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Abstract. Preconditioning the discrete system obtained by the Galerkin method for a bounded elliptic linear problem by the Choleski factors of the Gram matrix of basis we obtain an mesh independent spectral condition number. Moreover, the rate of convergence for the multigrid algorithm with Richardson relaxation for the preconditioned system is also mesh independent.

1. Introduction. Let $\mathcal{H}$ be an real separable Hilbert space, $\mathcal{A}$ an invertible linear operator on it and let $\{\mathcal{H}_k\}$, $k=0,1,2,\ldots$ an including finite dimension subspaces sequence in $\mathcal{H}$, where every subspace $\mathcal{H}_k$ is considered be spanned by a linear independent family $\{\phi_k^j \mid j=1,n_k\}$ in $\mathcal{H}$.

Given $f^* \in \mathcal{H}$, for the following problem in $\mathcal{H}$:

$$\mathcal{A}u = f^*, \tag{1.1}$$

using the Galerkin method corresponding to $\mathcal{H}_k$ - i.e. to find $u_k \in \mathcal{H}_k$ such that the reziduum $f^* - \mathcal{A}u_k$ be orthogonal onto $\mathcal{H}_k$ - we obtain the discrete approximation system in the real space $\mathbb{R}_k$, $\dim \mathbb{R}_k = \dim \mathcal{H}_k = n_k$:

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(1.2) $$A_k \bar{u}_k = \bar{f}_k^*$$

where the matrix A_k and the vector $\bar{f}_k^*$ have the entries $a_{ij}^* = \langle \mathcal{A}\phi_k^j, \phi_k^i \rangle$ and $(\bar{f}_k^*)_j = \langle f^*, \phi_k^j \rangle$, $i,j=1,n_k$. We consider in the following that the inner product in $\mathcal{H}_k$ is same as in $\mathcal{H}$ and the real space $\mathbb{R}_k$ is equipped with the Euclidean inner product denoted by the subscript k . Moreover, we look (1.2) as a operator equation on $\mathbb{R}_k$. The condition number of A_k is defined by

$$\mathrm{cond}(A_k) = \| A_k \|_k \, \| A_k^{-1} \|_k$$

where we denote by $\|A_k\|_k$ the spectral norm of A_k.
It is known that the condition number for the boundary value problems in the Sobolev spaces grows exponentially for the nodal basis of finite element. Particularly, Yserentant shows ([5]) that, when: $\mathcal{H}$ is the Sobolev space $H^1(\Omega)$ or a closed subspace of it; Ω is a plane bounded domain; $\mathcal{A}$ is the Riesz representation of a bilinear form corresponding to a elliptic boundary value problem with $\mathcal{H}$ the solution space; $\mathcal{H}_k$ is the space of finite element functions piecewise linear on a triangulation of $\bar{\Omega}$ and continuous on it; the hierarchical bases is used, i.e. $\phi_k^j = \phi_{k-1}^j$, $j=1,n_{k-1}$, $k=1,2,\ldots$, then the condition number grows quadratically with the number of levels k, for the preconditioned discrete matrix with the Choleski factors of the matrix C_k with the entries $c_{ij} = \langle \phi_k^i, \phi_k^j \rangle$, $i,j=1,n_o$ and $c_{ij} = \delta_{ij}$ otherwise, i.e.:

$$\mathrm{cond}(L_k^{-1} A_k L_k^{-*}) \leq \frac{\beta}{\alpha} \frac{K_1}{K_2} (k+1)^2 ,$$

where β and α are the constants of boundness and ellipticity and k_1, k_2 are constants what contains the influence of the finite element spaces. By L_k^{-*} is denoted the inverse of the transpose of L_k, where L_k^* are the Choleski factors: $C_k = L_k L_k^*$.

In this paper we generalize this result obtaining mesh independent condition number for the preconditioned discrete matrix with the Choleski factors of the Gram matrix of basis and some property for the multigrid rate of convergence.

Now, notting that the entries of the solution vector of the problem (1.2) are the coefficients of the function $u_k \in \mathcal{H}_k$ in the given basis, u_k being the discrete approximation solution of (1.1), we define the bijection operator j^k on $\mathbb{R}_k$ onto $\mathcal{H}_k$ by

$$(1.3) \qquad j^k e_k^j = \phi_k^j \ , \quad j=1,n_k,$$

where $\{e_k^j \mid j=1,n_k\}$ is the natural basis of $\mathbb{R}_k$. Hence, $u_k = j^k \bar{u}_k$ and for every vector in $\mathbb{R}_k$ we denote the corresponding function in $\mathcal{H}_k$, by the operator j^k, suppriming the overbare.
Let P_k be the orthogonal projection operator corresponding to $\mathcal{H}_k$.
For every $k \geq 1$ we consider the following discretization diagram $\mathcal{D}_k \equiv \mathcal{D}_k(P_k, P_{k-1}, j_k, j_{k-1}, I_k^{k-1})$:

$$(\mathcal{D}_k) \qquad \begin{tikzcd} \mathcal{H} \arrow[r, "P_k"] \arrow[d, leftrightarrow, "id"'] & \mathcal{H}_k \arrow[r, "J_k"] & \mathbb{R}_k \arrow[d, "I_k^{k-1}"] \\ \mathcal{H} \arrow[r, "P_{k-1}"'] & \mathcal{H}_{k-1} \arrow[r, "J_{k-1}"'] & \mathbb{R}_{k-1} \end{tikzcd}$$

where j_k, j_{k-1}, I_k^{k-1} are the adjoint linear operators of j^k, j^{k-1} defined by

$$(1.4) \qquad \langle j_k u_k, \bar{v}_k \rangle_k = \langle u_k, j^k \bar{v}_k \rangle$$

- analogously for k-1 level - and an full rank linear operator I_{k-1}^{k} defined on $\mathbb{R}_{k-1}$ into $\mathbb{R}_k$, by

(1.5) $$\langle I_{k-1}^{k}\bar{u}_{k-1}, \bar{v}_k\rangle_k = \langle \bar{u}_{k-1}, I_k^{k-1}\bar{v}_k\rangle_{k-1}$$

for every $u_k \in \mathcal{H}_k$, $\bar{v}_k \in \mathbb{R}_k$, $\bar{u}_{k-1} \in \mathbb{R}_{k-1}$.

This diagram produces for every linear operator $\mathcal{A}$ on $\mathcal{H}$ the linear operator A_k on $\mathbb{R}_k$ defined by

(1.6) $$A_k = J_k \mathcal{A}_k J^k$$

where $\mathcal{A}_k$ is the linear operator $\mathcal{A}_k = P_k \mathcal{A} P_k$, the $\mathcal{H}_k$- discrete approximation of $\mathcal{A}$. When j^k is defined by (1.3), then the matrix representation of A_k in the natural basis is the Galirkin matrix and then exists a pair of the intergrid transfer operators (I_{k-1}^{k}, I_k^{k-1}) determined by the representation of the $\mathcal{H}_{k-1}$ - basis in $\mathcal{H}_k$ such that the diagram $(\mathcal{D}_k)$ is commutative

(1.7) $$J_{k-1}P_{k-1} = I_k^{k-1} J_k P_k$$

Moreover, the Nicolaides variational relation holds (see [1]):

(1.8) $$A_{k-1} = I_k^{k-1} A_k I_{k-1}^{k}$$

Now, let f^* and $\mathcal{A}$ be the Riesz representations of an bounded linear functional f on $\mathcal{H}$ and respective of an bounded elliptic bilinear form on $\mathcal{H}$, i.e. there exist α and β such that

(1.9) $$|a(u,v)| \leq \beta \|u\| \|v\|$$

(1.10) $$a(u,u) \geq \alpha \|u\|^2 \quad , \text{ for every } u, v \in \mathcal{H}.$$

Then (1.1) is equivalent with the following problem: given f, to find $u \in \mathcal{H}$ such that for every $v \in \mathcal{H}$ holds:

$$(1.11) \qquad a(u,v)=f(v)$$

We note that for every $\bar{u}_k, \bar{v}_k \in \mathbb{R}_k$ we have on level k:

$$\langle A_k \bar{u}_k, \bar{v}_k \rangle_k = \langle \mathcal{A}_k u_k, v_k \rangle = a(u_k, v_k)$$

where with the our convention $u_k = J^k \bar{u}_k \in \mathcal{H}_k$; and the discrete problem on $\mathbb{R}_k$ (1.2) corresponds to the discrete problem on $\mathcal{H}_k$: $\mathcal{A}_k u_k = f_k^*$, where $f_k^* = P_k f^*$, and $\mathcal{A}_k$ being invertible operator on $\mathcal{H}_k$ because (1.10) holds on $\mathcal{H}_k$.

We mention the following result due to [1] : if one of the hypothesis is fullfied: i) there exists are uniform equivalence relation of $\mathcal{H}_k$ - and $\mathbb{R}_k$ - norms:

$$c_1 \| J^k \bar{u}_k \| \leq \gamma_k \| \bar{u}_k \|_k \leq c_2 \| J^k \bar{u}_k \|$$

ii) the family $\{\phi_k^j \mid j=1,n_k\}$ is orthonormale in $\mathcal{H}$, then the condition number of the discretized matrix is mesh independent: $\mathrm{cond}(A_k) \leq C$, where C not depend of level k . But, when $\mathcal{H}$ is an - or included into Sobolev space $H^m(\Omega)$, $m \geq 1$, and $\mathcal{H}_k$ is an finite element subspace of it, then i) is not found with c_1 independent of k- because a such uniform equivalence relation hold with $L_2(\Omega)$-norm and ii) is no practical modality to be used.

2. Mesh independent preconditioning. The discretization diagram $\mathcal{D}_k$ permits to consider the following linear operator on $\mathbb{R}_k$ defined by:

$$G_k = J_k J^k \tag{2.1}$$

where J_k is the adjoint of J^k defined by (1.4).

2.1. Lemma. The matrix representation of the linear operator G_k in the natural basis of $\mathbb{R}_k$ is the Gram matrix corresponding to the family $\{\phi_k^j\ j=1,n_k\}$.

Proof. Let c_{ij}^k, $i,j=1,n_k$ the entries of the matrix representation of G_k; because

$$\begin{aligned} c_{ij}^k &= \langle G_k e_k^j, e_k^i\rangle_k = \langle J_k J^k e_k^j, e_k^i\rangle_k \\ &= \langle J^k e_k^j,\ J^k e_k^i\rangle = \langle \phi_k^j, \phi_k^i\rangle \end{aligned}$$

this entries are the Gram matrix of the considered family entries.

For to simplify the notations we use a common notation both the linear operator on $\mathbb{R}_k$ and their matrix representation in the natural basis.

Now, corresponding to Choleski factorization into low and upper diagonal matrix, let following factorization in $\mathbb{R}_k$ of the linear operator G_k

$$G_k = L_k L_k^* \tag{2.2}$$

where $L_k : \mathbb{R}_k \to \mathbb{R}_k$ has a lowdiagonal matrix representation in the natural basis and L_k^* is their adjoint. A immediate consequence of this factorization is: for every $\bar{u}_k \in \mathbb{R}_k$, we have:

$$\|u_k\| \equiv \|J^k \bar{u}_k\| = \|L_k^* \bar{u}_k\|_k \tag{2.3}$$

because $\|u_k\|^2 = \langle J^k \bar{u}_k, J^k \bar{u}_k\rangle = \langle J_k J^k \bar{u}_k, \bar{u}_k\rangle_k$

$$=\langle G_k\bar{u}_k,\bar{u}_k\rangle_k=\langle L_kL_k^*\bar{u}_k,\bar{u}_k\rangle_k$$
$$=\|L_k^*\bar{u}_k\|_k^2 .$$

The preconditioned system considered here is:

(2.4) $\hat{A}_k\hat{u}_k=\hat{f}_k^*$, where

(2.5) $\hat{A}_k=L_k^{-1}A_kL_k^{-*}$

(2.6) $\hat{f}_k^* = L_k^{-1}\bar{f}_k^*$

and $\bar{u}_k$ is solution of (1.2) iff $\hat{u}_k=L_k^*\bar{u}_k$ is solution of (2.4).

<u>2.1. Theorem</u>. The spectral condition number of the preconditioned Galerkin matrix $\hat{A}_k$ is mesh independent

(2.7) $\operatorname{cond}(\hat{A}_k) \le \frac{\beta}{\alpha}$

indeed, the following estimations hold:

(2.8) $\|\hat{A}_k\|_k\le\beta$, $\|\hat{A}_k^{-1}\|_k\le 1/\alpha$

<u>Proof</u>. If (2.8) holds, then we obtain immediate (2.7). Using (2.5), (2.3) and (1.12) we have; for every $\bar{u}_k,\bar{v}_k\in\mathbb{R}_k$:

$$|\langle\hat{A}_kL_k^*\bar{u}_k,L_k^*\bar{v}_k\rangle_k|=|\langle A_k\bar{u}_k,\bar{v}_k\rangle_k|=|a(u_k,v_k)|$$
$$\le\beta\|u_k\|\|v_k\|=\beta\|L_k^*\bar{u}_k\|_k\cdot\|L_k^*\bar{v}_k\|_k$$

This implies the first inequality in (2.8). For the last, let:

$$
\begin{aligned}
\|\hat{A}_k^{-1}\bar{u}_k\|_k^2 &= \|L_k^*(A_k^{-1}L_k\bar{u}_k)\|_k^2 = \|J^k(A_k^{-1}L_k\bar{u}_k)\|^2 \\
&\leq \frac{1}{\alpha}|a(J^kA_k^{-1}L_k\bar{u}_k, J^kA_k^{-1}L_k\bar{u}_k)| \\
&= \frac{1}{\alpha}|\langle \mathcal{A}_k J^kA_k^{-1}L_k\bar{u}_k, J^kA_k^{-1}L_k\bar{u}_k\rangle| \\
&= \frac{1}{\alpha}|\langle L_k\bar{u}_k, A_k^{-1}L_k\bar{u}_k\rangle_k| \\
&\leq \frac{1}{\alpha}|\langle \bar{u}_k, \hat{A}_k^{-1}\bar{u}_k\rangle_k| \\
&\leq \frac{1}{\alpha}\|\hat{A}_k^{-1}\bar{u}_k\|_k\cdot\|\bar{u}_k\|_k
\end{aligned}
$$

from (1.7), (1.10) and (2.3). Then, for every $\bar{u}_k \in \mathbb{R}_k$, $\|\hat{A}_k^{-1}\bar{u}_k\|_k \leq \frac{1}{\alpha}\|\bar{u}_k\|_k$ and by this we have proved the theorem. ∎

2.1. Remark. The preconditioned system verifies the Nicolaides variational relations

(2.9) $$\hat{I}_k^{k-1} = (\hat{I}_{k-1}^k)^*; \quad \hat{A}_{k-1} = \hat{I}_k^{k-1}\hat{A}_k\hat{I}_{k-1}^k$$

where the "preconditioned" transfer operators are defined by

(2.10) $$\hat{I}_{k-1}^k = L_k^* I_{k-1}^k L_{k-1}^{-*}, \quad \hat{I}_k^{k-1} = L_{k-1}^{-1} I_k^{k-1} L_k$$

and this is easy to see from (1.8) and (2.5).

The next lemma offers an important tool in multigrid estimations.

2.2. Lemma. For every $\bar{u}_k \in \mathbb{R}_k$ holds

(2.11) $$\|\hat{I}_k^{k-1}\bar{u}_k\|_{k-1} \leq \|\bar{u}_k\|_k$$

i.e. $\|\hat{I}_k^{k-1}\| \leq 1$, and for every linear operator $\mathcal{D}_{k-1}$ on $\mathbb{R}_{k-1}$:

(2.12) $\qquad \|\hat{I}^{k}_{k-1}D_{k-1}\hat{I}^{k-1}_{k}\|_k \leq \|D_{k-1}\|_{k-1}$

Proof. Firstly we observe that

$$\hat{I}^{k}_{k-1}\hat{I}^{k-1}_{k} = L^{*}_{k}(J^{k})^{-1}\{(J^{k}I^{k}_{k-1})(L_{k-1}L^{*}_{k-1})^{-1}(I^{k-1}_{k}J_{k})\}(J_{k})^{-1}L_{k}$$
$$= L^{*}_{k}(J^{k})^{-1}\{P_{k-1}J^{k-1}(J_{k-1}J^{k-1})^{-1}J_{k-1}P_{k-1}\}(J_{k})^{-1}L_{k}$$
$$= L^{*}_{k}(J^{k})^{-1}P_{k-1}(J_{k})^{-1}L_{k}$$

where we are used the commutativity of the diagram $\mathcal{D}_k$ restricted to $\mathcal{H}_k$: $J_{k-1}P_{k-1}=I^{k-1}_{k}J_{k}$ and the both representations of the Gram operator $G_{k-1}=J_{k-1}J^{k-1}=L_{k-1}L^{*}_{k-1}$. With this, for every $\bar{u}_k \in \mathbb{R}_k$,

$$\|\hat{I}^{k-1}_{k}\bar{u}_k\|^2_{k-1} = \langle \hat{I}^{k}_{k-1}\hat{I}^{k-1}_{k}\bar{u}_k, \bar{u}_k\rangle_k$$
$$= \langle L^{*}_{k}(J^{k})^{-1}P_{k-1}J^{-1}_{k}L_k\bar{u}_k, \bar{u}_k\rangle_k$$
$$= \|P_{k-1}(J^{-1}_{k}L_k\bar{u}_k)\|^2$$
$$\leq \|J^{-1}_{k}L_k\bar{u}_k\|^2 = \langle (J_kJ^{k})^{-1}L_k\bar{u}_k, L_k\bar{u}_k\rangle_k$$
$$= \langle (L_kL^{*}_{k})^{-1}L_k\bar{u}_k,\ L_k\bar{u}_k\rangle_k = \|\bar{u}_k\|^2_k$$

Hence (2.11) holds. Now, for every $\bar{u}_k, \bar{v}_k \in \mathbb{R}_k$,

$$|\langle \hat{I}^{k}_{k-1}D_{k-1}\hat{I}^{k-1}_{k}\bar{u}_k, \bar{v}_k\rangle_k| = |\langle D_{k-1}\hat{I}^{k-1}_{k}\bar{u}_k, \hat{I}^{k-1}_{k}\bar{v}_k\rangle_{k-1}|$$
$$\leq \|D_{k-1}\|_{k-1}\|\hat{I}^{k-1}_{k}\bar{u}_k\|_{k-1}\|\hat{I}^{k-1}_{k}\bar{v}_k\|_{k-1}$$
$$\leq \|D_{k-1}\|_{k-1}\|\bar{u}_k\|_k\|\bar{v}_k\|_k$$

that proves (2.12). ∎

3. Multigrid convergence. As described in the classical literature of multigrid, one iteration of the two-grid algorithm for (2.4) consist in a smoothing step followed by a coarse-grid conection ([3],[4]):

i) smoothing step: given $\hat{u}_k^j$, compute

$$\hat{u}_k^{j,i+1}=(I_k-\hat{D}_k\hat{A}_k)\hat{u}_k^{j,i}+\hat{D}_k\hat{f}_k^* \ , \ i=0,\dots,\nu-1$$

where $\hat{u}_k^{j,0}=\hat{u}_k^j$; let $\tilde{u}_k=\hat{u}_k^{j,\nu}$;

ii) coarse grid correction step: compute

$$\hat{u}_k^{j+1}=\tilde{u}_k^j+\hat{I}_{k-1}^k\hat{A}_{k-1}^{-1}\hat{I}_k^{k-1}(\hat{f}_k^*-\hat{A}_k\tilde{u}_k)$$

This algorithm defines the linear iteration operator on $\mathbb{R}_k$

$$\hat{M}_k=(I_k-\hat{B}_k\hat{A}_k)\hat{R}_k^{\nu} \tag{3.1}$$

where I_k is the identity on $\mathbb{R}_k$ and

$$\hat{B}_k=\hat{I}_{k-1}^k\hat{A}_{k-1}^{-1}\hat{I}_k^{k-1} \tag{3.2}$$

$$\hat{R}_k= I_k-\hat{D}_k\hat{A}_k \tag{3.3}$$

We suppose that a is a bilinear symmetric form, hence $\hat{A}_k$ is symmetric and positive operator on $\mathbb{R}_k$ and in this case we consider the Richardson relaxation process as smoother, i.e.

$$\hat{D}_k=\omega_k I_k \ , \text{with} \ \ \omega_k=1/\beta \ \ \text{for every } k\geq 1 \tag{3.4}$$

For nonsymmetric problems considering $\hat{D}_k=\omega_k^2\hat{A}_k^*$ we obtain same type estimations (see [1]).

3.1. Lemma. For the preconditioned system (2.4) the following smoothing and approximation properties hold:

(3.5) $\|\hat{A}_k \hat{R}_k^{\nu}\|_k \leq \beta g(\nu)$, where

(3.6) $g(\nu) = 3/[8(\nu+1/2)]$ and

(3.7) $\|\hat{A}_k^{-1} - \hat{B}_k\|_k \leq 2/\alpha$

Proof. With $\{\lambda_j\}$ $j=1,n_k$ the eigenvalues of $\hat{A}_k$ and ρ denoting the spectral radius, we obtain

$$\|\hat{A}_k \hat{R}_k^{\nu}\|_k = \max_j [\lambda_j (1-\omega_k \lambda_j)^{\nu}] \equiv \rho(\hat{A}_k \hat{R}_k)$$

$$\leq \frac{1}{\omega_k} \sup_{0 \leq x \leq 1} [x(1-x)^{\nu}] \leq \beta g(\nu)$$

because $0 < \omega_k \lambda_j \leq 1$ from (2.8). The last inequality is given in [3]. Now, by (2.8) and (2.12),

$$\|\hat{A}_k^{-1} - \hat{B}_k\|_k \leq \|\hat{A}_k^{-1}\|_k + \|\hat{I}_{k-1}^k \hat{A}_{k-1}^{-1} \hat{I}_k^{k-1}\|_k \leq 2/\alpha .$$ ∎

Now we to able to give the convergence result of the same type as in [3].

3.1. Theorem of convergence (two-grid algorithm). Let ξ fixed in (0,1) interval. Then there exists ν_0 depending only ξ such that for every $\nu \geq \nu_0$

(3.8) $\|\hat{M}_k\|_k \leq \frac{2\beta}{\alpha} g(\nu) \leq \xi < 1$

i.e. the rate of convergence for the two-grid algorithm of the preconditioned system is bounded by a mesh independent constant.

Proof: It is suffice to choise ν_0 such that $g(\nu_0) \leq \frac{\alpha}{2\beta}\xi$. Then by the monotony of g and by lemma 3.1, for every $\nu \geq \nu_0$:

$$\|\hat{M}_k\|_k \leq \|\hat{A}_k^{-1} - \hat{B}_k\|_k \|\hat{A}_k \hat{R}_k\|_k \leq \frac{2\beta}{\alpha} g(\nu) \leq \xi < 1. \blacksquare$$

Now, let $\ell \geq 2$. With the finner level $\mathcal{H}_\ell$ and coarser level $\mathcal{H}_o$ we can define in a similar way as for two-grid algorithm, the multigrid algorithm changing the exact solver on coarse level k, k=1,...ℓ-1 an multigrid algorithm with the finner grid k. Let ν and γ^μ fixed. The multigrid iteration operator is defined by the following recursion ([4]):

$$\text{(3.9)} \quad \begin{aligned} &\hat{M}_1 = (I_1 - \hat{I}_o^1 \hat{A}_o^{-1} \hat{I}_1^o \hat{A}_1) \hat{R}_1^\nu \\ &\hat{M}_{k+1} = I_{k+1} - \hat{I}_k^{k+1} (I_k - \hat{M}_k^{\gamma^\mu}) \hat{A}_k^{-1} \hat{I}_{k+1}^k \hat{A}_{k+1} \hat{R}_{k+1}^\nu, \qquad k=1,\ldots,\ell-1 \end{aligned}$$

Notting with $\hat{M}_{k+1}^K$ the two-grid iteration operator on (k+1) grid, we obtain

$$\text{(3.10)} \quad \hat{M}_{k+1} = \hat{M}_{k+1}^K + \hat{I}_k^{k+1} \hat{M}_k^{\gamma^\mu} \hat{A}_k^{-1} \hat{I}_{k+1}^k \hat{A}_{k+1} \hat{R}_{k+1}^\nu$$

With $\delta_\nu = \frac{\beta}{\alpha} g(\nu)$

$\sigma_\nu = \frac{2\beta}{\alpha} g(\nu) \geq \|\hat{M}_{k+1}^k\|_{k+1}$ from theorem 3.1,

we obtain

$$\|\hat{M}_{k+1}\|_{k+1} \leq \sigma_\nu + \delta_\nu \|\hat{M}_k\|_k^{\gamma^\mu}$$

from (3.10) via (2.12), (2.8) and (3.5). In [1] we proved the following theorem :

3.2. Theorem of convergence (multigrid algorithm). If $\sigma_\nu + \delta_\nu < 1$, then for every $\mu \geq 1$ there exists ν_0 depending only μ such that for every $\nu \geq \nu_0$

(3.11) $$\|\hat{M}_\ell\|_\ell \leq \eta < 1$$

i.e. the multigrid algorithm for the preconditioned system converges with the rate of convergence mesh independent, where η is the solution of the equation

(3.12) $$f(\eta) \equiv \delta_\nu \eta^\mu - \eta + \sigma_\nu = 0$$

solution which lies in (0.1) interval.

Here ν_0 is the smallest integer for which $g(\nu_0) < \frac{\alpha}{3\beta}$. ∎

We turn out to initial problem (1.2) considering for it the two-grid linear iteration operator M_k

(3.13) $$M_k = (I_k - B_k A_k) R_k$$

where $B_k = I_{k-1}^k A_{k-1}^{-1} I_k^{k-1}$ and $R_k = I_k - \omega_k D_k A_k$.

It is easy to verify that for $D_k = G_k^{-1}$, where G_k is the Gram matrix, we obtain the following connexion between the initial and preconditioned two-grid iteration operators:

(3.14) $$\hat{M}_k = L_k^* M_k L_k^{-*}$$

Because the Grammian is a symmetric and positive definite matrix we can introduce the "G_k-energy" inner product and the corresponding norm on $\mathbb{R}_k$

$$|||\bar{u}_k|||_k = \|L_k^* \bar{u}_k\|_k$$

Denoting by $|||M_k|||$ the induced operator norm by this norm, we obtai the interesting result :

$$\|| M_k \|| = \sup_{\bar{v}_k \neq 0} \frac{\|| M_k \bar{v}_k \||_k}{\|| \bar{v}_k \||_k} = \sup \frac{\|| L_k^{-*} \hat{M}_k L_k^{*} \bar{v}_k \||_k}{\|| \bar{v}_k \||_k}$$

$$= \sup \frac{\| \hat{M}_k (L_k^{*} \bar{v}_k) \|_k}{\| L_k^{*} \bar{v}_k \|_k} = \| \hat{M}_k \|_k$$

Hence the following lemma holds:

3.2. Lemma. For any bounded and elliptic problem we can construct an two-grid algorithm for his Galerkin discretisations using an adequate relaxation process, such that the rate of convergence is mesh independent; i.e. chossing the relaxation by $R_k = I_k - \omega_k G_k^{-1} A_k$ where G_k is the Grammian of the finite element basis, and $\omega_k = 1/\beta$ we have

$$\rho(M_k) \leq \|| M_k \|| = \| \hat{M}_k \|_k \leq \frac{2\beta}{\alpha} g(\nu)$$

for every $\nu \geq \nu_0$ and ν_0 given by the theorem 3.1. ∎

4. Eigenvalue problem. We define the linear operator on $\mathcal{R}_k$ onto $\mathcal{H}_k$, $\hat{J}^k$ by

(4.1) $\qquad \hat{J}^k = J^k L_k^{-*}$.

Then, the preconditioned Galerkin matrix $\hat{A}_k$ is obtained by:

(4.2) $\qquad \hat{A}_k = \hat{J}_k \mathcal{A}_k \hat{J}^k$

where $\hat{J}_k$ is the adjoint of $\hat{J}^k$.

4.1. Theorem. The "preconditioned" discretization diagram $\hat{\mathcal{D}}_{F}$ is a commutative diagram

$$(4.3) \qquad \hat{J}_{k-1}P_{k-1} = \hat{I}_{k}^{k-1}\hat{J}_{k}P_{k}$$

where the linear transformation $\hat{J}_{e}$ is a unitary transformation of $\mathcal{H}_e$ onto $\mathbb{R}_e$

$$(4.4) \qquad \hat{J}^{e} = \hat{J}_{e}^{-1}$$

with $e = k-1$, k, such that for every linear operator $\mathcal{A}$ on $\mathcal{H}$ there exists a unique linear operator $\hat{A}_{e}$ on $\mathbb{R}_e$ fullfiing the following properties:

i) The matrix representation in the natural basis of $\hat{A}_{e}$ is the preconditioned Galerkin matrix.

ii) The Nicolaides variational relation holds:

$$(4.5) \qquad \hat{A}_{k-1} = \hat{I}_{k}^{k-1}\hat{A}_{k}\hat{I}_{k-1}^{k}$$

iii) The $\mathcal{H}_e$ - discrete approximation operator $\mathcal{A}_{e} = P_{e}\mathcal{A}P_{e}$, of $\mathcal{A}$ and the preconditioned Galerkin matrix have same spectrum, i.e.

$$(4.6) \qquad \sigma(\mathcal{A}_{e}) = \sigma(\hat{A}_{e})$$

iv) $\lambda \in \sigma(\hat{A}_{e})$ iff is a solution of the following eigenvalue problem

$$(4.7) \qquad A_{e}\bar{u}_{e} = \lambda G_{e}\bar{u}_{e}$$

where A_e is the Galerkin matrix and G_e is the Grammian of the $\mathcal{H}_e$- basis.

Proof. Because $\mathcal{D}_k$ is a commutative diagram then (4.3), (4.4), (4.5) and i) hold as we proved before. By (4.4) $\hat{A}_e=\hat{J}_e \mathcal{A}_e \hat{J}_e^{-1}$, i.e. $\hat{A}_e$ and $\mathcal{A}_e$ are similar operators, hence they are same spectrum. Now, because $\lambda G_e - A_e = L_e(\lambda I_e - \hat{A}_e)L_e^*$ we obtain: $\lambda G_e \bar{u}_e = A_e \bar{u}_e$ iff $\hat{A}_e \hat{u}_e = \lambda \hat{u}_e$ with $\hat{u}_e = L_e^* \bar{u}_e$ what proves (v). ∎

Remark. Let $\{\hat{u}_k^i\}$ be an orthonormal eigenvectors system for the symmetric positive linear operator $\hat{A}_k$. Then the corresponding $\mathcal{H}_k$ - eigenfunctions system of $\mathcal{A}_k$ $\{\tilde{u}_k^i\}$, is $\mathcal{H}_k$-orthonormal because $\tilde{u}_k^i = \hat{J}_k^{-1}\hat{u}_k^i$, $i=1,n_k$ and the corresponding eigenvectors system of the eigenvalue problem (4.7), $\{\bar{u}_k^i\}$ is G_k-energy inner product orthonormal because $\langle \hat{u}_k^i, \hat{u}_k^j\rangle_k = \langle L_k^* \bar{u}_k^i, L_k^* \bar{u}_k^j\rangle_k = \langle G_k \bar{u}_k^i, \bar{u}_k^j\rangle_k$. Moreover, the spectrum of the Galerkin matrix coincides with the spectrum of the $\mathcal{H}_k$ - discrete approximation operator $\mathcal{A}_k$ iff the Grammian G_k is the identity operator. ∎

4.1. Lemma. The two-grid algorithms on $\mathcal{H}_k$ and $\mathbb{R}_k$ described by the linear iteration operators

$$(4.8) \qquad \mathcal{M}_k=(i_k-\mathcal{P}_{k-1}\mathcal{A}_{k-1}^{-1}\mathcal{P}_{k-1}\mathcal{A}_k)(i_k-\omega_k\mathcal{A}_k)^s$$

and respectively by $\hat{M}_k$ from (3.1) are equivalente in the sense that their have same rate of convergence.

Proof. It easy to observe that the similarity relation holds:

$$(4.9) \qquad \hat{M}_k=\hat{J}_k\mathcal{M}_k\hat{J}_k^{-1}$$

and this proves lemma. ∎

5. Comments. For the hierarchical finite element bases we can split the Gram matrix on level k in

$$G_k = \left[\begin{array}{c|c} G_{k-1} & B_{k-1} \\ \hline B_{k-1}^* & D_{k-1} \end{array}\right]$$

where D_{k-1} is a diagonal matrix when the new functions introduced in the $\mathcal{H}_k$ - basis have disjoint supports. There exists the following recurence between the Choleski factors

$$L_k = \left[\begin{array}{c|c} L_{k-1} & O \\ \hline F_{k-1} & E_{k-1} \end{array}\right] \qquad \text{with}$$

$$L_{k-1}F_{k-1}^* = B_{k-1}$$

$$E_{k-1}E_{k-1}^* = D_{k-1} - B_{k-1}^* G_{k-1}^{-1} B_{k-1}$$

Because D_{k-1} is a diagonal matrix and B_{k-1} is a sparse matrix, we can compute by recurence the Choleski factors using the last relations.

Now, we remark that we no used any regularity property of the continuous problem and approximation of finite element property in our considerations.

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