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A NOTE ON THE CLASSES $(BCP)_\theta$

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1.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded, linear operators on $\mathcal{H}$.

The purpose of this note is to give a new proof of the fact that $(BCP)_\theta \subset A_{\aleph_0}$ for all $0 \le \theta < 1$ (see below the terminology). Our proof (see Theorem 1) does not involve the more sophisticated techniques of ([1], Chapter VIII) and it uses systematically the minimal co-isometric dilation of a given contraction.

The notation and terminology employed herein agree with that in [1]. Nevertheless, we begin by reviewing some useful definitions from the theory of dual algebras.

If $T \in \mathcal{L}(\mathcal{H})$, then $\sigma_e(T)$ denotes the essential (Calkin) spectrum of T. It is well-known that $\mathcal{L}(\mathcal{H})$ is the dual space of the Banach space (τc) of trace-class operators on $\mathcal{H}$ equipped with the trace-norm. The duality is implemented by the bilinear form

$$\langle T, L\rangle = \mathrm{tr}(TL), \qquad T \in \mathcal{L}(\mathcal{H}), \qquad L \in (\tau c)$$

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ that contains $1_{\mathcal{H}}$ and is closed in the weak-* topology on $\mathcal{L}(\mathcal{H})$ is called a dual algebra. It follows from general principles (cf. [3]) that if $\mathcal{A}$ is a dual algebra, then $\mathcal{A}$ can be identified with the dual space of $Q_{\mathcal{A}} = (\tau c)/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ denotes the preannihilator of $\mathcal{A}$ in (τc), under the pairing

$$\langle T,[L]\rangle = tr(TL), \quad T\in\mathcal{A}, \quad [L]\in Q_{\mathcal{A}}$$

It is also easy to see (cf [3]) that the weak* topology that accrues to $\mathcal{A}$ by virtue of being the dual space of $Q_{\mathcal{A}}$ is identical with the relative weak* topology that $\mathcal{A}$ inherits as a subspace of $\mathcal{L}(\mathcal{H})$.

If x and y are vectors from $\mathcal{H}$, then the rank-one operator $x \otimes y$, defined as usual by $(x \otimes y)(z)=(z,y)x$, $z\in\mathcal{H}$, belongs to (τc) and satisfies $tr(x \otimes y)=(x,y)$. Thus, if $\mathcal{A}$ is a dual algebra, then $[x \otimes y]\in Q_{\mathcal{A}}$.

Let $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ be a dual algebra and let n be any cardinal number such that $1\le n\le\aleph_0$. Then $\mathcal{A}$ is said to have property (A_n) provided every $n \times n$ system of simultaneous equations of the form

$$[L_{ij}] = [x_i \otimes y_j] \quad , \qquad 0\le i,j<n$$

(where $\{[L_{ij}]\}$ are arbitrary but fixed elements from $Q_{\mathcal{A}}$) has a solution $\{x_i\}_{0\le i<n}$, $\{y_i\}_{0\le i<n}$, consisting of a pair of sequences of vectors from $\mathcal{H}$.

The following definitions, introduced in [1], involve properties that dual algebras may have that are related to the properties (A_n).

Suppose $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ is a dual algebra and θ is a nonnegative real number. Then $\mathcal{X}_\theta(\mathcal{A})$ denotes the set of all $[L]$ in $Q_{\mathcal{A}}$ such that there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ satisfying the following conditions:

α) $\limsup_i (\|[x_i \times y_i]-[L]\|) \le \theta$

β) $\|x_i\|\le 1$, $\|y_i\|\le 1$, $1\le i<\infty$

and

γ) $\|[x_i \otimes z]\| + \|[z \otimes x_i]\| + \|[y_i \otimes z]\| + \|[z \otimes y_i]\| \to 0 \qquad (\ ) z$

Suppose now that $0\le\theta<\gamma$. Then a dual algebra ( ) is said to have property $X_{\theta,\gamma}$ if the closed absolutely convex hull of the set $\mathcal{X}_\theta(\mathcal{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $Q_{\mathcal{A}}$. It was proved (see [1] , Theorem 3.7) that if $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ is a dual algebra that have property $X_{\theta,\gamma}$ for some $0\leq\theta<\gamma$, then $\mathcal{A}$ has property $(A_{\aleph_0})$.

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and let $\mathbb{T}=\partial\mathbb{D}$.

A set $\Lambda\subset\mathbb{D}$ is said to be dominating for $\mathbb{T}$ if almost every point of $\mathbb{T}$ is a nontangential limit of a sequence of points from Λ.

The spaces $H^p=H^p(\mathbb{T})$, $1\leq p\leq\infty$ are the usual function spaces.

If $T\in\mathcal{L}(\mathcal{H})$ is an absolutely continuous contraction, we denote by $\mathcal{A}_T$ the dual algebra generated by T in $\mathcal{L}(\mathcal{H})$ and we write $Q_T=Q_{\mathcal{A}_T}$.

For such T, the Sz.-Nagy-Foiaş functional calculus Φ_T is a weak* continuous, norm decreasing algebra homomorphism of H^∞ into $\mathcal{L}(\mathcal{H})$ (cf [4]). The class $\mathbb{A}=\mathbb{A}(\mathcal{H})$ is defined to be the set of all absolutely continuous contractions $T\in\mathcal{L}(\mathcal{H})$, for which Φ_T is an isometry. If $T\in\mathbb{A}$, then one knows (cf [3]) that Φ_T is a weak* homeomorphism between H^∞ and $\mathcal{A}_T$.

If $T\in\mathbb{A}$ and $\lambda\in\mathbb{D}$, then there exists an element $[C_\lambda]\in Q_T$ such that

$$\langle f(T),[C_\lambda]\rangle=f(\lambda),\quad f\in H^\infty.$$

For any cardinal number n satisfying $1\leq n\leq\aleph_0$ the class $\mathbb{A}_n$ consists of all those T in $\mathbb{A}$ for which the dual algebra $\mathcal{A}_T$ has property (A_n).

We close this introductory section by defining an important class of operators, introduced in [2] . If T is any contraction in $\mathcal{L}(\mathcal{H})$ and $\mu\in\mathbb{D}$, let us write T_μ for the Möbius transform

$$T_\mu=(T-\mu I)(I-\bar{\mu}T)^{-1}$$

Then for each $0 \leq \theta < 1$, the class $(BCP)_\theta$ is defined to consist of all completely nonunitary contractions T in $\mathcal{L}(\mathcal{H})$ for which the set

$$\{\mu \in \mathbb{D} : \inf \sigma_e((T_\mu^* T_\mu)^{1/2}) \leq \theta \text{ or } \inf \sigma_e((T_\mu T_\mu^*)^{1/2}) \leq \theta\}$$

is dominating for $\mathbb{T}$. Let us denote

$$L_\theta = \{\mu \in \mathbb{D} : \inf \sigma_e((T_\mu^* T_\mu)^{1/2}) \leq \theta\}$$

and

$$R_\theta = \{\mu \in \mathbb{D} : \inf \sigma_e((T_\mu T_\mu^*)^{1/2}) \leq \theta\}$$

so $T \in (BCP)_\theta$ for some $0 \leq \theta < 1$ if and only if $L_\theta(T) \cup R_\theta(T)$ is dominating for $\mathbb{T}$.

As it was shown in [2] (see also [1], Chapter VIII). $(BCP)_\theta$ operators belong in the class $\mathbb{A}_{\aleph_0}$. In the next section, we shall give a more direct proof of the fact that for every $T \in (BCP)_\theta$, Λ_T has property $X_{\theta+\varepsilon,1}$, where $0 < \varepsilon < 1-\theta$. This enables one to show, using the above quoted result ([1], Theorem 3.7), that $(BCP)_\theta \subset \mathbb{A}_{\aleph_0}$.

2.

The main result of this section is the following

<u>Theorem 1</u>

Suppose $T \in \mathbb{A}$ and $0 \leq \theta < 1$. If $\mu \in \mathbb{D}$ satisfies $\inf \sigma_e((T_\mu^* T_\mu)^{1/2}) \leq \theta$, and $0 < \varepsilon < 1-\theta$, then there exists a sequence $(y_n)_{n=1}^\infty$ of vectors from the unit ball of $\mathcal{H}$ satisfying the following conditions:

$$a)\ \|[C_\mu]-[y_n\otimes y_n]\|\leq\theta+\varepsilon,\qquad (\forall)\, n\in\mathbb{N}$$

and

$$b)\ (\forall)\ z\in\mathcal{H},\ \lim_{n\to\infty}(\|[y_n\otimes z]\|+\|[z\otimes y_n]\|)=0.$$

Proof

Since $T\in\mathbb{A}$, it is easy to see that $T_\mu\in\mathbb{A}$ and that for all x,y in $\mathcal{H}$ we have:

$$1)\ \|[C_\mu]-[x\otimes y]\|_{Q_T}=\|[C_o]-[x\otimes y]\|_{Q_{T_\mu}}$$

and

$$2)\ \|[x\otimes y]\|_{Q_T}=\|[x\otimes y]\|_{Q_{T_\mu}}.$$

These comments show that we may assume that $\mu=0$.

Since $\inf\sigma_e((T_\mu^*T_\mu)^{1/2})\leq\theta$, it follows easily from the spectral theorem that we may choose an orthonormal sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{H}$ such that $\|Tx_n\|\leq\theta+\varepsilon$ for all $n\geq 1$.

Let $V\in\mathcal{L}(\mathcal{K})$ be the minimal isometric dilation of T^*. The space $\mathcal{K}$ may be decomposed as a direct sum $\mathcal{K}=\ell^2(\mathcal{F})\oplus\mathcal{R}$, where $\mathcal{F},\mathcal{R}$ are Hilbert spaces and $\ell^2(\mathcal{F})$ is the space of square summable sequences (indexed by Z^+) in $\mathcal{H}$. With respect to this decomposition, we have $V=V_1\oplus U$, where V_1 is a unilateral shift and U is an absolutely continuous unitary operator. We also have $T=V^*|_{\mathcal{H}}$ (see [4], Chapters 2 and 3).

Let us consider the orthogonal projection P of $\mathcal{K}$ onto $\operatorname{Ker} V^*$. Then we have

$$\|Px_n\|^2=\|x_n\|^2-\|VV^*x_n\|^2=1-\|Tx_n\|^2\geq 1-(\theta+\varepsilon)^2$$

Let $z_n = \frac{Px_n}{\|Px_n\|}$ and $y_n = P_{\mathcal{H}} z_n$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. We show that $(y_n)_{n=1}^{\infty}$ satisfies a) and b).

First of all, let us remark that

$$\|y_n\|^2 = \frac{\|P_{\mathcal{H}} P x_n\|^2}{\|Px_n\|^2} \geq \frac{|(P_{\mathcal{H}} P x_n, x_n)|^2}{\|Px_n\|^2} = \|Px_n\|^2 \geq 1-(\theta+\varepsilon)^2 = \gamma^2$$

hence

$$\|y_n - z_n\|^2 = 1 - \|y_n\|^2 \leq (\theta+\varepsilon)^2.$$

Thus, we obtain

$$\|[C_0] - [y_n \otimes y_n]\|_{Q_T} = \|[C_0] - [P_{\mathcal{K}} z_n \otimes P_{\mathcal{K}} z_n]\|_{Q_{V^*}} =$$

$$= \|[z_n \otimes z_n] - [P_{\mathcal{K}} z_n \otimes z_n]\|_{Q_{V^*}} \leq \|y_n - z_n\| \leq \theta+\varepsilon$$

for all $n \geq 1$, hence $(y_n)_{n=1}^{\infty}$ satisfies a).

Let $x \in \mathcal{H}$ and write $x = x^1 \oplus x^2$, where $x^1 \in \ell^2(\mathcal{F})$ and $x^2 \in \mathcal{R}$. Then we have

$$\|[x \otimes y_n]\|_{Q_T} = \|[x \otimes y_n]\|_{Q_{V^*}} =$$

$$= \sup_{\substack{f \in H^\infty \\ \|f\|_\infty \leq 1}} |(f(V^*)x, y_n)| = \sup_{\substack{f \in H^\infty \\ \|f\|_\infty \leq 1}} |(f(V_1^*)x^1, z_n)| =$$

$$= \|[x^1 \otimes z_n]\|_{Q_{V_1^*}}.$$

Since $z_n \xrightarrow[W]{} 0$ and $V_1^* \in C_0$, this last term tends to 0 (cf [1], Proposition 6.5).

Now, let us show that $\|[y_n \otimes x]\| \to 0$ as $n \to \infty$.

First, we remark that for every $f \in H^\infty$ and for all $n \geq 1$, we have:

$$(f(T)y_n,x)=(y_n,\ \tilde{f}(T^*)x) = (P_H z_n,\ \tilde{f}(T^*)x)=$$

$$=(z_n,\tilde{f}\ (T^*)x)=((I-VV^*)x_n',\tilde{f}(T^*)x)=$$

$$=((I-T^*T)x_n',\ \tilde{f}(T^*)x)=(D_T^2x_n',\ \tilde{f}(T^*)x),$$

where $D_T=(I-T^*T)^{1/2}$, $\tilde{f}(z)=\overline{f(\bar{z})}$, $f\in H^\infty$, $x_n'=\frac{x_n}{\|Px_n\|}$

Suppose now that the sequence $\{\|[y_n \otimes x]\|\}_{n=1}^{\infty}$ does not converge to 0. Then one may find a subsequence $(n_j)_{j=1}^{\infty}$ and a sequence $(g_j)_{j=1}^{\infty}$ in the unit sphere of H^∞ such that $g_j \xrightarrow[W^*]{} 0$ and such that

$$\delta=\inf_j|((g_j(T)y_{n_j},x)|>0$$

(see the proof of [3], Lemma 4.5).

Using the fact that $\{\|D_{T^*}T^{*n}x\|\}_{n=1}^{\infty}$ converges to 0, we fix a positive integer N such that $\|D_{T^*}T^{*N}x\|<\frac{\delta\gamma}{6}$. Then one may write, as in the above mentioned proof

$$g_j(z)=p_j(z)+z^Nm_j(z),\quad z\in D$$

where $p_j,\ m_j\in H^\infty$, $\|p_j\|_\infty\to 0$ as $j\to\infty$ and $\|m_j\|_\infty\leq 3$, for all j. Thus, we obtain

$$\delta\leq|(g_j(T)y_{n_j},x)|\leq|(p_j(T)y_{n_j},x)|+$$

$$+|(m_j(T)T^Ny_{n_j},x)|\leq\|p_j\|_\infty\|x\|+\|D_{T^*}T^{*N}x\|\,\|m_j\|_\infty\frac{1}{\gamma}\leq$$

$\leq\|p_j\|_\infty\|x\|+\frac{\delta}{2}$, which is impossible for j sufficiently large. The proof is complete.

Theorem 1 enables to us to give a new proof of the following known result:

<u>Theorem 2</u> ([1], Theorem 5.2).

Suppose $0\leq\theta<1$ and $T\in(BCP)_\theta$. Then $T\in\mathbb{A}_{\aleph_0}$

Proof

Fix $T\in(BCP)_\theta$ and take $0<\varepsilon<1-\theta$. First, we show that $T\in\mathbb{A}$. Indeed, for each $\mu\in L_\theta(T)\cup R_\theta(T)$ one can find a vector y_μ in the unit ball of $\mathcal{H}$ such that

$$|f(\mu)-(f(T)y_\mu,y_\mu)|\leq(\theta+\varepsilon)\|f\|_\infty \quad \text{for all } f \text{ in } H^\infty.$$

The vector $y_\mu\in\mathcal{H}$, with the above mentioned properties can be constructed by the same techniques as in the proof of Theorem 1.

Taking the supremum as μ varies over the dominating set $L_\theta\cup R_\theta$, we obtain

$$(1-\theta-\varepsilon)\|f\|_\infty\leq\|f(T)\| \qquad (\forall)\ f\in H^\infty$$

Since this inequality is valid for all $f\in H^\infty$, we can replace f by f^n, obtaining

$$(1-\theta-\varepsilon)\|f\|_\infty^n\leq\|f(T)\|^n.$$

Taking n-th roots and then letting n tend to infinity yelds $\|f\|_\infty\leq\|f(T)\|$ hence $T\in\mathbb{A}$. Now, if $\mu\in L_\theta$, then it follows from Theorem 1 that $[C_\mu]\in\mathcal{X}_{\theta+\varepsilon}(\mathcal{A}_T)$. The dual case $\mu\in R_\theta$ is similar, hence, for all $\mu\in L_\theta\cup R_\theta$, we have $[C_\mu]\in\mathcal{X}_{\theta+\varepsilon}(\mathcal{A}_T)$. Since the set $L_\theta(T)\cup R_\theta(T)$ is dominating for T, it follows from ([1], Prop.1.21) that $\overline{aco}\{[C_\mu];\ \mu\in L_\theta(T)\cup R_\theta(T)\}=(Q_T)_1$, hence $\mathcal{A}_T$ has property $X_{\theta+\varepsilon,1}$ and consequently $T\in\mathbb{A}_{\aleph_0}$ (Cf. [1], Theorem 3.7).

References

1. H.Bercovici, C.Foiaş and C.Pearcy, Dual algebras with applications to invariant subspaces and dilation theory, CBMS Regional Conference Series in Math.No.56, A.M.S. Providence, 1985.
2. ______________________, Factoring trace class operator-valued functions with applications to the class A . J.Operator Theory,14(1985),351-389.
3. S.Brown, B.Chevreau and C.Pearcy, Contractions with rich spectrum have invariant subspaces, J.Operator Theory 1(1979), 123-136.
4. B.Sz.-Nagy and C.Foiaş. Harmonic analysis of operators on Hilbert space, North Holland, Amsterdam, 1970.

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