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Introduction. The aim of this paper is to indicate how to decompose the base space of a standard H-cone of functions in disjoint parts which are parabolic or elliptic (with respect to the H-cone of restrictions). The accepted notions of parabolicity (proposed in [2]) and ellipticity are global ones.

Using increasing families of absorbent sets, the parabolic and elliptic parts are structured (Theorem 3.2).

The decomposition theorems are consequences of a new characterization of the parabolicity (Theorem 2.3).

§1. Preliminaries.

Throughout this paper $\mathcal{S}$ will be a standard H-cone of functions on X.

Recall that a subset A of X is called *absorbent set* (in X) if $A=\{x\in X / s(x)=0\}$ for some $s\in\mathcal{S}$. A subset A of X is absorbent iff $1_{X\setminus A}\in\mathcal{S}$. Every absorbent set is finely open and naturally closed.

The following remark will not be used further on.

Remark. ([2], Theorem 1 and [7], Exercise 7.2.11). Suppose that X is semi-saturated and $\mathcal{S}$ satisfies the natural sheaf property (For these notions see [5].). Then $A\subseteq X$ is absorbent iff it is a finely open and naturally closed subset of X.

Proof. Let $A_1:=X\setminus A$ and $x\in A$. From Proposition 4.3.12 in [4], it follows that the H-measure $B^{A_1^*}_{\varepsilon_x}$ is supported by $A_1=b(A_1)$. On the other hand, Theorem 3.6 in [5] implies that $B^{A_1^*}_{\varepsilon_x}$ is carried by the natural boundary of A_1. Hence $B^{A_1^*}_{\varepsilon_x}\equiv 0$, for every $x\in A$ and therefore $A=\{x\in X / B_1^{X\setminus A}(x)=0\}$. □

For $M\subseteq X$, we denote by $\overline{M}^f$ the fine closure of M.

The following assertions are direct consequences of the standard H-cone properties and of the quasi-Lindelöf principle for the fine topology (see (2.9) in [1]).

1.1 Remark. ([2], Lemma 2.1). If $(A_i)_{i\in I}$ is a family of absorbent sets in X, then :

a) $\overline{\bigcup_{i\in I} A_i}^f$ and $\bigcap_{i\in I} A_i$ are absorbent sets in X.

b) There exists a sequence $(i_n)\subseteq I$ such that $\overline{\bigcup_i A_i}^f=\overline{\bigcup_n A_{i_n}}^f$ and $\bigcap_i A_i=\bigcap_n A_{i_n}$.

If M is a nonempty subset of X, we denote by $\mathcal{S}_M$ the convex cone of the restrictions to M of the elements of $\mathcal{S}$. If M is subbasic (i.e. $M \subseteq b(M)$), then $\mathcal{S}_M$ is a standard H-cone of functions on M isomorphic with the standard H-cone $B^M(\mathcal{S})$ (see [4], Corollary 5.2.6 c)).

From now on, we refer to the (subbasic) set M, understanding that the H-cone of functions on M is $\mathcal{S}_M$.

The fine topology on M (with respect to $\mathcal{S}_M$) is obviously the induced topology on M by the fine topology on X.

1.2 Remark. ([2], Lemma2.2). If A, B are absorbent sets in X, $B \subsetneq A$ (i.e. $B \subseteq A$ and $B \neq A$) and $D \subseteq A \setminus B$, then: D is an absorbent set in $A \setminus B$ iff $D \cup B$ is an absorbent set in A.

§2. Parabolic and Elliptic Parts. Another Characterization of the Parabolicity.

The base space X of the standard H-cone of functions $\mathcal{S}$ is called parabolic ([2], Definition 1.4) if there exists a strictly increasing family $(A_t)_{t\in[0,1]}$ of absorbent sets in X such that :

(2.1) $$\bigcap_{u>t} A_u = A_t = \overline{\bigcup_{s<t} A_s}^f \quad \text{, for all } t\in[0,1].$$

Particularly we have $A_0 = \emptyset$ and $A_1 = X$.

If A, B are absorbent sets in X with $B \subsetneq A$ let :

$$\mathcal{A}_{B,A} := \{D \,/\, D \text{ is absorbent in } X,\ B \subsetneq D \subsetneq A\} .$$

We write $\mathcal{A}_A$ instead of $\mathcal{A}_{\emptyset,A}$.

The following definition is of the same type as that given in [8], §1.

The base space X of $\mathcal{S}$ is called elliptic if $\mathcal{A}_X = \emptyset$.

When we call a (subbasic) subset M of X parabolic or elliptic, we consider it as the base space of the standard H-cone of functions $\mathcal{S}_M$.

2.1 Remark. If $X' \subsetneq X$ is a finely dense subset of X then :

a) The absorbent sets in X' are in bijection, in the natural way, with the absorbent sets in X.

b) X' is parabolic (resp. elliptic) iff X is parabolic (resp. elliptic).

2.2 Lemma. a) If X is elliptic and $M \subseteq X$ is subbasic, then M is elliptic.

b) Let M be a subbasic subset of X. Then M is elliptic iff for every absorbent set A in X either $M \subseteq A$ or $M \subseteq X \setminus A$.

c) Let $(M_i)_{i \in I}$ be a family of elliptic subsets of X with $\bigcap_{i \in I} M_i \neq \emptyset$. Then $\bigcup_{i \in I} M_i$ is elliptic.

d) If $G \subseteq X$ is elliptic and finely open, then: G is a maximal elliptic subset of X iff there exist two absorbent sets in X, A and B, $B \subsetneq A$ such that $G = A \setminus B$.

Proof. The assertions a) and b) are clear.

c) Suppose that $\bigcup_{i \in I} M_i$ is not elliptic and let $A \in \mathcal{A}_X$ such that $\bigcup_{i \in I} M_i \cap A \neq \emptyset$ and $\bigcup_{i \in I} M_i \not\subseteq A$. Hence there exists $i_0 \in I$ with $M_{i_0} \cap A \neq \emptyset$, therefore $M_{i_0} \subseteq A$ by assertion b). The hypothesis $\bigcap_{i \in I} M_i \neq \emptyset$ implies now that $\bigcup_{i \in I} M_i \subseteq A$, which gives a contradiction.

d) From b) easily results that $A \setminus B$ is a maximal elliptic subset of X if A and B are absorbent and $A \setminus B$ is elliptic.

Let G be a maximal elliptic finely open subset of X. We take:

$$A := \bigcap \{ C \,/\, C \text{ is absorbent, } G \subseteq C \}$$

and

$$B := \overline{\bigcup}^f \{ C \,/\, C \text{ is absorbent, } C \subseteq (X \setminus G) \cap A \} .$$

Obviously $G \subseteq A \setminus B$ and G being elliptic, again assertion b) implies that $\mathcal{A}_{B,A} = \emptyset$ hence $G = A \setminus B$ from the maximality of G. □

<u>2.3 Theorem.</u> If $\mathcal{S}$ is a standard H-cone of functions on X, the following statements are equivalent :

a) X is parabolic.

b) X has no finely open elliptic subsets.

c) For every pair (A,B) of absorbent sets in X with $B \subsetneq A$, we have $\mathcal{A}_{B,A} \neq \phi$.

<u>Proof.</u> From Remark 1.2 and Lemma 2.2 c) and d) we already know that b) $\Longleftrightarrow$ c).

" a) $\Longrightarrow$ b)" Let $(A_t)_{t\in[0,1]}$ be the family which describes the parabolicity of X and suppose that there exists a finely open elliptic subset G of X, $G \neq \phi$. If we define $t_0 := \inf\{t\in[0,1] \,/ G\cap A_t \neq \phi\}$, then Lemma 2.2 b) implies that $G \subseteq A_{t_0} \setminus \bigcup_{s<t_0} A_s$, which contradicts (2.1).

" c) $\Longrightarrow$ a)" One can suppose that $\mathcal{S}$ is generated by a bounded potential kernel V which is absolutely continuous with respect to a finite measure μ on X. This is possible because the natural extension $\mathcal{S}_1$ of $\mathcal{S}$ on the base set X_1 is saturated hence $\mathcal{S}_1$ is generated in this way (see [4]; we use also Remark 2.1).

First, let us show that:

(2.2) for every $\beta\in(0,\mu V(X))$ there exists $A\in\mathcal{A}_X$ s.t. $\mu V(A)=\beta$.

If $\mathcal{A}$ is a class of absorbent sets in X, we define:

$$_\beta\mathcal{A} := \{B\in\mathcal{A} \,/\, \mu V(B)\leq\beta\}$$

and

$$^\beta\mathcal{A} := \{B\in\mathcal{A} \,/\, \mu V(B)\geqslant\beta\} .$$

Hypothesis c) implies that at least one of the sets $_\beta\mathcal{A}_X$ and $^\beta\mathcal{A}_X$ is nonempty. Suppose that $_\beta\mathcal{A}_X \neq \phi$. Using Remark 1.1, the set $_\beta\mathcal{A}_X$ results inductively ordered, hence by Zorn's lemma there exists a maximal absorbent set $A \in {_\beta\mathcal{A}_X}$. Suppose that $\mu V(A) < \beta$. We have

$\mathcal{A}_{A,X} \neq \phi$ and ${}_{\beta}\mathcal{A}_{A,X} = \phi$, from the maximality of A in ${}_{\beta}\mathcal{A}_{X}$. Therefore ${}^{\beta}\!\mathcal{A}_{A,X} \neq \phi$ and we can choose a minimal element A' of ${}^{\beta}\!\mathcal{A}_{A,X}$. We obtain $\mathcal{A}_{A,A'} = \phi$, contradicting the hypothesis c). Hence $\mu V(A) = \beta$.

From this moment we can reproduce the proof of Theorem 3.7 in [2]. More precisely, for every dyadic rational number $t \in [0,1]$, an absorbent set (in X) A_t can be constructed by induction and using (2.2), such that:

(2.3) $$\mu V(A_t) = t \cdot \mu V(X) \text{ and if } s < t \text{ then } A_s \subsetneq A_t .$$

The desired family $(A_t)_{t\in[0,1]}$ which verifies (2.1) is obtained extending the construction of A_t for every $t \in [0,1]$, such that (2.3) remains valid.□

A direct consequence of the above theorem is the following:

2.4 Corollary. a) If X is parabolic and $G \subseteq X$ is finely open, then G is parabolic.

b) Let $(M_i)_{i\in I}$ be a family of parabolic (subbasic) subsets of X. Then $\bigcup_{i\in I} M_i$ is parabolic.

c) If G_1 , G_2 are finely open subsets of X such that G_1 is parabolic and G_2 is elliptic, then $G_1 \cap G_2 = \phi$.

§3. Decomposition Theorems. Examples.

We can formulate the following decomposition theorem:

3.1 Theorem. Let $\mathcal{S}$ be standard H-cone of functions on X and P be the greatest finely open parabolic subset of X. Denoting by E the union of all open elliptic subsets of X, we have: $X = P \cup \overline{E}^{f}$ and $P \cap E = \phi$.

Proof. Obviously, from Corollary 2.4 there exists the greatest finely open parabolic subset P of X and $P\cap E=\emptyset$. If D is a finely open subset of X, $D\not\subseteq P$, then from Theorem 2.3 there exists a finely open elliptic subset of D , hence $D\cap E\neq\emptyset$ and the proof is finished.□

If J is a totally ordered set, we denote by $\mathcal{J}(J)$ the set of all ideals of J (i.e. $I\in\mathcal{J}(J)$ iff $I\subseteq J$ and $j\leq i$, $i\in I$ implies $j\in I$).

We present now the structure of the parabolic and elliptic parts, using absorbent sets.

3.2 Theorem. Let $\mathcal{S}$ be a standard H-cone of functions on X and P_a be the greatest parabolic absorbent set in X. If $P_a\neq X$, there exists a totally ordered set J, at most countable, such that:

a) For every $j\in J$, there exists a pair (B_j,C_j) of absorbent sets with: $P_a\subseteq B_j\subsetneq C_j$, $E_j:=C_j\setminus B_j$ is a (maximal) elliptic finely open subset of X and $i<j\Longleftrightarrow C_i\subseteq B_j$.

b) For every $I\in\mathcal{J}(J)$, let us define $B_I:=\overline{\bigcup_{j\in I}C_j}^f$, $C_I:=\bigcap_{j\in J\setminus I}B_j$ and $P_I:=C_I\setminus B_I$ (particularly $P_\emptyset=P_a$ and $P_J=X\setminus\overline{\bigcup}^f\{C_j\ /\ j\in J\}$). Then P_I is parabolic if it is nonempty.

c) $(E_j)_{j\in J}$ is the family of maximal elliptic finely open subsets of X.

d) E and P having the meaning of the above theorem, we have:

$$E=\bigcup\{E_j\ /\ j\in J\} \tag{3.1}$$

and

$$P\subseteq\bigcup\{P_I\cup P_I'\ /\ I\in\mathcal{J}(J)\}\ , \tag{3.2}$$

where for every $I\in\mathcal{J}(J)$, P_I' denotes the totally thin set $\overline{\bigcup_{j\in I}C_j}^f\setminus\bigcup_{j\in I}C_j$.

e) If J is a totally ordered set such that a) and c) are satisfied and if moreover $\mathcal{J}(J)$ is at most countable then

$$\overline{P}^f=\overline{\bigcup}^f\{P_I\ /\ I\in\mathcal{J}(J)\}.$$

Proof. From Corollary 2.4 b) and Remark 2.1 b), there exists the greatest parabolic absorbent set in X, denoted P_a.

Let $\mathcal{F}$ be the set of families $(B_j, C_j)_{j\in J}$, where J is a totally ordered set such that, for every $j\in J$, B_j and C_j are absorbent sets in X with $P_a \subseteq B_j \subsetneq C_j$, $\mathcal{A}_{B_j,C_j} = \phi$ and $i<j \Longleftrightarrow C_i \subseteq B_j$. Theorem 2.3 and $P_a \neq X$ imply that $\mathcal{F}$ is nonempty. The inclusion (with order) between the totally ordered sets induces an order on $\mathcal{F}$. This order relation is inductive and let $(B_j, C_j)_{j\in J}$ be a maximal element of $\mathcal{F}$. Assertion b) is a consequence of the maximality, using again Theorem 2.3.

Let D be a finely open elliptic subset of X and suppose that $D \not\subseteq \bigcup_{j\in J} E_j$. Hence by Lemma 2.2 b) $D \cap \bigcup_{j\in J} E_j = \phi$. If we define $I_o = \{j\in J \,/\, D\cap C_j = \phi\}$ then $I_o \in \mathcal{J}(J)$ and $D \subseteq P_{I_o}$. Therefore, P_{I_o} is not parabolic and this contradicts b). So that assertion c) and (3.1) are proved. Let us remark that under the conditions of a) we have ; b)$\Longleftrightarrow$c).

(3.2) is an obvious consequence of (3.1). As in the proof of Corollary 4.4 in [2] ,one can verify that P'_I is a totally thin set, for every $I \in \mathcal{J}(J)$.

We can suppose that X is saturated, hence the fine interior of any semi-polar set is empty. From (3.2) results now easily that assertion e) is valid.□

3.3 Remark. a) It is possible that J should not be well ordered (see Example 2 below), in contrast with the well ordered family of absorbent sets generated in [6].

b) Not all the sets P_I ,$I \in \mathcal{J}(J)$ are necessarily empty (see Example 3).

In the following examples, the standard H-cone of functions on X will be the cone of positive superharmonic functions on a $\mathfrak{P}$-harmonic space $(X, \mathcal{U})$ with countable base (in the sense of [7]), with $1\in\mathcal{U}(X)$ (see [4],page 113).

Examples. 1. Let $X=(-1,1)$ and for any open set U of X let $\mathcal{U}(U)$ be the set of lower semicontinuous, lower finite numerical functions u on U such that : $u|_{(-1,0]\cap U}$ is locally increasing (see [7],Theorem 2.1.2) and $u|_{(0,1)\cap U}$ is hyperharmonic for the Laplace equation. Obviously we have : $P_a=(-1,0]$ and $E=(0,1)$.

2. Let $X=(-1,1)$ and $F\subseteq X$ be a closed set of X such that $X\setminus F$ is dense in X. Then, following [9](see also [7], Exercice 3.1.17), there exists a Bauer space on X (which contains the constants) such that the absorbent sets (different from ϕ and X) are exactly the intervals $(-1,x]$ with $x\in F$. One can verify that we have obtained a $\mathfrak{P}$-Bauer space. We denote by J the set of all left extremities j of the components (j,j') of the open set $X\setminus F$. If we take $E_j=(j,j']\cap X$ for every $j\in J$, the family $(E_j)_{j\in J}$ satisfies the conditions a) and c) of Theorem 3.2. Obviously $P_I=\phi$, for every $I\in\mathcal{J}(J)$.

A suitable choice of the closed set F proves Remark 3.3 a).

3. Let X be the topological subspace of R^2

$$(-1,1)\times\{0\}\cup\{0\}\times(0,1).$$

For every open set U of X, we denote by $\mathcal{U}(U)$ the set of all lower semicontinuous, lower finite numerical functions u on U such that $(x\mapsto u(x,0))$ is hyperharmonic for the structure introduced in Example 2 and $(y\mapsto u(0,y))$ is locally increasing (where these two functions are defined). We use the notations of the above example, with $X_1=(-1,1)$and suppose that $0<\sup F$. If $A\subseteq X_1$ is a nonempty absorbent set in X_1 then $A\times\{0\}$ is absorbent in X and if $x\in F$, $x\geq 0$ then $(-1,x]\times\{0\}\cup\{0\}\times X_2$ is also absorbent in X, where X_2 denotes the interval $(0,1)$. Let us fix a nonvoid ideal $I_0\in\mathcal{J}(J)$ such that $0<\sup I_0$. Then we can take $B_j:=(-1,j]\times\{0\}$, $C_j:=((-1,j']\cap X_1)\times\{0\}$

if $j \in I_o$ and $B_j := (-1, j] \times \{0\} \cup \{0\} \times X_2$, $C_j := ((-1, j'] \cap X_1) \times \{0\} \cup \{0\} \times X_2$

if $j \in J \setminus I_o$. These are the absorbent sets which give the decomposition of X, with P_a empty and $P_{I_o} = \{0\} \times X_2$ nonempty. In particular, if we take $F = \{\frac{1}{2}\} \cup \{\frac{1}{2} \pm \frac{1}{n} \,/\, n \in N,\ n \geqslant 3\}$ and $I_o = \{-1\} \cup \{\frac{1}{2} - \frac{1}{n} \,/\, n \in N,\ n \geqslant 3\}$, then $P = P_{I_o}$ and $E = X \setminus (P \cup \{(\frac{1}{2}, 0)\})$.

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