mn}(z, f)\, \Psi_n(y, dz)\, \Psi'_m(x', dy) =$$

$$= \lim_m \int \varphi_{mn}(y, f)\, \Psi'_m(x', dy) = \Psi'_n(x', f).$$ (We used the assumption (iii)). Consequently, $\Psi\Psi_n = \Psi'_n$.

Let $f : S \longrightarrow R$ uniformly continuous and bounded. Let $n \in N$ and $f_n = f_{/E_n}$. We have $\Psi\Psi_n(f_n) = \Psi'_n(f_n) \in C_b(E')$.

$$\sup_{x' \in E'} |\Psi(x', f_{/E}) - \Psi(x', \Psi_n(f_n))| \le \sup_{x' \in E'} \Psi(x', |f_{/E} - \Psi_n(f_n)|) \xrightarrow[n \to \infty]{} 0$$

Consequently, the function $x' \mapsto \Psi(x', f_{/E})$, $x' \in E'$, is τ'-continuous. It follows that $x' \longmapsto \Psi(x', \ )$, $x' \in E'$, is

τ'-weak-continuous. The uniqueness of Ψ follows from Lemma 2.13. (because Ψ is determined by V, it follows that Ψ is determined by the set $\{f_{/E} \mid f : S \longrightarrow R,$ f bounded, uniformly continuous $\}$. Therefore $(E, \tau, \Psi_n/N)$ is a limit of the TAS $\mathcal{E}$.

From Theorem 2.9., we obtain that $(M(E), \hat{\Psi}_n/N)$ is a limit of the projective system $(M(E_n), \hat{\varphi}_{mn}/N)$.

REMARK 2.14. We can replace the assumption (iii) of Theorem 2.10. by the following assumption: $(E_n)_N$ is increasing and $\varphi_{nn+1}(x, \ ) = \varepsilon_x$, for every $n \in N$ and $x \in E_n$.

A very special case is that where the approximants are discrete.

DEFINITION 2.15. A TAS $(E_w, \tau_w, \varphi_{ww'}/W)$, where τ_w is the discrete topology for every $w \in W$, is called a discrete approach system (DAS).

The following theorem shows that the real line R endowed with the usual topology λ is the limit of a DAS.

THEOREM 2.16. For every $m \leq n$, denote $D_m = \{k2^{-m}/k \in Z\}$ and $\lambda_{mn}(x, \ ) = (1 - \langle 2^m x \rangle)\varepsilon_{[2^m x]2^{-m}} + \langle 2^m x \rangle \varepsilon_{[2^m x + 1]2^{-m}}$,

$x \in D_n$, where $[a]$ and $\langle a \rangle$ are the integer part and the fractional part of the real number a. The DAS $(D_n, \lambda_{mn}/N)$ has the limit $(R, \lambda, \lambda_n/N)$, where

$$\lambda_n(x, \ ) = (1 - \langle 2^n x \rangle)\varepsilon_{[2^n x]2^{-n}} + \langle 2^n x \rangle \varepsilon_{[2^n x + 1]2^{-n}}, \quad x \in R.$$

In addition, $(M(R), \hat{\lambda}_n/N)$ is a limit for the projective system $(M(D_n), \hat{\lambda}_{mn}/N)$.

Proof. It is easy to verify all the hypothesis of Theorem 2.10. with $\varepsilon_n = 2^{-n}$, $n \in N$. It remains to show, that

$(D_n, \lambda_{mn}/N)$ is a DAS. It is enough to show that $\lambda_{nn+1}\lambda_{mn} = \lambda_{mn+1}$, for every $m \leq n$.

Let $x \in E_{n+1}$. Think that x is represented in base 2. We have $\lambda_{nn+1}\lambda_{mn}(x, \ ) =$

$$= (1 - \langle 2^n x \rangle)\lambda_{mn}([2^n x]2^{-n}, \ ) + \langle 2^n x \rangle \lambda_{mn}([2^n x + 1]2^{-n}, \ )$$

If $\langle 2^n x \rangle = 0$ then $\lambda_{nn+1}(x, \ ) = \lambda_{mn}([2^n x]2^{-n}, \ ) = \lambda_{mn+1}(x, \ )$, because in this case $[2^n x] = 2^n x$.

If $\langle 2^n x \rangle = 2^{-1}$ then $\lambda_{nn+1}\lambda_{mn}(x, \ ) =$

$$= 2^{-1}\lambda_{mn}([2^n x]2^{-n}, \ ) + 2^{-1}\lambda_{mn}([2^n x + 1]2^{-n}, \ ) =$$

$$= 2^{-1}\{(1 - \langle 2^m x \rangle + 2^{m-n-1})\varepsilon_{[2^m x]2^{-m}} + (\langle 2^m x \rangle - 2^{m-n-1})\varepsilon_{[2^m x + 1]2^{-m}} +$$

$$+ (1 - \langle 2^m x + 2^{m-n-1} \rangle)\varepsilon_{[2^m x + 2^{m-n-1}]2^{-m}} +$$

$$+ \langle 2^m x + 2^{m-n-1} \rangle \varepsilon_{[2^m x + 2^{m-n-1} + 1]2^{-m}},$$

beacause in this case $[2^{m-n}[2^n x]] = [2^m x]$, $\langle 2^{m-n}[2^n x] \rangle = \langle 2^m x \rangle - 2^{m-n-1}$ and $[2^n x + 1] = 2^n x + 2^{-1}$.

There are two possibilities: $[2^m x + 2^{m-n-1}] = [2^m x]$ or $2^m x + 2^{m-n-1} = [2^m x + 1]$. We obtain in every case that $\lambda_{nn+1}\lambda_{mn}(x, \ ) = \lambda_{mn+1}(x, \ )$.

THEOREM 2.17. For every $m \leq n$ denote $K_m = (D_m)^m$ and $\pi_{mn}((x_0, x_1, \ldots, x_n), \ ) = \lambda_{mn}(x_0, \ ) \otimes \ldots \otimes \lambda_{mn}(x_m, \ )$, $(x_0, \ldots, x_n)$ K_n. Then $(K_n, \pi_{mn}/N)$ is a DAS and has the limit $(R^N, \mathcal{X}^N, \pi_n/N)$, where $\mathcal{X}^N$ is the product of the usual topologies and $\pi_n((x_i)_N, \ ) = \lambda_n(x_0, \ ) \otimes \ldots \otimes \lambda_n(x_n, \ )$ for every $(x_i)_N \in R^N$ (λ_{mn} and λ_n are defined in Theorem 2.16.).

Proof. Obvious.

DEFINITION 2.18. If $(E_w, \tau_w, \varphi_{ww'}/W)$ and $(E'_w, \tau'_w, \varphi_{ww'}/W)$

are two TAS's, such that $E'_w \in \sigma(\tau_w)$, $\tau'_w = \tau_{w/E'_w}$ and

$\varphi'_{ww'} = \varphi_{ww'/E'_{w'}} \times \sigma(\tau'_w)$ for every $w \leq w'$, then

$(E'_w, \tau'_w, \varphi'_{ww'}/W)$ is called a subsystem of the TAS

$(E_w, \tau_w, \varphi_{ww'}/W)$.

THEOREM 2.19. Let $\mathcal{E}=(E_w, \tau_w, \varphi_{ww'}/W)$ be a TAS and let W' be a countable cofinal set in W. Suppose that for every $w \in W'$, (E_w, τ_w) is a separable metrizable space. Let $(E, \tau, \psi_w/W)$ be a limit of the TAS $\mathcal{E}$ and let $\mathcal{E}'=(E'_w, \tau'_w, \varphi'_{ww'}/W)$ be a subsystem of $\mathcal{E}$. Suppose that E'_w is closed for every $w \in W'$. Denote $E' = \{x \in E \,/\, \psi_w(x, E'_w) = 1,\ w \in W\}$, $\tau' = \tau_{/E'}$ and $\psi'_w = \psi_{w/E'} \times \sigma(\tau'_w)$, $w \in W$. If $E' \neq \emptyset$ then $(E', \tau', \psi'_w/W)$ is a limit of the subsystem $\mathcal{E}'$, and $(M(E'), \hat{\psi}'_w/W)$ is a limit of the projective system $(M(E'_w), \hat{\varphi}'_{ww'}/W)$.

Proof. We can suppose that $W=W'$ is countable. For every $w \in W$, and $f' \in C_b(E'_w)$ there is $f \in C_b(E_w)$ with $f_{/E'} = f'$. We obtain that $\psi'_w(f') = \psi_w(f)_{/E'} \in C_b(E')$. Consequently, $(E', \tau', \psi'_w/W)$ is a space with approximations for the TAS $\mathcal{E}'$.

To prove that E' is closed in (E, τ), we choose a family of functions $(f^n_w)_{\substack{w \in W \\ n \in N}}$ such that $f^n_w \in C_b(E_w)$, $0 \leq f^n_w \leq 1$, $f^n_{w/E'_w} = 1$ and $f^n_{w/CB(E'_w, n^{-1})} = 0$, for every $w \in W$, $n \in N$. We have $E' = \bigcap_{\substack{w \in W \\ n \in N}} (\psi_w(f^n_w))^{-1}(\{1\})$.

Let $(E'', \tau'', \psi''_w/W)$ be a space with approximations for

$\mathcal{E}'$. For every $w\in W$ define $\eta_w(x,A) = \Psi''_w(x,A\cap E')$, $x\in E''$, $A\in\sigma(\tau_w)$. Clearly, η_w is a transition probability from $(E'',\sigma(\tau''))$ to $(E,\sigma(\tau))$. Obviously, $(E'',\tau'',\eta_w/W)$ is a space with approximations for $\mathcal{E}$.

Let η be the representation of E'' in $(E,\tau,\Psi_w/W)$. For every $w\in W$ denote $A_w = \{x\in E / \Psi_w(x,E'_w) = 1\}$. For $w<w'$, $x\in A_{w'}$, we have $\Psi_w(x,E'_w) = \int_{E'_{w'}} \varphi_{ww'}(y,E'_w)\Psi_{w'}(x,dy) =$

$= \Psi_{w'}(x,E'_{w'}) = 1$. Consequently, $A_{w'}\subset A_w$ and $E' = \bigcap_{w\in W} A_w$.

Let $x''\in E''$. For every $w\in W$ we have $1 = \Psi''_w(x'',E'_w) =$

$= \int \Psi_w(y,E'_w)\eta(x'',dy) = \eta(x'',A_w) + \int_{CA_w}\Psi_w(y,E'_w)\eta(x'',dy)$.

Therefore $\eta(x'',CA_w) = \int_{CA_w}\Psi_w(y,E'_w)\eta(x'',dy)$. Because

$\Psi_w(y,E'_w)<1$ for every $y\in CA_w$ we obtain $\eta(x'',CA_w)=0$.

Let $(w_n)_N$ be a serially ordered cofinal set in W ($w_m<w_n$ for every $m<n$). Then $\eta(x'',CE') = \lim \eta(x'',CA_{w_n}) = 0$.

Therefore $\Psi = \eta_{/E''\times\sigma(\tau')}$ is a transition probability from $(E'',\sigma(\tau''))$ to $(E',\sigma(\tau'))$. For every $f'\in C_b(E')$ there is $f\in C_b(E)$ such that $f_{/E'} = f'$ because (E,τ) is a metrizable space. Consequently, $\Psi(C_b(E'))\subset C_b(E'')$. Obviously, $\Psi\Psi'_w = \Psi''_w$, $w\in W$, and the uniqueness of Ψ follows from the uniqueness of η.

THEOREM 2.20. For every locally compact, separable and metrizable (l.c.s.m.) space (S,τ) there are a DAS $(S_n, \varphi_{mn}/N)$ with S_n denumerable, $n\in N$, and a family of transition probabilities $(\eta_n)_N$ such that $(S,\tau,\eta_n/N)$ is a limit of $(S_n,\varphi_{mn}/N)$. Also, $(M(S),\hat{\eta}_n/N)$ is a limit for the projective system $(M(S_n), \hat{\varphi}_{mn}/N)$.

Proof. We need the following lemma

LEMMA 2.21. Every l.c.s.m. space is homeomorphic to a closed set included in (R^N, λ^N).

Proof. Let S be an l.c.s.m. space, $\tilde{S}$ the Alexandroff's compactification and $\tilde{S}-S=\{c\}$. The space $\tilde{S}$ is metrizable. Consequently, there are a compact K included in Hilbert's cube $[1,2]^N$ endowed with the topology $\lambda^N/[1,2]^N$, and a homeomorphism $f: S \longrightarrow K$. Let $f(c)=c'$. Consider in (R^N, λ^N) the distance $d(x,y)=\sum_N 2^{-n}|x_n-y_n|(1+|x_n-y_n|)^{-1}$, where $x=(x_n)_N$, $y=(y_n)_N$. Define $g: K-\{c'\} \longrightarrow R^N$ by

$g(x)=(d(x,c'), x_n d(x,c')^{-1})_{n\in N}$, $x=(x_n)_N \in K-\{c'\}$.

Obviously, g is a homeomorphism between $K-\{c'\}$ and $Im(g)$; $Im(g)$ is closed in R^N. Consequently, $g\circ f$ is a homeomorphism between S and $Im(g)$.

We continue the proof of Theorem 2.20.

Let S' be a closed in (R^N, λ^N) such that (S, τ) is homeomorphic with S'. Let $(K_n, \pi_{mn}/N)$ be the DAS defined in Theorem 2.17. For every $n\in N$, let

$S_n=\{x\in K_n$ / there is $y \in S'$ such that $\pi_n(y,\{x\}) > 0\}$,

For every $m<n$, denote

$\varphi_{mn} = \pi_{mn}/S_n \times \mathcal{P}(S_n)$; $\psi_m = \pi_m/S' \times \mathcal{P}(S_m)$.

Let $m<n$, $x_n \in S_n$. There is $x \in S'$ such that $\psi_n(x,\{x_n\}) > 0$. We have

$$1 = \psi_m(x,S_m) = \sum_{y\in S_n} \psi_n(x,\{y\})\, \varphi_{mn}(y,S_m).$$

We obtain that $\varphi_{mn}(y,S_m)=1$ for every $y \in S_n$. Consequently,

φ_{mn} is a transition probability from S_n to S_m. Obviously $\varphi_{np}\varphi_{mn}=\varphi_{mp}$ for every $m<n<p$. It follows that $(S_n,\varphi_{mn}/N)$ is a subsystem of the DAS $(K_n,\pi_{mn}/N)$.

To prove that $(S',\tau',\psi_n/N)$ is a limit of the DAS $(S_n,\varphi_{mn}/N)$, we shall apply Theorem 2.20. Let $A_n=\{x\in R^N \,/\, \psi_n(x,S_n)=1\}$, $E=\bigcap_{n\in N} A_n$. For every $x\notin S'$, there is $k\in N$ such that $d(x,S')>2^{-k}$. We obtain that $\pi_{k+1}(x,S_{k+1})<1$, therefore $x\notin E$. Consequently, $E=S'$.

With the homeomorphism $f: S \longrightarrow S'$ define $\eta_n(x,\ )=\psi_n(f(x),\ )$, $n\in N$, $x\in S$. Obviously, $(S,\tau,\eta_n/N)$ is a limit of the DAS $(S_n,\varphi_{mn}/N)$.

If $(E_w,\tau_w,\varphi_{ww'}/W)$ is a TAS, then $(M,\pi_w/W)$ is a limit of the projective system $(M(E_w),\hat{\varphi}_{ww'}/W)$ (se the proof of Theorem 2.9.). Denote

$P=\{(\mu_w)_W\in M \,/\, \mu_w$ is a probability measure $\}$.

Endow P with the restriction of the topology of M.

LEMMA 2.22. If $(E_w,\tau_w,\varphi_{ww'}/W)$ is a TAS and (E_w,τ_w) is a Hausdorff compact space for every $w\in W$, then P is a compact simplex of M.

Proof. Because $P(E_w)=\{\mu\in M(E_w) \,/\, \mu$ probability measure$\}$ is a compact space for every $w\in W$, it follows that $\prod_W P(E_w)$ is compact as well. For every $w<w'$ and $f\in C(E_w)$ define the function $g_{ww'f}:\prod_W P(E_w)\longrightarrow R$ by

$g_{ww'f}((\mu_w)_W)=\mu_w(f)-\mu_{w'}\varphi_{ww'}(f)$. Then $g_{ww'f}$ is continuous and $P=\bigcap_{w<w'}\ \bigcap_{f\in C(E_w)}(g_{ww'f})^{-1}(\{0\})$ is closed.

Obviously, P is a base of the convex cone
$M_+=\{(\mu_w)_W\in M \,/\, \mu_w>0\}$. and $M=M_+-M_+$.

Let $\mu=(\mu_w)_W$, $\nu=(\nu_w)_W$ in M and $w<w'<w''$. Then

$$\mu_w\vee\nu_w=(\mu_{w'}\varphi_{ww'})\vee(\nu_{w'}\varphi_{ww'})\leq(\mu_{w'}\vee\nu_{w'})\varphi_{ww'}\leq$$

$$\leq(\mu_{w''}\vee\nu_{w''})\varphi_{w'w''}\varphi_{ww'}=(\mu_{w''}\vee\nu_{w''})\varphi_{ww''}.$$

For every $w\in W$ define $\eta_w=\lim\limits_{w'>w}(\mu_{w'}\vee\nu_{w'})\varphi_{ww'}$.

This limit exists because $((\mu_{w'}\vee\nu_{w'})\varphi_{ww'})_{w'>w}$ is an bounded increasing family. Obviously, $\eta=(\eta_w)_W\in M$, $\mu\vee\nu\leq\eta$. Suppose that $\eta'=(\eta'_w)_W\in M$ such that $\mu\vee\nu\leq\eta'$. We have

$(\mu_{w'}\vee\nu_{w'})\varphi_{ww'}\leq\eta'_{w'}\varphi_{ww'}=\eta'_w$ for every $w<w'$.

Consequently, $\eta\leq\eta'$ and $\mu\vee\nu=\eta$.

THEOREM 2.23. Let $(E_w, \tau_w, \varphi_{ww'}/W)$ be a TAS. Suppose that there is a countable cofinal subset W' in W such that (E_w, τ_w) is a metrizable compact space for every $w\in W'$. Denote by E the set of extreme points of P and let τ be the restriction of the topology of P . If $P\neq\emptyset$ then the approach system of measurable spaces $(E_w, \sigma(\tau_w), \varphi_{ww'}/W)$ has the limit $(E,\sigma(\tau),\psi_w/W)$, where $\psi_w((\mu_v)_W)=\mu_w$ for every $w\in W$, $(\mu_v)_W\in E$.

If, in addition, E is closed in P, then $(E,\tau,\psi_w/W)$ is a limit of the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$ and $(M(E), \hat{\psi}_w/W)$ is a limit of the projective system $(M(E_w), \hat{\varphi}_{ww'}/W)$.

Proof. For all the theorems in Choquet's theory which we use here, see [7], sec. 9.

Obviously, $(E,\tau,\psi_w/W)$ is a space with approximations for the TAS $(E_w, \tau_w, \varphi_{ww'}/W)$ and $(E, \sigma(\tau), \psi_w/W)$ is a space with approximations for the approach system of measurable spaces $(E_W, \sigma(\mathcal{T}_W), \varphi_{WW'}/W)$.

Let $(E', \sigma', \psi'_W/W)$ be a spaces with approximations for $(E_W, \sigma(\mathcal{T}_W), \varphi_{WW'}/W)$. Since for every $x' \in E'$ we have, $(\psi'_W(x', ~))_W \in$ $\in P$, there is a probability measure $\eta_{x'}$ on $(E, \sigma(\mathcal{T}))$ such that $x' = \int y\, \eta_{x'}(dy)$. Define $\psi(x', ~) = \eta_{x'}$. The function $x' \mapsto (\psi'_W(x', ~))_{W \in W}$, $x' \in E'$, is σ'-measurable and the function $\mu \longmapsto \eta_\mu$, $\mu \in P$, is $\sigma(\mathcal{T})$-measurable. Consequently, because $x' \mapsto \eta_{x'}$ is $\sigma' - \sigma(\mathcal{T})$-measurable, ψ is a transition probability from (E', σ') to $(E, \sigma(\mathcal{T}))$.

By the uniqueness of $\eta_{x'}$ it follows that ψ is the representation of (E', σ') in $(E, \sigma(\mathcal{T}))$.

If E is closed and $(E', \mathcal{T}', \psi'_W/W)$ is a space with approximations for $(E_W, \mathcal{T}_W, \varphi_{WW'}/W)$ the function $\mu \longmapsto \eta_\mu$, $\mu \in P$, is $\mathcal{T}$-continuous. Consequently, the function $x' \longmapsto \psi(x', ~)$, $x' \in E'$, is $\mathcal{T}'$-continuous and $(E, \mathcal{T}, \psi_W/W)$ is a limit of the TAS $(E_W, \mathcal{T}_W, \varphi_{WW'}/W)$.

If $(E, \mathcal{T})$ is a topological space denote by $(\tilde{E}, \tilde{\mathcal{T}})$ the Alexandroff's compactification of $(E, \mathcal{T})$, $\tilde{E} - E = \{c\}$. Obviously, $(\tilde{E}, \tilde{\mathcal{T}})$ is metrizable if and only if $(E, \mathcal{T})$ is a l.c.s.m. space. Denote $C_0(E) = \{f \in C(E) \;/\; \lim_{x \to c} f(x) = 0\}$.

LEMMA 2.24. Let $(E, \mathcal{T})$ and $(E', \mathcal{T}')$ be topological spaces and let φ be a transition measure from $(E', \sigma(\mathcal{T}'))$ to $(E, \sigma(\mathcal{T}))$, $0 \le \varphi \le 1$. Denote by $\tilde{\varphi}$ the transition probability from $(\tilde{E}', \sigma(\tilde{\mathcal{T}}'))$ to $(\tilde{E}, \sigma(\tilde{\mathcal{T}}))$ defined by

$$\tilde{\varphi}(x,A) = \chi_{E'}(x)\left[\varphi(x, A \cap E) + \varepsilon_c(A)(1 - \varphi(x,E))\right] + \chi_{\{c'\}}(x)\, \varepsilon_c .$$

The following are equivalent:

(i) $\varphi(C_0(E)) \subset C_0(E')$,

(ii) $\varphi(C(\tilde{E})) \subset C(\tilde{E}')$,

. . . .

If, in addition, (E,τ) is a l.c.s.m. space, then the following are equivalent:

(iii) $\varphi(C_b(E)) \subset C_b(E')$;

(iv) For every $\varepsilon > 0$, $x \in E'$, there are a compact set K in E and a neighbourhood U of x such that $\varphi(y, CK) < \varepsilon$ for every $y \in U$.

We have also that (i) implies (iii).

Proof. (i)$\Leftrightarrow$(ii). Suppose $\varphi(C_o(E)) \subset C_o(E')$. Let $\tilde{f} \in C(\tilde{E})$. We have $\tilde{\varphi}(\tilde{f}) = \varphi(\tilde{f}_{/E} - \tilde{f}(c)) + \tilde{f}(c)$. Because $\tilde{f}_{/E} - \tilde{f}(c) \in C_o(E)$ and $\tilde{\varphi}(\tilde{f})(c') = \tilde{f}(c)$ we obtain $\tilde{\varphi}(\tilde{f}) \in C(\tilde{E}')$.

Conversely, if $\varphi(C(\tilde{E})) \subset C(\tilde{E}')$, $\tilde{f} \in C(\tilde{E})$ and $\tilde{f}(c) = 0$ then $\varphi(f) = \tilde{\varphi}(\tilde{f})_{/E'}$, and $\lim\limits_{x \to c'} \tilde{\varphi}(x, \tilde{f}) = \lim\limits_{x \to c'} \varphi(x, f) = 0$.

We obtain $\varphi(f) \in C_o(E')$.

(i)$\Rightarrow$(iii). Assume, for a contradiction, that there are $x \in E'$, $a > 0$, a generalized sequence $(x_i)_I$ convergent to x, a sequence of compacts $(K_n)_N$ with $\bigcup\limits_N K_n = E$ and $K_n \subset \overset{\circ}{K}_{n+1}$, $n \in N$, such that $\varphi(x_i, CK_n) > a$ for every $i > i_n$. There are $m \in N$ and $f \in C_o(E)$ such that $\varphi(x, CK_m) < a/2$, $f_{/K_m} = 1$, $f_{/CK_{m+1}} = 0$, $0 \leq f \leq 1$.

For every $n > m$ and $i > i_n$, $\varphi(x_i, f) \leq$

$$\leq \varphi(x_i, K_{m+1}) \leq 1 - a < \varphi(x, K_m) < \varphi(x, f).$$

Consequently, $\varphi(x, f) - \varphi(x_i, f) > a/2$ and $\varphi(f) \notin C_o(E')$.

(iv)$\Rightarrow$(iii). Let $f \in C_b(E)$, $0 \leq f \leq 1$, $x \in E'$ and $\varepsilon > 0$. There are a compact K and a neighbourhood U of x such that $\varphi(y, CK) < \varepsilon/4$ for every $y \in U$. There is $f' \in C_o(E)$ such that $0 \leq f' \leq 1$ and $f'_{/K} = f_{/K}$. Let $y \in U$. We have

$|\varphi(x,f)-\varphi(y,f)| \leq 2\varphi(y,CK)+2\varphi(x,CK)+|\varphi(y,f')-\varphi(x,f')| \leq$

$\leq |\varphi(x,f')-\varphi(y,f')| + \varepsilon$. Consequently, $\varphi(f) \in C_b(E)$.

(iii) $\Rightarrow$ (iv). Let $x \in E'$, $\varepsilon > 0$. There is a compact set K' of E such that $\varphi(x,CK') < \varepsilon/2$. There are a compact set K and $f \in C_b(E)$ such that $K \supset K'$, $f_{/K'}=1$, $f_{/CK}=0$, $0 \leq f \leq 1$.

There is a neighbourhood U of x such that

$|\varphi(x,E)-\varphi(y,E)| < \varepsilon/4$ and $|\varphi(x,f)-\varphi(y,f)| < \varepsilon/4$, for every $y \in U$. We obtain that

$\varphi(y,CK) \leq \varphi(y,E)-\varphi(y,f) \leq \varphi(x,E)-\varphi(x,f)+\varepsilon/2 \leq \varphi(x,CK')+\varepsilon/2 \leq \varepsilon$

for every $y \in U$.

THEOREM 2.25. Let $\mathcal{E}=(E_w, \tau_w, \varphi_{ww'}/W)$ be a TAS such that $\varphi_{ww'}(C_o(E_w)) \subset C_o(E_{w'})$ for every $w<w'$. Then

(i) $\tilde{\mathcal{E}}=(\tilde{E}_w, \tilde{\tau}_w, \tilde{\varphi}_{ww'}/W)$ is a TAS;

(ii) If $(\tilde{E}, \tilde{\tau}, \tilde{\varphi}_w/W)$ is a limit of the TAS $\tilde{\mathcal{E}}$ there exists a unique point $c \in \tilde{E}$ such that $\tilde{\varphi}_w(c, \;)=\varepsilon_{c_w}$ for every $w \in W$.

(iii) If $(\tilde{E}, \tilde{\tau})$ and $(\tilde{E}_w, \tilde{\tau}_w)$, $w \in W$, are metric compact spaces and $(\tilde{E}, \tilde{\tau}, \tilde{\varphi}_w/W)$ is a limit of the TAS $\tilde{\mathcal{E}}$, then the TAS $\mathcal{E}$ has the limit $(E, \tau, \varphi_w/W)$, where $E=\tilde{E}-\{c\}$, $\tau=\tilde{\tau}_{/E}$ and $\varphi_w=\tilde{\varphi}_{w/E\times\sigma(\tau_w)}$ for every $w \in W$.

Proof. (i) Obviously, for every $w<w'<w''$, $\tilde{\varphi}_{w'w''}\tilde{\varphi}_{ww'}=\tilde{\varphi}_{ww''}$ and $\tilde{\varphi}_{ww'}(C(\tilde{E}_w)) \subset C(\tilde{E}_{w'})$. (See Lemma 2.24.)

(ii) $(\{1\}, \mathcal{P}(\{1\}), \varepsilon_{c_w}/W)$ is a space with approximations for $\tilde{\mathcal{E}}$ and let ψ be its representation in $\tilde{E}$.

Denote $A_w=\{x \in \tilde{E} \,/\, \tilde{\varphi}_w(x,\{c_w\})=1\}$, $w \in W$. Let $w'>w$. For every $x \in \tilde{E}$, we have $\tilde{\varphi}_w(x,\{c_w\}) =$

$=\int \tilde{\varphi}_{ww'}(y,\{c_w\})\,\tilde{\varphi}_{w'}(x,dy) = \tilde{\varphi}_{ww'}(c_{w'},\{c_w\})\,\tilde{\varphi}_{w'}(x,\{c_{w'}\}) =$

$= \tilde{\Psi}_{w'}(x,\{c_{w'}\})$. It follows that $A_w = A_{w'}$ for every $w, w' \in W$. Denote $A = A_w$. Because $\Psi\tilde{\Psi}_w = \varepsilon_{c_w}$, it follows that $A \neq \emptyset$. If there are $x_1, x_2 \in A$, $x_1 \neq x_2$, then $\Psi_1(1, \ ) = \varepsilon_{x_1}$ and $\Psi_2(1, \ ) = \varepsilon_{x_2}$ are two representations of $\{1\}$ in E. Consequently, A has only one element, $A = \{c\}$.

(iii) For every $w \in W$ and $f \in C_b(E_w)$ we have $\Psi_w(f) \in C_b(E)$, in accordance with Lemma 2.24. Therefore $(E, \tau, \Psi_w/W)$ is a space with approximations for the TAS $\mathcal{E}$. Let $(E', \tau', \Psi'_w/W)$ be another space with approximations for $\mathcal{E}$. For every $w \in W$, $x \in E'$ and $A \in \sigma(\tilde{\tau}_w)$ define $\tilde{\Psi}'_w(x,A) = \Psi'_w(x, A \cap E_w)$. Then $(E', \tau', \tilde{\Psi}'_w/W)$ is a space with approximations for $(\tilde{E}_w, \tilde{\tau}_w, \tilde{\Psi}_{ww'}/W)$. Let Ψ' be the representation of $(E', \tau', \tilde{\Psi}'_w/W)$ in $(\tilde{E}, \tilde{\tau}, \tilde{\Psi}_w/W)$.

For every $x \in E'$ we have $\tilde{\Psi}'(x,\{c\}) \leq \Psi'\tilde{\Psi}_w(x, c_w) = \tilde{\Psi}'(x,\{c_w\}) = 0$. Consequently, $\Psi'(x,\{c\}) = 0$ for every $x \in E'$.

Define $\Psi = \Psi'_{/E' \times \sigma(\tau)}$. Obviously, $\Psi\Psi_w = \Psi'_w$, $w \in W$. We prove that $\Psi(C_b(E)) \subset C_b(E')$ using Lemma 2.24. Let $\varepsilon > 0$, $x \in E'$. There are compact sets K and K', $K, K' \subset E$, such that $K \subset \mathring{K}'$ and $\Psi(x,K) > 1 - \varepsilon/2$. There is $f \in C(\hat{E})$, $0 \leq f \leq 1$, with $f_{/K} = 1$ and $f_{/CK'} = 0$. We have $\Psi(f_{/E}) = \Psi'(f) \in C_b(E')$. Therefore there is a neighbourhood V of x such that $\Psi(y,f) > 1 - \varepsilon$ for every $y \in V$. Consequently, $\Psi(y, CK') < \varepsilon$ for every $y \in V$.

We need the product of TASes because we shall study stochastic processes defined on the space of its trajectory.

Let I be a set of indices. If $(E_i, \sigma_i)_I$ and $(E'_i, \sigma'_i)_I$ are families of measurable spaces and if η_i is a transition probability for every $i \in I$, $\eta_i : E'_i \times \sigma_i \to [0,1]$, then denote by $\underset{I}{\otimes}\eta_i$ the transition probability from $(\prod_I E'_i, \underset{I}{\otimes}\sigma'_i)$ to $(\prod_I E_i, \underset{I}{\otimes}\sigma_i)$ defined by

$\underset{I}{\otimes}\eta_i((x_i)_I, \ ) = \underset{I}{\otimes}\eta_i(x_i, \ )$ for every $(x_i)_I \in \prod_I E'_i$.

For every family of TASes $(E_{iw}, \tau_{iw}, \varphi_{iww'}/W_i)_I$ denote $U = \bigcup_{J \subset I} \{(i, w_i)_J / w_i \in W_i$ for every $i \in J\}$. Endow U with the directed order relation $\leq$ defined by $(i, w_i)_J \leq (i, w'_i)_{J'}$ if $J \subset J'$ and $w_i \leq w'_i$ for every $i \in J$. For every $u = (i, w_i)_J \in U$ denote $E_u = \prod_J E_{iw_i}$, $\tau_u = \underset{J}{\otimes}\tau_{iw_i}$. For every $u = (i, w_i)_J$ and $u' = (i, w'_i)_{J'}, u \leq u'$, denote $\varphi_{uu'}$ the transition probability from $(E_{u'}, \sigma(\tau_{u'}))$ to $(E_u, \sigma(\tau_u))$ defined by $\varphi_{uu'}((x_i)_{J'}, \ ) = \underset{J}{\otimes}\varphi_{iw_iw'_i}(x_i, \ )$, $(x_i)_{J'} \in E_{u'}$.

DEFINITION 2.26. If $(E_u, \tau_u, \varphi_{uu'}/U)$ is a TAS then it is called the product of family of TASes $(E_{iw}, \tau_{iw}, \varphi_{iww'}/W_i)_I$

Generally, $(E_u, \tau_u, \varphi_{uu'}/U)$ is not a TAS.

LEMMA 2.27. Let the complete separable metric space (E, d). If $\mu, \nu \in M(E)$ and $f \in C_b(E)$ then $|\mu(f) - \nu(f)| \leq \|f\| \, L(\mu, \nu)$, where $\|f\| = \sup_{x \in E} |f(x)|$.

Proof. We have $|\mu(f) - \nu(f)| \leq$

$$\leq \left| \int_{[0, \|f\|]} [\mu \circ f^{-1}([x, \|f\|]) - \nu \circ f^{-1}([x, \|f\|])] dx \right| +$$

$$+ \left| \int_{[-\|f\|, 0]} (\mu \circ f^{-1}([-\|f\|, x]) - \nu \circ f^{-1}([-\|f\|, x])) dx \right| \leq 2\|f\| L(\mu, \nu)$$

LEMMA 2.28. Let I be a countable set. Let $(E_i, \tau_i)_I$

be a family of metrizable separable complete spaces and let $(E_i', \mathcal{T}_i')_I$ be a family of metrizable spaces. If for every $i \in I$ η_i is a transition probability from $(E_i', \sigma(\mathcal{T}_i'))$ to $(E_i, \sigma(\mathcal{T}_i))$ such that the map $x \mapsto \eta_i(x, \ )$, $x \in E_i'$, is weakly continuous, then the map $y \mapsto (\bigotimes_I \eta_i)(y, \ )$, $y \in \prod_I E_i'$, is also weakly continuous.

Proof. Denote $E = \prod_I E_i$, $\mathcal{T} = \bigotimes_I \mathcal{T}_i$, $E_i' = \prod_I E_i'$, $\mathcal{T}' = \bigotimes_I \mathcal{T}_i'$, $\eta = \bigotimes_I \eta_i$. Obviously, $\sigma(\mathcal{T}) = \bigotimes_I \sigma(\mathcal{T}_i)$. Let $n \mapsto i_n$ be a bijection between N and I. For every $n \in N$ let d_n be a complete metric compatible with the topology $\mathcal{T}_{i_n}$, $d_n \leq 1$. Define the metric d on E by $d(x,y) = \sum_N 2^{-n} d_n(x_{i_n}, y_{i_n})$ for every $x = (x_i)_I$, $y = (y_i)_I$ in E. The metric d is compatible with the topology $\mathcal{T}$. (E,d) is a complete separable metric space.

Let $f \in C_b(E,d)$ uniformly continuous. Let a sequence $(x_n)_N$ of E convergent to x. Denote $I_n = \{i_0, i_1, \dots, i_n\}$. Let $a = (a_i)_I$ be a fixed point in E. Define for every $n \in N$ the functions $f_n : E \to R$ by $f_n(x) = f((x_i)_{I_n}, (a_i)_{I - I_n})$, $x = (x_i)_I$ and $g_n : \prod_{I_n} E_i \to R$ by $g_n(y) = f(y, (a_i)_{I - I_n})$, $y \in \prod_{I_n} E_i$. Obviously , f_n is uniformly continuous.

For every $n \in N$ denote

$b_n = \sup \{|f(x) - f(y)| \,/\, x = (x_i)_I, y = (y_i)_I, (x_i)_{I_n} = (y_i)_{I_n}\}$.

We have $\eta(f_n) = \bigotimes_{I_n} \eta_i(g_n)$. We show by induction that $\bigotimes_{I_n} \eta_i(g_n) \in C_b(\prod_{I_n} E_i')$. Let $x = (x_i)_{I_{n+1}}$, $x' = (x_i')_{I_{n+1}}$ in $\prod_{I_n} E_i'$.

We have $\left| \underset{I_{n+1}}{\otimes} \eta_i(x, g_{n+1}) - \underset{I_{n+1}}{\otimes} \eta_i(x', g_{n+1}) \right| \leq$

$\leq \int \left| \int g_{n+1}(y,z) \underset{I_n}{\otimes} \eta_i((x_i)_{I_n}, dy) - \int g_{n+1}(y,z) \underset{I_n}{\otimes} \eta_i((x'_i)_{I_n}, dy) \right|$

$\eta_{i_{n+1}}(x_{n+1}, dz)$ +

$+ \int \left| \int g_{n+1}(y,z) \eta_{i_{n+1}}(x_{i_{n+1}}, dz) - \int g_{n+1}(y,z) \eta_{i_{n+1}}(x'_{n+1}, dz) \right|$
$\underset{I_n}{\otimes} \eta_i((x'_i)_{I_n}, dy)$.

If we suppose that $\underset{I_n}{\otimes} \eta_i(g_n) \in C_b(\prod_{I_n} E'_i)$ it follows that the first term converges to 0 when $(x'_i)_{I_n} \longrightarrow (x_i)_{I_n}$.

The second term can be majorized by
$2 \| f \| \, L(\eta_{i_{n+1}}(x_{i_{n+1}}, \ ), \eta_{i_{n+1}}(x'_{i_{n+1}}, \ ))$ and consequently, it converges to 0 when $x'_{i_{n+1}} \longrightarrow x_{i_{n+1}}$. Therefore

$\underset{I_n}{\otimes} \eta_i(C_b(\prod_{I_n} E_i)) \subset C_b(\prod_{I_n} E'_i)$ for every $n \in N$.

Since the sequence of uniformly continuous functions $(\eta(f_n))_N$ is uniformly convergent to $\eta(f)$, it follows that $\eta(f) \in C_b(E')$.

PROPOSITION 2.29. If the family of TASes $(\mathcal{E}_i)_I = (E_{iw}, \tau_{iw}, \varphi_{iww'}/W_i)_I$ is countable and (E_{iw}, τ_{iw}) is a complete separable metrizable space for every $i \in I$, $w \in W_i$, then $(E_u, \tau_u, \varphi_{uu'}/U)$ is a TAS (the product of the family $(\mathcal{E}_i)_I$).

Proof. Obvious.

The following theorem gives conditions under which the limit of a product of TASes is the product of its limits.

THEOREM 2.30. Let $(\mathcal{E}_i)_I=(E_{in}, \tau_{in}, \varphi_{imn}/N)_I$ be a countable family of TASes such that for every $i\in I$ there exists a complete separable metric space (S_i,d_i) and a convergent series with positive terms $\sum_N \varepsilon_{in}$ and for every $m\in N$ we have:

(i) the topological space (E_{im}, τ_{im}) is a closed subspace of the metric space (S_i,d_i),

(ii) for every convergent sequence $(x_n)_N \in \prod_N E_{in}$, the sequence $(\varphi_{imn}(x_n, \,))_{n>m}$ of $M(E_{im})$ is weakly convergent,

(iii) if the sequence $(x_n)_N \in \prod_N E_{in}$ converges to $x\in E_{im}$ then $\lim_n \varphi_{imn}(x_n, \,)=\varepsilon_x$,

(iv) $\sup_{x \in E_{m+1}} \varphi_{imm+1}(x, B(x, \varepsilon_{im})\cap E_{im}) > 1-\varepsilon_{im}$.

Let $(E_i, \tau_i, \psi_{in}/N)$ be the limit of the TAS $\mathcal{E}_i$, $i\in I$.

The TAS product of the family $(\mathcal{E}_i)_I$, $(E_u, \tau_u, \varphi_{uu'}/U)$ has the limit $(\prod_I E_i, \otimes_I \tau_i, \psi_u/U)$, where $\psi_u((x_i)_I, \,)=\otimes_J \psi_{in_i}(x_i, \,)$ for every $u=(i,n_i)_J\in U$ and $(x_i)_I\in\prod_I E_i$. Also, if $(u_n=(i,k_{ni})_{J_n})_N$ is an increasing sequence of U with $\bigcup_N J_n=I$ and $\lim_n k_{ni}=\infty$ then the TAS $(E_{u_n}, \tau_{u_n}, \varphi_{u_m u_n}/N)$ has the limit $(\prod_I E_i, \otimes_I \tau_i, \psi_{u_n}/N)$. In addition, the projective system $(M(E_{u_n}), \hat{\varphi}_{u_m u_n}/N)$ has the limit $(M(\prod_I E_i), \hat{\psi}_{u_n}/N)$ and the projective system $(M(E_u), \hat{\varphi}_{uu'}/U)$ has the limit $(M(\prod_I E_i), \hat{\psi}_u/U)$.

Proof. Denote $E=\prod_I E_i$, $\tau=\otimes_I \tau_i$, $S=\prod_I S_i$. Let $n\mapsto i_n$ a bijection between N and I. Denote by d the metric on S defined by $d(x,y)=\sum_{N} 2^{-n}[d_{i_n}(x_{i_n},y_{i_n})\wedge 1]$

for every $x=(x_i)_I$ and $y=(y_i)_I$ of S.

Obviously, $(E, \tau, \Psi_u/U)$ is a space with approximations for the TAS $(E_u, \tau_u, \varphi_{uu'}/U)$. (See Lemma 2.28.). Because of Theorem 2.10., we can suppose that (E_i, τ_i) is a subspace of (S,d),

$E_i=\{x\in S$ / there is a sequence $(x_n)_N\in\prod_N E_i$ convergent to x and $\Psi_{im}(x, \ )=\lim_n \varphi_{imn}(x_n, \ )$ where $x=\lim_n x_n\in E_i$.

There is an increasing sequence $(u_n=(i,k_{ni})_I)_N$ of U such that $\sum_{i\in I}\sum_{k\geq k_{ni}} \varepsilon_{ik}<2^{-n}$ and $\prod_{i\in I}\prod_{k\geq k_{ni}}(1-\varepsilon_{ik})>1-2^{-n}$. Let $(v_n)_N$ be an increasing sequence of $\mathcal{U}$ with $v_n=(i,k'_{ni})_I\geq u_n$, $n\in N$. We want to apply Theorem 2.10. for the TAS $(E_{v_n}, \tau_{v_n}, \varphi_{v_m v_n}/N)$. The approximants of this TAS are included in (S,d). Let $n\in N$ and $x=(x_i)_I\in E_{v_{n+1}}$. Obviously, $B(x,2^{-n+1})\supset\prod_I B(x_i,2^{-n})$

Consequently, $\varphi_{v_n v_{n+1}}(x,B(x,2^{-n+1})\cap E_{v_n})\geq$

$$\geq\prod_I \varphi_{ik'_{ni}k'_{n+1i}}(x_i,B(x_i,2^{-n})\cap E_{v_n})$$

$$\geq\prod_I \varphi_{ik'_{ni}k'_{n+1i}}(x_i,B(x_i,\sum_{k\geq k'_{ni}}\varepsilon_{ik})\cap E_{v_n})\geq$$

$$\geq\prod_I\prod_{k\geq k'_{ni}}(1-\varepsilon_{ik})\geq 1-2^{-n+1}$$. (See Lemma 2.11.).

If the sequence $(x_n)_N\in\prod_N E_{v_n}$ converges to $x\in S$ then the sequence $(\varphi_{v_m v_n}(x_n, \ ))_{n>m}$ converges to $\Psi_{v_m}(x, \ )$

for every $m \in N$. (See Lemma 2.28.).

It follows that $(E, \tau, \Psi_{v_n}/N)$ is the limit of the TAS $(E_{v_n}, \tau_{v_n}, \varphi_{v_m v_n}/N)$.

Let $(E', \tau', \Psi'_u/U)$ be a space with approximations for the TAS product of the family $(\mathcal{E}_i)_I$. For every $x \in E'$ there exists a probability measure $\mu_x \in M(E)$ such that $\mu_x \Psi_{v_n} = \Psi'_{v_n}(x, \ )$, $n \in N$. Since on the set of functions

$\{ \Psi'_u(f) \ / \ f \in C_b(E_u) \ , \ u=(i,k_i)_J \in U \ , \ J \text{ finite} \}$

μ_x is known, it follows that μ_x is unique. Obviously, μ_x does not depend on the sequence $(v_n)_N$.

For every $u \in U$, there is $u' \in U$ such that $u' \geq u$ and $u' \geq u_0$. We have $\mu_x \Psi_u = \mu_x \Psi_{u'} \varphi_{uu'} = \Psi'_{u'} \varphi_{uu'}(x, \ ) = \Psi'_u(x, \ )$. The map $x \mapsto \mu_x$, $x \in E'$, defines a transition probability Ψ from $(E', \sigma(\tau'))$ to $(E, \sigma(\tau))$. Because for every $u \in U$, the map $x \longmapsto \Psi\Psi_u(x, \ ) = \Psi'_u(x, \ )$, $x \in E'$, is weakly continuous, it follows that the map $x \longmapsto \Psi(x, \ )$, $x \in E'$, is also weakly continuous. Therefore Ψ is the representation of E' in E.

The other assertions of the Theorem are obvious.

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