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FINE POTENTIALS AND SUPERMEAN FUNCTIONS ON STANDARD H-CONES

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N. BOBOC and Gh. BUCUR

In the paper [4] we caracterised the elements of a standard H-cone of functions on a semisaturated set X by a topological property (lower semicontinuity) and an algebraic order property (supermean property).

Now we obtain similar results replacing lower semicontinuity by fine lower semicontinuity but this time using the axiom of nearly continuity. It turn out that in fact the axiom of nearly continuity is also necessary for the validity of the remained results.

In connection with the above characterization we introduced and studed the nation of fine potential. The existence of a strictly positive fine potential on X is in fact equivalent with the axiom of nearly continuity.

1. The fine topology and the quasi-continuity

In the following $\mathcal{S}$ will be a standard H-cone of functions on a semisaturated set X. As for the terminology one can see [2] [6] and [8]. We recall the following notions

a) A map $\varphi:\mathcal{P}(X)\longrightarrow \overline{R}_+$ is called a weight if it is increasing ($\varphi(A)\leq\varphi(B)$ if $A\subset B$) and $\varphi(\emptyset)=0$. We say that φ is a fine (resp. countable subadditive, right continuous) weight if we have

$\varphi(A)=\varphi(\overline{A}^f)$ where $\overline{A}^f$ means the fine closure of A (resp. $\varphi(\bigcup_{n=1}^{\infty} A_n)\leq \sum_{n=1}^{\infty}\varphi(A_n)$, $\varphi(A)=\inf\{\varphi(G)\mid G \text{ open}, G\supset A\}$).

For any $x\in X$ we denote by ψ_x the map on $\mathcal{P}(X)$ defined by

$$\psi_x(A)=R^A 1(x), \qquad A\in\mathcal{P}(X)$$

It is obvious that the map ψ_x is a fine weight which is countable subadditive and right continuous.

b) A function $f:X\longrightarrow\overline{R}$ is termed <u>quasi-continuous</u> (resp. <u>quasi lower semicontinuous</u> or <u>quasi upper semicontinuous</u>) with respect to the weight φ if for any positive number $\varepsilon>0$, there exists an open set G_ε such that $\varphi(G_\varepsilon)<\varepsilon$ and the restriction to $X\setminus G_\varepsilon$ of f is a continuous (resp. lower semicontinuous or upper semicontinuous) function on $X\setminus G_\varepsilon$.

c) A function $f:X\longrightarrow\overline{R}$ is termed <u>quasi-continuous</u> (resp. <u>quasi lower semicontinuous</u> or <u>quasi upper semicontinuous</u>) if there exists a decreasing sequence $(G_n)_n$ of open subsets of X such that the restriction to $X\setminus G_n$ of f is a continuous (resp. lower semicontinuous or upper semicontinuous) function on $X\setminus G_n$ for any $n\in N$ and

$$\bigwedge_n B^{X\setminus G_n} 1=0$$

<u>Proposition 1.1</u>. If f is a quasi-continuous (resp. quasi lower semicontinuous or quasi upper semicontinuous) function on X then there exists a semipolar set M such that the function f is quasi-continuous (resp. quasi-lower semicontinuous or quasi-upper semicontinuous) with respect to any weight ψ_x for all $x\in X\setminus M$.

<u>Proof</u>. Let $(G_n)_n$ be a decreasing sequence of open subsets

of X such that $\bigwedge_n B^{G_n}1 = 0$ and such that the restriction to $X \setminus G_n$ of the function f is continuous (resp. lower semicontinuous or upper semicontinuous) for any $n \in N$. We know (see [6], Theorem 33.7) that the set $M := \bigwedge_n B^{G_n}1 < \inf_n B^{G_n}1$ is semipolar and therefore the function f is quasi-continuous (resp. ...) with respect to any weight ψ_x, $x \in X \setminus M$.

<u>Proposition 1.2</u>. A function $f: X \longrightarrow \overline{R}$ is quasi-continuous (resp. quasi-lower semicontinuous or quasi-upper semicontinuous) iff there exists a countable subset A of X which is dense in X such that f is quasi-continuous (resp. quasi-lower semiconti-nuous or quasi-upper semicontinuous) with respect to any weight ψ_x, $x \in A$.

<u>Proof</u>. The "only if" part follows from the previous proposition using the fact that any semipolar set has no interior point. For the "if" part we consider $A = \{x_n \mid n \in N\}$ a dense subset of X such that f is quasi-continuous (resp. - - -) with respect to any weight ψ_{x_n}, $n \in N$. For any $n \in N$ we consider an open subset G_n of X such that the restriction to $X \setminus G_n$ of f is continuous (resp. - - -) and

$$\psi_{x_i}(G_n) < \frac{1}{2^n} \qquad (\forall) \quad i \in \{1, 2, \ldots, n\}.$$

Let now $(D_n)_n$ be the decreasing sequence of open subsets of X defined by

$$D_n = \bigcup_{j > n} G_j$$

From the above consideration we have

$$\Psi_{x_i}(D_n) \leq {\sum_{j>n}}' \Psi_{x_i}(G_j) < \frac{1}{2^n} \qquad (\forall) \quad i \leq n.$$

Hence

$$\bigwedge_n B^{D_n} 1(x_i) = 0 \qquad (\forall)\ i \in N, \ \bigwedge_n B^{D_n} 1 = 0 \text{ on } X.$$

With the same proof as in Proposition 3.9 from [4] we give the following result:

<u>Proposition 1.3</u>. Let $(s_n)_n$ be a sequence of continuous functions from $\mathcal{S}$. Then the function $s := {\sum_n}' s_n$ is quasi-continuous with respect to any weight Ψ_x , $x \in [s < \infty]$.

<u>Theorem 1.4</u>. Suppose that $\mathcal{S}$ satisfies the axiom of nearly continuity and let X_1 be the saturated of X. Then there exists a Borel, fine closed and semipolar subset A of X_1 such that any semipolar subset of $X_1 \setminus A$ is polar.

<u>Proof</u>. From [6], Theorem 5.5.8, we may consider a Borel, fine closed and semipolar subset A of X_1 such that the set $X_1 \setminus A$ is a Green set for the pair $(\mathcal{S}, \mathcal{S}^*)$.

We know ([6], Theorem 5.5.8) that a subset M of $X_1 \setminus A$ is semipolar (resp. polar) iff it is co-semipolar (resp. co-polar). On the other hand, since $\mathcal{S}$ satisfies the axiom of nearly continuity we deduce ([6], Theorem 5.6.3) that $\mathcal{S}^*$ satisfies the axiom of polarity and therefore any semipolar subset of $X_1 \setminus A$ is co-polar and polar as well.

For a standard H-cone of functions $\mathcal{S}$ on X we denote by $\mathcal{G}_o$ the set of all fine open subsets G of X such that there exist $s, t \in \mathcal{S}$, s,t finite, $s \leq t$ and such that

$$G = [s < t]$$

Remark. If $G \in \mathcal{G}_o$ then there exists $s', t' \in \mathcal{S}$, $s' \leq 1$, $t' \leq 1$, $s' \leq t'$ such that $G = [s' < t']$.

Indeed, let $s, t \in \mathcal{S}$, s, t finite, $s \leq t$ such that $G = [s < t]$. Since

$$G = \bigcup_{n=1} [s \wedge n < t \wedge n]$$

we deduce the assertion taking

$$s' = \sum_{n=1}' \frac{1}{n \cdot 2^n} \cdot s \wedge n, \qquad t' = \sum_{n=1}' \frac{1}{n \cdot 2^n} \cdot t \wedge n .$$

Proposition 1.5. The family $\mathcal{G}_o$ is closed under the finite intersection and countable union.

Proof. For a sequence (G_n) from $\mathcal{G}_o$ we consider the sequences $(s_n)_n$, $(t_n)_n$ in $\mathcal{S}$ such that $s_n, t_n \leq 1$, $s_n \leq t_n$ $G_n = [s_n < t_n]$ for any $n \in N$.

Obviously we have

$$G_1 \cap G_2 = [s_1 + s_2 < (s_1 + t_2) \wedge (s_2 + t_1)]$$

$$\bigcup_{n=1} G_n = \Big[\sum_{n=1}' \frac{1}{2^n} s_n < \sum_{n=1}' \frac{1}{2^n} t_n\Big].$$

Lemma 1.6. If K is a compact subset of X and G is a fine open subset of X such that $K \subset G$ then there exists a sequence $(G_n)_n$ in $\mathcal{G}_o$ such that

$$\overline{G}_n^f \subset G_{n+1}, \quad K \subset \bigcup_{n=1}^{\infty} G_n \subset G .$$

Proof. Let p be a bounded continuous generator of $\mathcal{S}$. Using ([9], Proposition 1-7) we may choose an open set D_n in X such that $X \setminus G \subset D_n$ and

$$B^{D_n} p < B^{X \setminus G} p + \frac{1}{n} \quad \text{on} \quad K.$$

Since $B^{X\setminus G}p<p$ on G we deduce that we have

$$K\subset\bigcup_{n=1}^{\infty}\left[B^{D_n}p<p\right]\subset G$$

and therefore

$$K\subset[q<p]\subset G$$

where q is the element of $\mathcal{S}$ defined by

$$q=\sum_{n=1}^{\infty}\frac{1}{2^n}B^{D_n}p$$

The proof is finished taking, for any $n\in N$,

$$G_n:=\left[p-q>\frac{1}{n}\right]=\left[(q+\frac{1}{n})\wedge p<p\right].$$

Theorem 1.7. Suppose that $\mathcal{S}$ satisfies the axiom of nearly continuity and that X is a souslinean and semisaturated set. Then any positive Borel and fine lower semicontinuous function on X is quasi-lower semicontinuous.

Proof. First we show that any finite element s of $\mathcal{S}$ is quasi-continuous. Indeed, taking a nearly continuous and finite generator p of $\mathcal{S}$ it follows, using the axiom of nearly continuity that the element $s\wedge np$ is nearly continuous for any $n\in N$. From Proposition 1.3 we deduce that $s\wedge np$ is quasi-continuous for any $n\in N$. By standard arguments we get that s is a quasi-continuous function on X:

Let now $s,t\in\mathcal{S}$, $s\leq t\leq 1$ and let G be the element of the family $\mathcal{G}_o$ given by $G=[s<t]$. Using the above considerations we deduce that, for any $n\in N$, the function $f_n:X\to R$ defined by

$$f_n=\inf\{1,n(t-s)\}$$

is quasi-continuous. Since the sequence $(f_n)_n$ increases to the characteristic function 1_G of the set G it follows that 1_G is a quasi-lower semicontinuous function on X.

We show now that for any Borel and fine open subset G of X the function 1_G is quasi-lower semicontinuous.

Since for any $x \in G$ there exists a fine open neighbourhood V_x such that $\overline{V}_x \subset G$ we deduce that we have

$$B^{X\setminus V_x} p = p \quad \text{on} \quad X\setminus G, \quad B^{X\setminus V_x} p < p \quad \text{on} \quad V_x$$

where p is a finite generator of $\mathcal{S}$.

Hence for any point $x \in G$ there exists a fine open neighbourhood U_x ($U_x = B^{X\setminus V_x} p < p$) such that $U_x \in \mathcal{G}_o$, $\overline{U}_x \subset G$. Using the fact that the fine topology on X is quasi-Lindelöf ([9], Theorem 2.3) we deduce that there exists a sequence $(Ux_n)_n$ in $\mathcal{G}_o$ such that the difference between G and $G_o := \bigcup_n Ux_n$ is a semipolar subset of X. Using Theorem 1.4 we choose a Borel, fine closed and semipolar subset A of G such that any semipolar subset of $G\setminus A$ is polar.

Let μ be a positive finite measure on A such that a subset M of A is polar iff $\mu(M)=0$ (See [9], Theorem 1-10). Since X is souslinean we deduce that there exists a sequence $(K_n)_n$ of compact subsets of A such that $\mu(A\setminus \bigcup_n K_n)=0$. Hence the set $A\setminus \bigcup_n K_n$ is polar. From Lemma 1.6 we may consider a sequence $(G_n)_n$ in $\mathcal{G}_o$ such that

$$K_n \subset G_n \subset G$$

Obviously the set $G_o^* := \bigcup_n G_n$ belongs to $\mathcal{G}_o$, $\bigcup_n K_n \subset G_o^* \subset G$. From the preceding considerations we deduce that $G_o \cup G_o^*$ belongs to $\mathcal{G}_o$, $G_o \cup G_o^* \subset G$ and the difference $G\setminus(G_o \cup G_o^*)$ is a polar subset of G. Since the characteristic function of any polar subset is quasi-continuous we get that the function 1_G is quasi-lower semicontinuous.

We finish the proof observing that any Borel and fine lower semicontinuous function f on X is the limit of the following increasing sequence $(f_n)_n$ of quasi-lower semicontinuous functions on X,

$$f_n := \frac{1}{2^n} \sum_i 1_{[\frac{i}{2^n} < f]}$$

2. Fine potentials and fine supermean property

In the sequel we suppose that $\mathcal{S}$ is a standard H-cone of functions on a semisaturated set X.

Definition. An element $p \in \mathcal{S}$ is termed a fine potential on X if for any increasing sequence $(G_n)_n$ of fine open subsets of X such that $\bigcup_{n=1} G_n = X$ and such that $\overline{G}_n^f \subset G_{n+1}$, for any $n \in N$, we have $\bigwedge_n B^{X \setminus G_n} p = 0$.

Remark. It is obvious that the set of all fine potentials is a solid and convex subcone of $\mathcal{S}$ with respect to the natural order relation and for any sequence $(p_n)_n$ of fine potentials on X the function $f := \sum_n p_n$ is a fine potential on X whenever it is finite of a dense subset of X.

Whenever we have an increasing sequence $(G_n)_n$ of fine open sets of X such that for any $n \in N$, $\overline{G}_n^f \subset G_{n+1}$ then there exists an increasing sequence $(D_n)_n$ of Borel and fine open subsets of X such that $G_n \subset D_n \subset G_{n+1}$, $\overline{D}_n^f \subset D_{n+1}$. Indeed, we take, for any $n \in N$, $D_n := X \setminus b(X \setminus G_n)$. Hence an element $p \in \mathcal{S}$ is a fine potential if and only if for any sequence $(G_n)_n$ of fine open and Borel subsets of X such that $\overline{G}_n^f \subset G_{n+1}$, for any $n \in N$, and such that $\bigcup_n G_n = X$ we have $\bigwedge_n B^{X \setminus G_n} p = 0$.

From the above considerations one can deduce the following assertion:

Proposition 2.1. There exists a strictly positive and fine potential on X iff any universally continuous element of $\mathcal{S}$ is a fine potential on X.

Proposition 2.2. If $p \in \mathcal{S}$ is a fine potential on X then the natural extention of p to the saturated set X_1 of X is also a fine potential on X_1.

Proof. Let $p \in \mathcal{S}$ be a fine potential on X and let $(G_n)_n$ be a sequence of fine open subsets in X_1 such that $\overline{G}_n^f \subset G_{n+1}$ and $\bigcup_{n=1} G_n = X_1$. Since for any fine open set G of X_1 we have $\overline{G \cap X}^f \cap X = \overline{G}^f \cap X$ we deduce that the sequence $(G_n')_n$ of open subsets on X defined by $G_n' = G_n \cap X$ is increasing to X and moreover $\overline{G}_n'^f \subset G_{n+1}'$ for any $n \in N$ where this time $\overline{G}_n'^f$ means the fine closure in X of the set G_n'.

We have

$$B^{X_1 \setminus G_{n+1}} p \le B^{X_1 \setminus \overline{G}_n^f} p = B^{X \setminus \overline{G}_n^f} p \le B^{X \setminus G_n'} p \ ,$$

$$\bigwedge_n B^{X_1 \setminus G_n} p \le \bigwedge_n B^{X \setminus G_n'} p = 0 \ .$$

i.e. p is a fine potential on X_1.

Theorem 2.3. The following assertions are equivalent:

1) There exists a strictly positive fine potential on X.

2) $\mathcal{S}$ satisfies the axiom of nearly continuity.

3) For any universally continuous element $p \in \mathcal{S}$ and any decreasing sequence $(F_n)_n$ of fine closed subsets of X such that $F_{n+1} \subset b(F_n)$ for any $n \in N$ and such that $\bigcap_n F_n = \emptyset$ we have

$$\bigwedge_n B^{F_n} p = 0 .$$

Proof. Obviously we have $3) \Rightarrow 1)$. For the relation $1) \Rightarrow 2)$ we remark that from Propositions 2.1, 2.2 there exists a 1-continuous and fine potential p on X, $p>0$, which is also a fine potential on X_1. Let now q be an universally bounded element of $\mathcal{S}$. Since any bounded element of $\mathcal{S}$ has a nonempty carrier (see [5], Proposition 1.2) we may consider a kernel V defined on $\overline{X}_1$ such that: for any positive, bounded, Borel function f on $\overline{X}_1$ we have:

$$Vf \in \mathcal{S}, \quad \text{carr } Vf \subset \overline{[f>0]}, \quad V1 = q .$$

We show that $V1_K = 0$ for any compact subset K of $\overline{X}_1 \setminus X_1$. Let $(G_n)_n$ be a sequence of open subsets of $\overline{X}_1$ such that $\overline{G}_{n+1} \subset G_n$ and such that $\bigcap_n G_n = K$. Since q is dominated by αp for a suitable positive number α, we get

$$V1_K = B^{X \cap G_n} V1_K \leq \alpha B^{X \cap G_n} p \qquad (\forall)\ n \in N ,$$

$$\bigwedge_n B^{X \setminus G_n} p \leq \bigwedge_n B^{X \cap \overline{G}_n} p = 0, \quad V1_K = 0 .$$

Hence $q = V(1_{X_1})$. To show that q is nearly continuous it will be sufficent to prove (see [7]) that

$$B^K V(1_K) = V1_K .$$

for any compact subset K of X_1 .

Let now K be a compact subset of X_1 and let G be a fine open neighbourhood of K. Using Lemma 1.6 we may consider a sequence $(D_n)_n$ of fine open subsets of X_1 such that $\overline{D}_n^f \subset D_{n+1} \subset G$ for any $n \in N$ and such that $K \subset \bigcup_n D_n$. We consider also a sequence $(U_n)_n$ of open subsets of X_1 such that $\overline{U}_{n+1} \subset U_n$ for any $n \in \mathbb{N}$ and such that $\bigcap_n U_n = K$

Obviously the sequence $(F_n)_n$ of fine closed subsets of X_1 defined by $F_n = \overline{U}_n \setminus D_n$ satisfies the relations

$$F_{n+1} \subset \mathring{F}_n^f \qquad (\forall)\ n \in N, \quad \bigcap_{n \in N} F_n = \emptyset$$

Hence we have $\bigwedge_n B^{F_n} p = 0$ and therefore $\bigwedge_n B^{F_n} q = 0$. We have also

$$V(1_K) = B^{U_n} V(1_K) \leq B^{\overline{U}_n \setminus D_n} V(1_K) + B^{D_n} V(1_K)$$

$$B^{F_n} V(1_K) + B^G V(1_K) \leq B^{F_n} q + B^G V(1_K) \ .$$

for any $n \in N$ and hence

$$V(1_K) \leq \bigwedge_n B^{F_n} q + B^G V(1_K) = B^G V(1_K)$$

Since the fine open nieghbourhood G of K is arbitrary we get $V(1_K) = B^K V(1_K)$.

2) $\Rightarrow$ 3) Let p be an universally continuous element of $\mathcal{S}$ and let $(F_n)_n$ be a sequence of fine closed subsets of X such that $F_{n+1} \subset b(F_n)$ for any $n \in N$ and such that $\bigcap_n F_n = \emptyset$.

For any $s \in \mathcal{S}$ we have

$$B^{F_n} s = s \quad \text{on} \quad b(F_n), \qquad B^{F_n} s = s \quad \text{on} \quad \bigcup_{i > n} F_i \ ,$$

$$B^{F_n}(B^{F_i} s) = B^{F_i} s = s \quad \text{on} \quad F_{i+1} \qquad (\forall) \quad i \geq n \ ,$$

and therefore we deduce

$$B^{F_n}(B^{F_i}s) \geq B^{F_{i+1}}s \qquad (\forall) \quad i \geq n .$$

On the other hand since the sequence $(F_n)_n$ is decreasing and $\bigcap_n F_n=\emptyset$ we get

$$(\bigwedge_n B^{F_n}p)(x)=\inf_n B^{F_n}p(x) \qquad (\forall) \quad x\in X.$$

The set X being semisaturated there exists, for any $n\in N$ and any $x\in X$, a finite measure μ_n^x on X such that

$$\mu_n^x(s)=B^{F_n}s(x) \qquad (\forall) \quad s\in \mathcal{S}$$

From the above considerations we have:

$$\mu_x^n(\bigwedge_i B^{F_i}p)=\inf_i \mu_n^x(B^{F_i}p)=\inf_i B^{F_n}(B^{F_i}p)(x) \geq$$

$$\geq \inf_i B^{F_{i+1}}p(x)=(\bigwedge_i B^{F_i}p)(x) \qquad (\forall) \quad n\in N$$

Hence $B^{F_n}(\bigwedge_i B^{F_i}p)=\bigwedge_i B^{F_i}p$ for any $n\in N$. Since $\mathcal{S}$ satisfies the axiom of nearly continuity we deduce that $\bigwedge_i B^{F_i}p$ is a nearly continuous element of $\mathcal{S}$ whose fine carrier is contained in any subset F_n. Using the fact that any nonzero nearly continuous element of $\mathcal{S}$ has a nonempty fine carrier on X we get $\bigwedge_i B^{F_i}p=0$.

<u>Theorem 2.4</u>. Suppose that $\mathcal{S}$ satisfies the axiom of nearly continuity. If $p\in\mathcal{S}$ is finite then:

a) p is a fine potential on X iff p is nearly continuous;

b) if X is souslinean and p is bounded then the following assertions are equivalent.

1) p is a fine potential on X.

2) p is a natural potential on X.

3) for any increasing sequence $(G_n)_n$ of Borel and fine open

subsets of X such that $\bigcup_n G_n = X$ we have $\bigwedge_n B^{X\setminus G_n} p = 0$.

Proof. a) Using Theorem 2.3 we deduce that any universally (hence any nearly continuous) element of $\mathcal{S}$ is a fine potential on X.

Suppose now that p is a fine potential on X and let q be a nearly continuous element of $\mathcal{S}$, $q>0$ on X. Since the convex cone of all nearly continuous elements of $\mathcal{S}$ is a band in $\mathcal{S}$ with respect to the specific order on $\mathcal{S}$ we may decompose the element p as a form

$$p=p'+p'' , \quad p',p'' \in \mathcal{S}$$

where p' is nearly continuous and p" has now specific and nearly continuous ($\neq 0$) minorant. We show that $p''=0$. Indeed, p" is a fine potential and for any $n \in N$ we have

$$p''=R(p''-nq)+q' , \quad q' \in \mathcal{S} , \; q' \leq nq$$

Since $\mathcal{S}$ satisfies the axiom of nearly continuity it follows that q' is nearly continuous. Hence we get $q'=0$ and therefore

$$R(p''-nq)=p'' \qquad (\forall) \quad n \in N.$$

Let us denote by F_n the fine closed subset of X defined by

$$F_n = nq \leq p'' .$$

From the above considerations we have

$$B^{F_n} p'' = B^{F_n}(R(p''-nq)) = R(p''-nq) = p''$$

for any $n \in N$. On the other hand we have, for any $n \in N$,

$$F_{n+1} = \left[(1+n)q \leq p''\right] \subset \left[nq < p''\right] \subset (\overset{\circ}{F_n})^f .$$

Since p" is a fine potential on X we get $0=\bigwedge_n B^{F_n}p"=p"$.

b) Suppose now that X is also a souslinean space and that p is a natural potential on X. We show that for any increasing sequence $(G_n)_n$ of Borel and fine open subset of X such that $\bigcup_n G_n=X$ we have $\bigwedge_n B^{X\setminus G_n}p=0$. Let $(G_n)_n$ be a such type of sequence. From Theorem 1.7 we deduce that for any $n\in N$ there exists a sequence $(G_{n,m})_m$ of open subsets of X such that $G_{n,m}\supset G_{n,m+1}$ for any $m\in N$ and sucht that $\bigwedge_m B^{G_{n,m}\setminus G_n}1=0$. For any sequence $(m_n)_{n\in N}$ of natural numbers we have

$$B^{(\bigcup_{n=1}^{k} G_{n,m_n})\setminus G_k}1\leq \sum_{n=1}^{k} B^{G_{n,m_n}\setminus G_n}1 ,$$

$$B^{X\setminus G_k}p\leq B^{X\setminus \bigcup_{n=1}^{k} G_{n,m_n}}p+\sum B^{G_{n,m_n}\setminus G_n}p$$

Obviously the sequence $(D_k)_{k\in N}$ of open subsets of X defined by $D_k:=\bigcup_{n=1}^{k} G_{n,m_n}$ is increasing to X and therefore, p being a natural potential on X, we have

$$\bigwedge_k B^{X\setminus D_k}p=0 \ , \quad \bigwedge_k B^{X\setminus G_k}p\leq\bigwedge_k \sum_{n=1}^{k} B^{G_{n,m_n}\setminus G_n}p$$

SInce p is bounded we get

$$\bigwedge_k B^{X\setminus G_k}p=0.$$

The assertion 3) $\Rightarrow$ 1) follows from the remark to the definition of a fine potential.

Theorem 2.5. Suppose that X is souslinean. If $\mathcal{S}$ satisfies the axiom of nearly continuity then for any fine open subset G of X the H-cone $\mathcal{S}'(G)$ ([3]) satisfies also the axiom of nearly continuity.

Proof. Let p be a universaly continuous element of $\mathcal{S}$. It will be sufficient to show that $p-B^{X\setminus G}p$ is a fine potential on G with respect to $\mathcal{S}'(G)$. We consider a sequence $(G_n)_n$ of fine open and Borel subsets of G which increases to G. Let D be an open neighbourhood of $X\setminus G$ and let $(D_n)_n$ be the sequence of fine open and Borel subsets of X defined by $D_n = D \cup G_n$ for any $n \in N$. Obviously the sequence (D_n) is increasing to X. Using the hypothesis and the fact that p is a bounded fine potential on X we get, from Theorem 2.4.,

$$\bigwedge_n B^{X\setminus D_n} p = 0$$

On the other hand, from [2], I, Proposition 2.3, we have, for any $n \in N$,

$$B^{G\setminus G_n}(p-B^{X\setminus G}p) \leq B^{X\setminus G_n}p - B^{X\setminus G}p \ ,$$

$$B^{X\setminus G_n}p \leq B^{X\setminus (D\cup G_n)}p + B^{D}p \ .$$

and therefore

$$\bigwedge_n B^{G\setminus G_n}(p-B^{X\setminus G}p) \leq B^{D}p - B^{X\setminus G}p \text{ on } G \ .$$

The open set D being arbitrary the element $p-B^{X\setminus G}p$ is a fine potential on G.

Theorem 2.6. Suppose that X is souslinean with respect to $\mathcal{S}$. The following assertions are equivalent:

1) $\mathcal{S}$ possesses a strictly positive fine potential on X.

2) Any positive Borel and fine lower semicontinuous function f on X belongs to $\mathcal{S}$ if

a) f is finite on a finely dense subset of X.

b) for any $x \in X$ and any open subset D of X, $x \in D$ there exists a fine neighbourhood V of x such that $V \subset D$, $\varepsilon_x^{X \setminus V}(f) \leq f(x)$ where $\varepsilon_x^{X \setminus V}$ means the H-measure on X given by

$$s \to B^{X \setminus V} s(x) .$$

3) Any positive and fine lower semicontinuous function f on X belongs to $\mathcal{S}$ if:

a) f is finite on a finely dense subset of X.

b) for any $x \in X$ there exists a fundamental system $\mathcal{V}_x$ of fine neighbourhoods of x such that $\varepsilon_x^{X \setminus V} f \leq f(x)$ for any $V \in \mathcal{V}_x$.

<u>Proof</u>. 1) $\Rightarrow$ 2) Let p be a positive, Borel and fine lower semicontinuous function on X which satisfies the condition a) and b) from the statement 2. Using Theorem 2.3 we deduce that $\mathcal{S}$ satisfies the axiom of nearly continuity and therefore from Theorem 1.7 it follows that f is quasi-lower semicontinuous on X. Using [4] Proposition 1.4 we get the existence of a sequence $(s_n)_n$ of $\mathcal{S}$ such that the function $f_n := s_n + f$ is lower semicontinuous for any $n \in N$ and $\bigwedge_n s_n = 0$. Since for any $n \in N$ the function f_n is lower semicontinuous and for any $x \in X$ and any open subset D of X with $x \in D$ there exists a fine neighbourhood V of x $V \subset D$ such that $\varepsilon_x^{X \setminus V}(f_n) \leq f_n(x)$ we deduce, using [4], Theorem 3.5, that $f_n \in \mathcal{S}$. Let s be the element of $\mathcal{S}$ defined by $s = \bigwedge_n f_n$. Obviously we have

$$f \leq s \leq f + \inf_n s_n \quad .$$

Since $\bigwedge_n s_n = 0$ the set

$$P := [\inf s_n > 0]$$

is a Borel and polar subset of X and we have f=s on $X\setminus P$.

We show that for any $x_o \in P$ we have also $f(x_o)=s(x_o)$. Since the set $\{x_o\}$ is polar we deduce that for any decreasing sequence $(D_n)_n$ of open neighbourhoods of x_o which is a fundamental system of neighbourhoods of x_o we have

$$\sup_n B^{X\setminus D_n} s(x_o) = \lim_{n\to\infty} B^{X\setminus D_n} s(x_o) = s(x_o) .$$

If we take for any $n \in N$ a fine neighbourhood V_n of x_o such that $V_n \subset D_n$ and such that $\varepsilon_{x_o}^{X\setminus V_n} f \leq f(x_o)$ we deduce that we have

$$\varepsilon_{x_o}^{X\setminus V_n}(s) = B^{X\setminus V_n} s(x_o) \geq B^{X\setminus D_n} s(x_o)$$

and therefore, since the measure $\varepsilon_{x_o}^{X\setminus V_n}$ does not charge any polar subset of $X\setminus\{x_o\}$ we deduce the following relation

$$B^{X\setminus D_n} s(x_o) \leq \varepsilon_{x_o}^{X\setminus V_n}(s) = \varepsilon_{x_o}^{X\setminus V_n}(f) \leq f(x_o)$$

Hence $s(x_o) \leq f(x_o)$, s=f.

The assertion 2) $\Rightarrow$ 3) follows using [4], Theorem 4.4.

3) $\Rightarrow$ 1). Let p be an universally continuous element of $\mathcal{S}$ and let $(G_n)_n$ be an increasing sequence of fine open subsets of X such that $\overline{G}_n^f \subset G_{n+1}$ for any $n \in N$ and such that $\bigcup_n G_n = X$. We have

$$q := \bigwedge_n B^{X\setminus G_n} p = \inf_n B^{X\setminus G_n} p$$

and for any $x \in X$ there exists $n_o \in N$ with $x_o \in G_{n_o}$. If we take a fine neighbourhood V of x_o such that $\overline{V} \subset G_{n_o}$ we have

$$B^{X\setminus V} B^{X\setminus \overline{G}_n^f} p = B^{X\setminus \overline{G}_n^f} p \qquad (\forall) \quad n \geq n_o$$

and therefore

$$B^{X\setminus V}q(x_o) = \varepsilon_{x_o}^{X\setminus V}q = \inf_n \varepsilon_{x_o}^{V}(B^{X\setminus G_n}q) =$$

$$= \inf_n B^{X\setminus V}(B^{X\setminus \bar{G}_n^{\mathcal{S}}}q)(x_o) = \inf_n B^{X\setminus \bar{G}_n^{\mathcal{S}}}q(x_o) = q(x_o) \quad .$$

Since the function p-q is a positive fine lower semicontinuous function on X which satisfies the assertion b) from the above statement 3 we deduce that $p-q \in \mathcal{S}$ and therefore $q \prec p$. From the previous considerations we deduce that the fine carrier ([7]) of q is empty and therefore q=0. Hence p is a fine potential on X and therefore there exists a strictly positive fine potential on X.

References

1. N.Boboc, Gh.Bucur: Natural localization and natural sheaf property in Standard H-cones of functions. Preprint Series in Math. 32, 1982, INCREST Bucuresti.
2. N.Boboc, Gh.Bucur: Natural localization and natural sheaf property in standard H-cones of functions I (II). Preprint Series in Math. 44, 1984 (45, 1984) INCREST, Bucuresti.
3. N.Boboc, Gh.Bucur, Natural localization and natural sheaf property in Standard H-cones of functions. Rev.Roum.Math. Pures et Appl. 1, 1-21, 1985, 2, 193-219, 1985).
4. N.Bobob, Gh.Bucur, Potentials and Supermedean functions on fine open sets in Standard H-cones. Preprint Series in Math. 59, 1984 INCREST, Bucuresti or Rev.Roum.Math.Pures et Appl. 9, 745-774, 1986.
5. N.Boboc, Gh.Bucur, Green potentials on standard H-cones. Preprint Series in Math. 35, 1985 INCREST, Bucuresti.
6. N.Boboc, Gh.Bucur, A.Cornea, Order and Convexity in Potential Theory; H-cones. Lecture Notes in Math. 853. Springer Verlag, Berlin-Heidelberg-New YOrk, 1981.
7. N.Boboc, Gh.Bucur, A.Cornea: Carrier theory and negligible sets on standard H-cones of functions, Rec.Roum.Math.Pures et Appl. 9, 163-197, 1980.
8. M.Brelot: Recherches axiomatiques sur un théoréme de Choquet concernant l'effilement Nagoya Math. Journal 29, 39-46, 1967.
9. Gh.Bucur, W.Hansen: Balayage, quasi-balayage and fine decomposition properties in standard H-cones of functions. Rev. Roum.Math.Pures et Appl. 29, 1, 19-41, 1984.

10. C.Constantinescu, A.Cornea, Potential Theory on Harmonic Spaces, Berlin-Heidelberg-New York, Springer 1972.

11. B.Fuglede, Localization in fine potential theory and uniform approximation by subharmonic functions. Journal of functional analysis 49, 57-72, 1982.

12. T.J.Lyons: Cones of lower semicontinuous functions and a characterization of finely hyperharmonic functions, Math. Ann. 261, 293-297, 1982.

13. J.Lukes, J. Malý: Fine hyperharmonicity without axiom D. Math. Ann. 261, 299-306, 1982.

14. J.Malý,Quasitopological Methods in Fine Potential Theory, Preprint 1984, Praha.