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ON INDUCTIVE LIMITS OF CERTAIN C^* - ALGEBRAS OF THE FORM $C(X) \otimes F$

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The study of inductive limits of C^*-algebras of the form $C(X) \otimes F$ (with F finite-dimensional C^*-algebra) has been suggested by E.G.Effros in [5]. Clearly, for this problem, the structure of the *-homomorphisms between algebras of the above form is important. This question has been considered in [1], [2], [8], [9], [10], [11] and [12].

After some preliminaries in section 1, we consider in section 2 *-homomorphisms $\Phi: C(X) \otimes A \to C(Y) \otimes B$ compatible (2.3.) with a map $\Theta: Y \to K(X)$ ($K(X)$ the closed subsets of X) which generalize the homomorphisms compatible with a covering considered in [8]. Our results are more precise in the following two situations:

1^o. $\Theta(y) = \varphi^{-1}(y)$, $\varphi: X \to Y$ a continuous surjection;

2^o. $\Theta(y) = \{\varphi(y)\}$, $\varphi: Y \to X$ continuous (2.7.)

Given a homomorphism, we find conditions that insure the existence of a Θ as in 1^o above with which it is compatible (2.8.). We also improve one of our previous results (Proposition 2.5. in [8]) concerning homomorphisms compatible with a p-fold covering (2.9.).

In section 3 the homomorphisms $C(X)\otimes A\to C(Y)\otimes B$ are unital, A, B, are finite-dimensional and the compact spaces X, Y are metrizable(excepting Proposition 3.1.). Our results describe the local structure of such homomorphisms in terms of continuous "quasifields" of finite-dimensional C^*-algebras (3.1. and 3.4.). Using classes of inner equivalent injective homomorphisms between continuous quasifields of finite-dimensional C^*-algebras (see 3.3.) we study the set of classes of inner equivalent homomorphisms (injective homomorphisms) from $C(X)$ to $C(Y)\otimes B$ (3.4.). A similar analysis is done for the set of all *-homomorphisms (injective *-homomorphisms) from $C(X)\otimes A$ to $C(Y)\otimes B$ which are compatible with a given continuous sujective map from X to Y, the fibre of which satisfies a certain continuity property (3.6.).

Section 4 contains the main result of this paper. Consider a system:

$$C(X_1)\otimes A_1 \xrightarrow{\Phi_1} C(X_2)\otimes A_2 \xrightarrow{\Phi_2} \cdots$$

with X_k, A_k as in section 3. We give conditions under which the above inductive limit is "trivial", in the sense that it coincides with the tensor product of a commutative C^*-algebra with an A.F.-algebra. The assumptions on the spaces X_k involve the vanishing of certain non-abelian cohomologies (this occurs for X_k contractible, for instance). Moreover, it is required that $\Phi_k(C(X_k)\otimes 1_{A_k})\subset C(X_{k+1})\otimes 1_{A_{k+1}}$ (see 4.3.). For such trivial inductive limits we also consider the isomorphism problem (4.4.).

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§1.

For A and B unital C^*-algebras, Hom(A,B) (resp. $\mathrm{Hom}_i(A,B)$) will denote the set of all unital *-homomorphisms (resp. all unital injective *-homomorphisms) from A to B endowed with the topology of pointwise convergence. Z(A) denotes the center and U(A) the group of all unitaries of A. $\Phi, \Psi \in \mathrm{Hom}(A,B)$ are called <u>inner equivalent</u>, $\Phi \sim \Psi$, if $\Phi = \mathrm{Ad}\,u \circ \Psi$ for some $u \in U(B)$. For $M \subset \mathrm{Hom}(A,B)$, we denote by $M/\sim$ the corresponding set of classes of inner equivalent *-homomorphisms.

For a compact topological space X we use the canonical identification $C(X) \otimes A = C(X,A)$. If $f \in C(X) \otimes A$ and $F \subset X$, we denote $\|f_{|F}\| := \sup_{x \in F} \|f(x)\|$ if $F \neq \emptyset$ and $\|f_{|\emptyset}\| := 0$. For a finite-dimensional C^*-algebra $A = \bigoplus_{i \in I} A_i$ (where each A_i is a finite discrete factor) the inclusions $A_i \subset A$, $i \in I$, induce canonical embeddings $C(X) \otimes A_i \subset C(X) \otimes A$, $i \in I$, and we have $C(X) \otimes A = \bigoplus_{i \in I} C(X) \otimes A_i$.

If $\varphi : X \to Y$ is a continuous map between compact spaces, we denote by $\varphi^* : C(Y) \to C(X)$ the map $\varphi^*(f) = f \circ \varphi$, $f \in C(Y)$.

Let G be a topological group, G_c the sheaf of germs of continuous G-valued functions on X and $H^1(X, G_c)$ the corresponding cohomology set; for a contractible compact space X, $H^1(X, G_c)$ reduces to the trivial element ([7]).

§ 2.

Throughout this section X,Y will denote compact spaces and A a finite-dimensional C^*-algebra.

<u>2.1</u>. Consider $A = \bigoplus_{i \in I} A_i$, where I is a finite set and each A_i is a finite discrete factor.

Denote $K(X) := \{F \mid F$ is a non-empty closed (i.e. compact) subset of X$\}$. Consider $\Phi \in \mathrm{Hom}(C(X) \otimes A, C(Y) \otimes B)$, where

B is a unital C^*-algebra. For any $y \in Y$, let $X_{y,\Phi} \in K(X)$ be such that $\{g \in C(X) \mid g|X_{y,\Phi} = 0\}$ is the kernel of the unital *-homomorphism:

$$C(X) \ni g \to \Phi(g \otimes 1_A)(y) \in B$$

Then, for each $y \in Y$, $X_{y,\Phi} \in K(X)$ is determined by the condition:

$$\|\Phi(g \otimes 1_A)(y)\| = \|g|X_{y,\Phi}\|, \quad g \in C(X).$$

In a similar way one sees that for any $y \in Y$ and $i \in I$ there is a unique closed subset $X^i_{y,\Phi}$ of X such that:

$$\|\Phi(f_i)(y)\| = \|f_i|X^i_{y,\Phi}\|, \quad f_i \in C(X) \otimes A_i.$$

Note that $X^i_{y,\Phi}$ can be the empty set. Clearly:

$$X_{y,\Phi} = \bigcup_{i \in I} X^i_{y,\Phi}$$

2.2. For any $f = \bigoplus_{i \in I} f_i \in \bigoplus_{i \in I} C(X_i) \otimes A_i$ and $y \in Y$ we have:

$$(1) \quad \|\Phi(f)(y)\| = \max_{i \in I} \|f_i|X^i_{y,\Phi}\|$$

$$(2) \quad \|\Phi(f)(y)\| \leq \|f|X_{y,\Phi}\|$$

since:

$$\|\Phi(f)(y)\| = \|\sum_{i \in I} \Phi(f_i)(y)\| = \max_{i \in I} \|\Phi(f_i)(y)\| =$$

$$= \max_{i \in I} \|f_i|X^i_{y,\Phi}\| \leq \max_{i \in I} \|f_i|X_{y,\Phi}\| = \|f|X_{y,\Phi}\|.$$

Moreover:

(3) Φ is injective $\Leftrightarrow \bigcup_{y\in Y} X^i_{y,\Phi}=X$ for any $i\in I$.

Indeed, by (1) we have:

$$||\Phi(f)||=\max_{i\in I} ||f_i|\bigcup_{y\in Y} X^i_{y,\Phi}||$$

and each $\bigcup_{y\in Y} X^i_{y,\Phi}$ is closed.

2.3. Consider a map $\Theta:Y\to K(X)$. We say that a $*$-homomorphism $\Phi\in \mathrm{Hom}\,(C(X)\otimes A, C(Y)\otimes B)$, where B is any unital C^*-algebra, is <u>Θ-compatible</u> if:

(1) $X_{y,\Phi}\subset\Theta(y)$, $y\in Y$.

This is equivalent to:

(2) $||\Phi(f)(y)||\leq||f_{|\Theta(y)}||$, $f\in C(X)\otimes A$, $y\in Y$.

Indeed, (1)$\Rightarrow$(2) by 2.2.(2). Conversely, for any $g\in C(X)$ and $y\in Y$ we have $||g|X_{y,\Phi}||=||\Phi(g\otimes 1_A)(y)||\leq||g|\Theta(y)||$ and since $X_{y,\Phi}$ is closed in X it follows that $X_{y,\Phi}\subset\Theta(y)$.

The above argument also shows that $X_{y,\Phi}$ is the smallest non-empty closed subset F of X such that $||\Phi(f)(y)||\leq$ $\leq||f_{|F}||$ for any $f\in C(X)\otimes A$.

2.4. Consider $\Phi\in\mathrm{Hom}(C(X)\otimes A, C(Y)\otimes B)$, where $A=\bigoplus_{i\in I} A_i$, I is a finite set and each A_i is a finite discrete factor, and a map $\Theta:Y\to K(X)$. Then, the following are equivalent:

(1) $||\Phi(f)(y)||=||f|\Theta(y)||$, $f\in C(X)\otimes A$, $y\in Y$.

(2) $X^i_{y,\Phi} = \Theta(y)$, $y \in Y$, $i \in I$.

Indeed (2) $\Rightarrow$ (1) by 2.2.(1). Conversely, for every $i \in I$ and $y \in Y$, we have $\|f_i|X^i_{y,\Phi}\| = \|\Phi(f_i)(y)\| = \|f_i|\Theta(y)\|$, $f_i \in C(X)\otimes A_i$, and since each $X^i_{y,\Phi}$ is closed in X, we deduce $X^i_{y,\Phi} = \Theta(y)$.

2.5. Suppose moreover that $(\Theta(y))_{y \in Y}$ is a partition of X and that Φ is compatible with Θ. Then the following are equivalent:

(1) Φ is injective

(2) $\|\Phi(f)(y)\| = \|f|\Theta(y)\|$, $f \in C(X)\otimes A$, $y \in Y$.

Indeed, (2) $\Rightarrow$ (1) by 2.2.(3) and 2.4. Conversely, suppose there are $i_o \in I$, $y_o \in Y$ such that:

$$X^{i_o}_{y_o,\Phi} \subsetneq \Theta(y_o)$$

Since Φ is compatible with Θ, we have $X^{i_o}_{y,\Phi} \subset \Theta(y)$, $y \in Y$. Then, using 2.2.(3) and the fact that $(\Theta(y))_{y \in Y}$ is a partition of X, one has:

$$X = \bigcup_{y \in Y} X^{i_o}_{y,\Phi} \subsetneq \bigcup_{y \in Y} \Theta(y) = X$$

a contradiction. Hence $X^i_{y,\Phi} = \Theta(y)$, $y \in Y$, $i \in I$, and the conclusion is obtained using again 2.4.

2.6. Proposition. Consider $\Phi \in \mathrm{Hom}\,(C(X)\otimes A,\ C(Y)\otimes B)$ and a map $\Theta: Y \to K(X)$ and suppose there is a unital embedding $B \subset M_n$, for some $n \in \mathbb{N}$. Then Φ is Θ-compatible if and only if:

$$(1)\quad tr(\Phi(g\otimes 1_A)(y)) \in n\cdot co\, g(\Theta(y)),\ g\in C(X),\ y\in Y$$

where tr denotes the usual trace on M_n.

Proof: For any $y\in Y$, consider the unital finite-dimensional $*$-representation $C(X)\otimes A \ni f\to\Phi(f)(y)\in M_n$. Since this is a direct sum of irreducible $*$-representations, it follows that for any $x\in X_{y,\Phi}$ there is a unital $*$-representation $\Pi_{x,y}$ of A such that:

$$(2)\quad \Phi(f)(y) = \bigoplus_{x\in X_{y,\Phi}} \Pi_{x,y}(f(x)) \in M_n$$

for all $f\in C(X)\otimes A$. In particular, in this case, each $X_{y,\Phi}$ is a finite set.

Suppose that Φ is Θ-compatible. Using the above discussion, for $g\in C(X)$ and $y\in Y$ we get:

$$tr(\Phi(g\otimes 1_A)(y)) = \sum_{x\in X_{y,\Phi}} g(x)\cdot \dim \Pi_{x,y} =$$

$$= n\cdot\Big(\sum_{x\in X_{y,\Phi}} g(x)\cdot n^{-1}\cdot \dim\Pi_{x,y}\Big) \in n\cdot co\, g(\Theta(y))$$

since $X_{y,\Phi}\subset\Theta(y)$ and Φ being unital, $\sum_{x\in X_{y,\Phi}} n^{-1}\cdot\dim\Pi_{x,y}=1$.

Conversely, assume (1) and suppose there is $y_0\in Y$ such that $X_{y_0,\Phi}\not\subset\Theta(y_0)$. Then there is $x_0\in X_{y_0,\Phi}\setminus\Theta(y_0)$ and $g_0\in C(X)$ such that $g_0(x_0)=1$ and $g_0\,|\,\Theta(y_0)\cup(X_{y_0,\Phi}\setminus\{x_0\})=0$.

Using (1) and (2) we have:

$$tr(\Phi(g_0\otimes 1_A)(y_0)) = \sum_{x\in X_{y_0,\Phi}} g_0(x)\cdot\dim\Pi_{x,y_0} =$$

$$= \dim \Pi_{x_0, y_0} \not\subset \{0\} = n \cdot co\, g_0(\Theta(y_0))$$

a contradiction.

2.7. Consider in particular the map $\Theta: Y \to K(X)$ given by $\Theta(y) := \{\varphi(y)\}$, $y \in Y$, where $\varphi: Y \to X$ is a continuous map. Then Φ is Θ-compatible if and only if:

$$(1)\quad \Phi(g \otimes 1_A) = g \circ \varphi \otimes 1_B,\ g \in C(X).$$

Indeed, since $X_{y,\Phi} = \{\varphi(y)\}$, we have $\Phi(g \otimes 1_A)(y) = \Pi_{\varphi(y),y}(g(\varphi(y)).1_A) = g(\varphi(y)).1_B$, for any $g \in C(X)$ and $y \in Y$. Conversely, if (1) holds then for any $g \in C(X)$ and $y \in Y$ we have $\|g \mid X_{y,\Phi}\| = \|\Phi(g \otimes 1_A)(y)\| = \|g(\varphi(y))\|$ and since each $X_{y,\Phi}$ is closed, $X_{y,\Phi} = \{\varphi(y)\}$.

On the other hand let B be a finite-dimensional C^*-algebra and $\varphi: X \to Y$ a continuous surjective map. A *-homomorphism $\Phi: C(X) \otimes A \to C(Y) \otimes B$ is said to be <u>φ-compatible</u> if:

$$\Phi(g \circ \varphi \otimes 1_A) = g \otimes 1_B,\ g \in C(Y).$$

If Φ is injective, then φ is uniquely determined by Φ since we can use 2.5.; we have that $(X_{y,\Phi})_{y \in Y}$ is a partition of X and $\varphi^{-1}(y) = X_{y,\Phi}$, $y \in Y$.

Let B, Φ be as in Proposition 2.6. and consider the map $\Theta: Y \to K(X)$ given by $\Theta(y) := \varphi^{-1}(y)$, $y \in Y$, where $\varphi: X \to Y$ is a continuous surjection. In this situation the following assertions are equivalent:

(2) Φ is Θ-compatible

(3) Φ is φ-compatible

(4) $tr(\Phi(g \circ \varphi \otimes 1_A)(y)) = n \cdot g(y)$, $g \in C(Y)$, $y \in Y$.

(tr denotes the usual trace on M_n).

(2) => (3). For any $g \in C(Y)$ and $y \in Y$ we have:

$$\Phi(g \circ \varphi \otimes 1_A)(y) = \bigoplus_{x \in X_{y,\Phi}} \Pi_{x,y}(g(\varphi(x)) \cdot 1_A) = g(y) \cdot 1_B$$

since $X_{y,\Phi} \subseteq \varphi^{-1}(y)$ (we use the notation and remarks made in the proof of Proposition 2.6.).

(3) => (4) is obvious.

(4) => (2) By assumption, for any $g \in C(Y)$ and $y \in Y$ we have $n \cdot g(y) = \sum_{x \in X_{y,\Phi}} g(\varphi(x)) \cdot \dim \Pi_{x,y} = \sum_{t \in \varphi(X_{y,\Phi})} c_y(t) g(t)$,

where each $c_y(t) > 0$. Now fix $y_0 \in Y$, suppose there is $t_0 \in \varphi(X_{y_0,\Phi}) \setminus \{y_0\}$ and let $g_0 \in C(Y)$ be such that $g_0(t_0) = 1$, $g_0 | \{y_0\} \cup (\varphi(X_{y_0,\Phi}) \setminus \{t_0\}) = 0$; then $g = g_0$ and $y = y_0$ will contradict the above form of assumption (4). Hence $\varphi(X_{y,\Phi}) = \{y\}$, $y \in Y$.

2.8. The following proposition gives sufficient conditions for a homomorphism Φ to be compatible with some good φ.

Proposition. Let B be a finite-dimensional C^*-algebra and consider $\Phi \in Hom(C(X) \otimes A, C(Y) \otimes B)$. Assume that the cardinality of $X_{y,\Phi}$ is locally constant on Y and $(X_{y,\Phi})_{y \in Y}$ is a partition of X. Then the map $\varphi: X \to Y$, $\varphi(X_{y,\Phi}) = \{y\}$, $y \in Y$ is a covering map and Φ is φ-compatible.

Proof. Fix $y \in Y$. The assumptions imply that there are

$n \in \mathbb{N}$ and $U \in \mathcal{V}(y')$ such that $X_y := X_{y,\Phi}$ has exactly n elements for all $y \in U$. Say $X_{y'} = \{z_1(y'), \ldots, z_n(y')\}$ and let $V'_p = \overline{V'_p} \in \mathcal{V}(z_p(y'))$, $p = 1,2,\ldots,n$, with $V'_p \cap V'_q = \emptyset$ for $p \neq q$.

Now, for fixed $p \in \{1,2,\ldots,n\}$ we claim there is $W \in \mathcal{V}(y')$, $W \subset U$ such that $X_y \cap V'_p \neq \emptyset$ for any $y \in W$. Indeed, in the contrary case there is a net $(y_i)_{i \in I}$ in U which converges to y' such that $X_{y_i} \cap V'_p = \emptyset$. But for $g \in C(X)$, $g(z_p(y'))=1$, supp $g \subset V'_p$ we have:

$$1 = |g(z_p(y'))| \leq \| g|X_{y'} \| = \| \Phi(g \otimes 1_A)(y') \| =$$

$$= \lim_i \| \Phi(g \otimes 1_A)(y_i) \| = \lim_i \| g|X_{y_i} \| = 0$$

a contradiction which proves the claim. Therefore we can choose $V \in \mathcal{V}(y')$, $V \subset U$, such that $X_y \cap V'_p \neq \emptyset$, $y \in V$, $p=1,2,\ldots,n$.

We prove that φ is continuous. Indeed, if a net $(x_j)_{j \in J}$ in X converges to $x \in X$ but $\varphi(x_j) \not\to \varphi(x)$, then, X being compact, we may suppose that $\varphi(x_j) \to y_0 \neq \varphi(x)$.

For $g \in C(X)$, $g(x)=1$, $g|X_{y_0}=0$ we have:

$$0 = \| g|X_{y_0} \| = \| \Phi(g \otimes 1_A)(y_0) \| = \lim_j \| \Phi(g \otimes 1_A)(\varphi(x_j)) \| =$$

$$= \lim_j \| g|X_{\varphi(x_j)} \| \geq \lim_j |g(x_j)| = |g(x)| = 1$$

a contradiction.

For each $y \in V$, let $z_p(y)$ be the unique element of $X_y \cap V'_p$, $p=1,2,\ldots,n$. Each map $z_p : V \to V_p := z_p(V)$ is a bijection since

$\varphi \circ z_p = id_V$; note that $V_p = \varphi^{-1}(V) \cap V'_p \in V(z_p(y'))$. Moreover, each z_p is continuous. Indeed, if a net $(y_k)_{k\in K}$ in V converges to $\tilde{y} \in V$ and $z_p(y_k) \not\to z_p(\tilde{y})$, we may consider $z_p(y_k) \to \tilde{x}$ for some $\tilde{x} \in \overline{V_p} \subset \overline{V'_p} = V'_p$, $\tilde{x} \neq z_p(\tilde{y})$ and we have $\tilde{y} = \lim_k y_k =$ $= \lim_k \varphi(z_p(y_k)) = \varphi(\tilde{x})$, that is $x \in \varphi^{-1}(\tilde{y}) \cap V'_p = X_{\tilde{y}} \cap V'_p$ hence $\tilde{x} = z_p(\tilde{y})$, a contradiction.

Thus each $\varphi_p = \varphi|_{V_p} : V_p \to V$ is a homeomorphism with inverse z_p. Hence φ is a covering map.

Since $X_y = \varphi^{-1}(y)$, $y \in Y$, it follows from 2.7. that Φ is φ-compatible.

<u>2.9.</u> The next proposition gives the structure of homomorphisms compatible with a finite covering, which improves the result in ([8] , Proposition 2.5.) by replacing the absolute retract assumption with contractibility and by using a shorter argument.

<u>Proposition</u>. Let $\varphi: X \to Y$ be a p-fold covering map ($p \in \mathbb{N}$), where X, Y are compact metric spaces and assume Y is contractible. Then there is a partition $(U_i)_{i=1}^p$ of X into clopen sets and there exist homeomorphisms $z_i : Y \to U_i$ satisfying $\varphi \circ z_i = id_Y$ ($1 \le i \le p$) such that: if $\Phi: C(X) \otimes A \to C(Y) \otimes B$ is a φ-compatible *-homomorphism, then there are $u \in C(Y, U(B))$ and *-homomorphisms $\Psi_1, \Psi_2, \ldots, \Psi_p : A \to B$ such that:

$$\Phi(f)(y) = Adu(y)\left(\bigoplus_{k=1}^{p} \Psi_k(f(z_k(y)))\right)$$

for all $f \in C(X) \otimes A$ and $y \in Y$.

<u>Proof</u>: Since Y is simply connected, there is a homeomorphism $H: X \to Y \times \{1, 2, \ldots, p\}$ such that the diagram:

$$\begin{array}{ccc} X & \xrightarrow{H} & Y\times\{1,2,\dots,p\} \\ \varphi \downarrow & \swarrow \psi & \\ Y & & \end{array}$$

commutes, where ψ is the canonical projection. For each $1\leq i\leq p$ we define: $U_i = H^{-1}(Y\times\{i\})$, the homeomorphism $h_i : Y\to Y\times\{i\}$ given by $h_i(y) = (y,i)$, $y\in Y$ and $z_i : Y\to U_i$, $z_i := H^{-1}\circ h_i$.

Using Proposition 2.4. from [8] and the fact that Y is connected, we find *-homomorphisms $\Psi_1,\dots,\Psi_p : A\to B$, a <u>proper</u> open covering $(V_i)_{i\in I}$ of Y(see [7], p.17) and $u_i\in C(V_i, U(B))$ such that:

$$\Phi(f)(y) = Ad\,u_i(y)\Big(\bigoplus_{k=1}^{p} \Psi_k(f(z_k(y)))\Big)$$

for $f\in C(X)\otimes A$, $y\in V_i$, $i\in I$. (The set of Ψ's in [8], 2.4. depends on the local neighborhood but they can be chosen canonical [4], that is in a finite set, so that this locally constant choice of the Ψ's is actually constant). The continuous maps $g_{ij} : V_i\cap V_j\to G :=$ the topological group of all unitaries of the relative commutant of $\bigoplus_{k=1}^{p}(\Psi_k(A))$ in B, defined by $g_{ij}(y) := u_i(y)^* u_j(y)$, $y\in V_i\cap V_j$, $i,j\in I$, satisfy $g_{ij}\cdot g_{jk} = g_{ik}$ on $V_i\cap V_j\cap V_k$ and hence $\{V_i, g_{ij}\}_{i\in I}$ defines an element in $H^1(Y, G_c)$. Since Y is contractible, $H^1(Y, G_c)$ reduces to the distinguished element. Therefore, we may assume that, for any $i\in I$ there exists a continuous map $v_i : V_i\to G$ such that $g_{ij}(y) = v_i(y)v_j(y)^*$, $y\in V_i\cap V_j$, $i,j\in I$. We define $u : Y\to U(B)$ by $u(y) := u_i(y)v_i(y)$, $y\in V_i$, $i\in I$. Since $u_i(y)v_i(y) = u_j(y)v_j(y)$ for $y\in V_i\cap V_j$, $i,j\in I$, the map u is well-defined and continuous.

It is easy to verify that:

$$\Phi(f)(y) = \mathrm{Ad}\, u(y)\Big(\bigoplus_{k=1}^{p} \Psi_k(f(z_k(y)))\Big),$$

$f \in C(X) \otimes A$, $y \in Y$.

§3.

Throughout this section X,Y will denote compact metric spaces (excepting Proposition 3.1.) and A,B finite-dimensional C^*-algebras.

In this section we give a local description of homomorphisms from $C(X) \otimes A$ to $C(Y) \otimes B$ by considering separately the cases X=point and $A=\mathbb{C}$. We also consider certain classes of inner equivalent homomorphisms.

<u>3.1. Proposition</u>. Consider $\Phi \in \mathrm{Hom}(A, C(Y) \otimes B)$, where Y is a compact space. For every $y' \in Y$ there exist $V \in \mathcal{V}(y')$, $\Psi \in \mathrm{Hom}(A,B)$ and $u \in C(V, U(B))$ such that:

$$\Phi(a) = \mathrm{Ad}\, u(y)(\Psi(a)), \quad a \in A, \ y \in V.$$

<u>Proof</u>. It is enough to consider the case when $A = \bigoplus_{i=1}^{n} M_{k_i}$, $B = M_l$.

For every $y \in Y$, consider the unital finite-dimensional $*$-representation $A \ni a \to \Phi(a)(y) \in M_l$. Since this is a direct sum of irreducible $*$-representations, it follows that $(\exists)$ $p_i(y) \in \{0,1,2,\dots\}$ and $u'(y) \in U(l)$ such that:

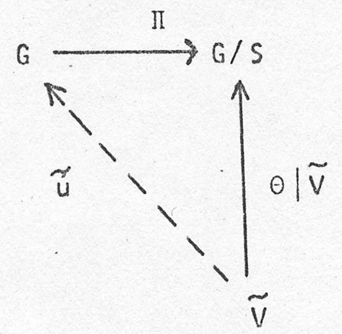
$$\Phi(a)(y) = \mathrm{Ad}\, u'(y)\Big(\bigoplus_{i=1}^{n} a_i \otimes 1_{p_i(y)}\Big)$$

for any $a = \overset{n}{\underset{i=1}{\oplus}} a_i \in \overset{n}{\underset{i=1}{\oplus}} M_{k_i}$ and $y \in Y$.

Since for any i, the map $Y \ni y \to tr(\Phi(1_{k_i})(y)) = k_i \cdot p_i(y) \in \{0,1,2,\dots,\}$ is continuous (here tr denotes the usual trace on M_1), $(\exists) V' \in \mathcal{V}(y')$ and $(\exists)\, \tilde{\Psi} \in Hom(A,B)$ such that:

$$\Phi(a)(y) = Ad\, u'(y)(\tilde{\Psi}(a)),\ a \in A,\ y \in V'.$$

We denote $G := U(B)$, $S := U(\tilde{\Psi}(A)^c)$ (here $\tilde{\Psi}(A)^c$ is the relative commutant of $\tilde{\Psi}(A)$ in B), $G/S := \{gS \mid g \in G\}$ and $\Pi : G \to G/S$ the canonical map. G/S will be embedded into the topological space $Hom(\tilde{\Psi}(A), B)$ by the formula $\Pi(g)(\tilde{\Psi}(a)) = Adg(\tilde{\Psi}(a))$, $g \in G$, $a \in A$. It follows that we can define a continuous map $\Theta : V' \to G/S$ by $\Theta(y)(\tilde{\Psi}(a)) = \Phi(a)(y)$, $y \in V'$, $a \in A$. Since S is a closed subgroup of the Lie group G, Π has smooth local sections. Thus, there is $\tilde{V} \in \mathcal{V}(y')$, $\tilde{V} \subset V'$ and $\tilde{u} \in C(\tilde{V}, G)$ such that the diagram:



commutes, which ends the proof.

<u>3.2.</u> We consider on $K(X)$ the topology given by the Pompeiu-Hausdorff metric $\tilde{d}$, defined by:

$$\tilde{d}(F,G) := \max(\sup_{x \in F} d(x,G),\ \sup_{y \in G} d(F,y))$$

$F, G \in K(X)$. Here d is a metric which gives the topology of X. Denote by $F(X)$ the set of all finite non-empty subsets of X. Then $F(X) \subset K(X)$ is endowed with the induced topology.

The proof of the following lemma is elementary and will be omitted.

Lemma. Let W be a metric space and a map $\Theta : W \to F(X)$. The following assertions are equivalent:

(1) $\Theta \in C(W, F(X))$

(2) the map $W \ni w \to \| f_{|\Theta(w)} \| \in \mathbb{R}$ is continuous for every $f \in C(X)$.

3.3. Let T be a compact space and for each $t \in T$ let $E(t)$ be a C^*-algebra. We say that $((E(t))_{t\in T}, \Gamma)$ is a continuous quasifield of C^*-algebras if Γ is a continuity structure for T and the $\{E(t)\}$ in the sense of J.M.G.Fell ([6]), i.e.: every $a \in \Gamma$ is a map defined on T such that $a(t) \in E(t)$ for any $t \in T$ and

(1) Γ is a $*$-algebra under the pointwise operations

(2) $\overline{\{a(t) \mid a \in \Gamma\}} = E(t)$, $t \in T$

(3) for any $a \in \Gamma$, the map $T \ni t \to \|a(t)\| \in \mathbb{R}$ is continuous.

Any continuous field of C^*-algebras ([3]) is a continuous quasifield.

Let $\mathcal{E}_i = ((E_i(t))_{t\in T}, \Gamma_i)$, $i=1,2$, be two continuous quasifields of C^*-algebras. We say that $\Psi = (\Psi_t)_{t\in T}$ is a homomorphism from $\mathcal{E}_1$ to $\mathcal{E}_2$ if: 1^o every Ψ_t is a $*$-homomorphism of C^*-algebras from $E_1(t)$ to $E_2(t)$; 2^o Ψ takes Γ_1 into Γ_2 (if we consider quasifields of unital C^*-algebras, each Ψ_t is assumed unital). We say that Ψ is injective if each Ψ_t is injective.

We denote by $\mathrm{Hom}(E_1, E_2)$ (resp. $\mathrm{Hom}_i(E_1, E_2)$) the set of all homomorphisms (resp. injective homomorphisms) from E_1 to E_2.

In the unital case we say that $\Psi^{(i)}=(\Psi_t^{(i)})_{t\varepsilon T}\varepsilon$ $\varepsilon\mathrm{Hom}(E_1, E_2)$, $i=1,2$, are <u>inner equivalent</u>, written $\Psi^{(1)}\sim\Psi^{(2)}$, if there is $u\varepsilon\Gamma_2$ such that $u(t)\varepsilon U(E_2(t))$ and $\Psi_t^{(1)}=\mathrm{Ad}\ u(t)\circ\Psi_t^{(2)}$ for any $t\varepsilon T$.

<u>3.4.</u> Let B be a C^*-algebra, $B\simeq M_{n_1}\oplus M_{n_2}\oplus\ldots\oplus M_{n_k}$, $n:=n_1+n_2+\ldots+n_k$, $\mathcal{F}_n(X):=\{F\varepsilon F(X)\,|\,F$ has at most n elements$\}$.

For any $\Theta\varepsilon C(Y,\mathcal{F}_n(X))$ consider $E_\Theta(y):=C(\Theta(y))$, $y\varepsilon Y$ (each $\Theta(y)$ is a discrete topological space) and $\Gamma_\Theta:=\{Y\ni y\to f|_{\Theta(y)}\varepsilon$ $\varepsilon E_\Theta(y)\,|\,f\varepsilon C(X)\}$. Using Lemma 3.2. we see that $E(\Theta):=((E_\Theta(y))_{y\varepsilon Y},\Gamma_\Theta)$ is a continuous quasifield of C^*-algebras.

Let $C:=\mathrm{Hom}(C(X), C(Y)\otimes B)$, $C_i:=\mathrm{Hom}_i(C(X), C(Y)\otimes B)$ and let F be the constant continuous field on Y, of fibre B. We define a map:

$$F : C\to \bigcup_{\Theta\varepsilon C(Y,\mathcal{F}_n(X))} \mathrm{Hom}_i(E(\Theta),F)$$

by:

$$F(\Phi):= (\Psi_{y,\Phi})_{y\varepsilon Y}$$

where $\Psi_{y,\Phi}(f|_{X_{y,\Phi}}):=\Phi(f)(y)$ for $f\varepsilon C(X)$, $y\varepsilon Y$ and $X_\Phi:Y\ni y\to X_{y,\Phi}\varepsilon\mathcal{F}_n(X)$.

<u>Proposition.</u> The map F is a bijection which induces in a canonical way a bijection of $C/\sim$ onto

$$\bigcup_{\Theta\varepsilon C(Y,F_n(X))} (\mathrm{Hom}_i(E(\Theta),F)/\sim)$$

Moreover, Frestricts to a bijection of C_i onto $\bigcup_{\Theta\varepsilon\tilde{C}(Y,F_n(X))} \mathrm{Hom}_i(E(\Theta),F)$ which induces a bijection of $C_i/\sim$ onto $\bigcup_{\Theta\varepsilon\tilde{C}(Y,F_n(X))} (\mathrm{Hom}_i(E(\Theta),F)/\sim)$, where $\tilde{C}(Y,F_n(X)):=\{f\varepsilon C(Y,F_n(X)) \mid \bigcup_{y\varepsilon Y} f(y)=X\}$

Proof: Consider $F(\Phi_i)=(\Psi^{(i)}_{y,\Phi_i})_{y\varepsilon Y}$, $i=1,2$, with $F(\Phi_1)=F(\Phi_2)$, that is $\Psi^{(1)}_{y,\Phi_1}=\Psi^{(2)}_{y,\Phi_2}$, $X_{y,\Phi_1}=X_{y,\Phi_2}$ for any $y\varepsilon Y$. Then $\Phi_1(f)(y)=\Psi^{(1)}_{y,\Phi_1}(f|X_{y,\Phi_1})=\Psi^{(2)}_{y,\Phi_2}(f|X_{y,\Phi_2})=$ $=\Phi_2(f)(y)$ for $f\varepsilon C(X)$, $y\varepsilon Y$, hence F is injective.

For the surjectivity of F consider $\Psi=(\Psi_y)_{y\varepsilon Y}\varepsilon$ $\varepsilon\,\mathrm{Hom}_i(E(\Theta),F)$, where $\Theta\varepsilon C(Y,F_n(X))$ and define $\Phi\varepsilon C$ by $\Phi(f)(y):=\Psi_y(f|\Theta(y))$, $f\varepsilon C(X)$, $y\varepsilon Y$. Using the definition of $X_{y,\Phi}(y\varepsilon Y)$ and the fact that each Ψ_y is injective, we have $||f|_{X_{y,\Phi}}||=||\Phi(f)(y)||=||f|\Theta(y)||$, $y\varepsilon Y$, which implies $X_{y,\Phi}=\Theta(y)$ for any $y\varepsilon Y$. It follows that $F(\Phi)=\Psi$.

Finally, using 2.2(3) it follows that $F(C_i)=$ $=\bigcup_{\Theta\varepsilon\tilde{C}(Y,F_n(X))} \mathrm{Hom}_i(E(\Theta),F)$.

3.5. Remark. Consider the continuous map $\varphi:\mathbb{T}\to\mathbb{T}$ given by $\varphi(y):=y^2$, $y\varepsilon\mathbb{T}(:=\{y\varepsilon\mathbb{C}\mid |y|=1\})$. Define $\Theta\varepsilon C(\mathbb{T},F_1(\mathbb{T}))$ by $\Theta(y):=\{\varphi(y)\}=\{y^2\}$, $y\varepsilon\mathbb{T}$ and two continuous maps $f,g:\mathbb{T}\to\mathbb{C}$ by $f(y)=1$, $g(y)=y$, $y\varepsilon\mathbb{T}$.

Then $f \in \Gamma_\Theta$ and $g.f \notin \Gamma_\Theta$, thus $((E_\Theta(y))_{y\in T}, \Gamma_\Theta)$ is not a continuous field of C^*-algebras (see [3], 10.1.9.).

3.6. Let $\varphi: X \to Y$ be a continuous surjective map such that $\varphi^{-1}(y)$ is a finite subset of X for any $y \in Y$ and the map:

$$Y \ni y \to \varphi^{-1}(y) \in F(X)$$

is continuous. This condition is satisfied if, for instance, φ is a covering map with a finite fibre.

Denote by $C(\varphi)$ the set of all φ-compatible $*$-homomorphisms from $C(X)\otimes A$ to $C(Y)\otimes B$ and by $C_i(\varphi)$ the set $C(\varphi) \cap \mathrm{Hom}_i(C(X)\otimes A, C(Y)\otimes B)$.

Let $E := ((E(y))_{y\in Y}, \Gamma)$ be the continuous field of C^*-algebras given by $E(y) := C(\varphi^{-1}(y))\otimes A$, $y \in Y$, $\Gamma :=$ $:= \{Y \ni y \to f|\varphi^{-1}(y) \in E(y) \mid f \in C(X)\otimes A\}$ (To see that E is indeed a continuous field use Lemma 3.2. and standard partition of unity arguments). Let F be the constant continuous field on Y, of fibre B. Define a map $G: C(\phi) \to \mathrm{Hom}(E, F)$ by $G(\Phi) := (\Psi_y)_{y\in Y}$ where $\Psi_y(f|\varphi^{-1}(y)) := \Phi(f)(y)$, $f \in C(X)\otimes A$, $y \in Y$.

Using 2.5. we easily obtain the following:

Proposition. The map G is a bijection which induces a bijection from $C(\varphi)/\sim$ onto $\mathrm{Hom}(E,F)/\sim$.

Moreover G maps $C_i(\varphi)$ onto $\mathrm{Hom}_i(E,F)$ and induces a bijection from $C_i(\varphi)/\sim$ onto $\mathrm{Hom}_i(E,F)/\sim$.

§4.

In this section we prove our main result concerning the stability under inductive limits of C^*-algebras of the form $C(X)\otimes A$ and isomorphisms of such C^*-algebras.

4.1. We first clarify the local structure of Θ-compatible homomorphisms with $\Theta(y)=\{\varphi(y)\}$ where $\varphi:Y\to X$ is continuous.

Proposition. Let X,Y be compact spaces, A,B finite-dimensional C^*-algebras, $\varphi:Y\to X$ a continuous map and consider $\Phi\in \mathrm{Hom}(C(X)\otimes A,\ C(Y)\otimes B)$ such that:

$$\Phi(g\otimes 1_A) = g\circ\varphi\otimes 1_B,\ g\in C(X).$$

Then, for each $y'\in Y$ there exist a neighborhood V of y' a continuous map $u:V\to U(B)$ and a *-homomorphism $\Psi\in \mathrm{Hom}(A,B)$ such that:

$$\Phi(f)(y)=\mathrm{Ad}\ u(y)(\Psi(f(\varphi(y))))$$

for $f\in C(X)\otimes A$, $y\in V$.

Proof: Fix $V\in \mathcal{V}(y')$, $\Psi\in \mathrm{Hom}(A,B)$ and $u\in C(V,U(B))$ given by Proposition 3.1. for the homomorphism $A\ni a\to\Phi(1_{C(X)}\otimes a)\in$ $\in C(Y)\otimes B$. Then, for any $g\in C(X)$, $a\in A$ and $y\in V$ we have:

$$\Phi(g\otimes a)(y)=(\Phi(g\otimes 1_A)(y))\cdot(\Phi(1_{C(X)}\otimes a)(y))=$$

$$=((g\circ\varphi)(y).1_B).(\mathrm{Ad}\ u(y)(\Psi(a)))=$$

$$=\mathrm{Ad}\ u(y)(\Psi(g\otimes a(\varphi(y))))$$

which completes the proof.

4.2. In the situation of the above proposition suppose that Y is connected. Then there are $\Psi\in \mathrm{Hom}(A,B)$, a proper open covering $(U_i)_{i\varepsilon I}$ of Y and $u_i \varepsilon C(U_i, U(B))$ such that:

$$\Phi(f)(y)=\mathrm{Ad}\, u_i(y)(\Psi(f(\varphi(y))))$$

for $f\varepsilon C(X)\otimes A$, $y\varepsilon U_i$, $i\varepsilon I$. For $y\varepsilon Y$, denote by $(\Phi(C(X)\otimes A)(y))^c$ the relative commutant of $\Phi(C(X)\otimes A)(y)$ in B. Since for any $y_1, y_2 \varepsilon Y$ there is a (inner) *-automorphism of B (depending on y_1 and y_2) which maps $\Phi(C(X)\otimes A)(y_1)$ onto $\Phi(C(X)\otimes A)(y_2)$, $(\Phi(C(X)\otimes A)(y_1))^c$ and $(\Phi(C(X)\otimes A)(y_2))^c$ are *-isomorphic and hence:

$$U((\Phi(C(X)\otimes A)(y_1))^c)\underset{-}{\sim}U((\Phi(C(X)\otimes A)(y_2))^c), y_1, y_2 \varepsilon Y$$

(as topological groups). Assume also that the cohomology set $H^1(Y, U((\Phi(C(X)\otimes A)(y))^c)_c)$ is reduced to the distinguished element for some $y\varepsilon Y$ (and hence for all $y\varepsilon Y$).

<u>Proposition</u> $\Phi \sim \varphi^* \otimes \Psi$.

<u>Proof</u>: Define continuous maps $g_{ij}: U_i \cap U_j \to G$, where G is the unitary group of the relative commutant of $\Psi(A)$ in B, by $g_{ij}(y)=u_i(y)^* u_j(y)$, $y\varepsilon U_i \cap U_j$, $i, y\varepsilon I$.

Since $g_{ij}\cdot g_{jk}=g_{ik}$ on $U_i \cap U_j \cap U_k$, $\{U_i, g_{ij}\}_{i\varepsilon I}$ defines an element in $H^1(Y, G_c)$. As $H^1(Y, G_c)$ is trivial, we may assume that for any $i\varepsilon I$ there is a continuous map $v_i: U_i \to G$ such that $g_{ij}(y)=v_i(y)v_j(y)^*$, $y\varepsilon U_i \cap U_j$, $i, j\varepsilon I$. Define $u: Y \to U(B)$ by $u(y):=u_i(y)v_i(y)$, $y\varepsilon U_i$, $i\varepsilon I$. Since $u_i(y)v_i(y)=u_j(y)v_j(y)$ for $y\varepsilon U_i \cap U_j$, $i, j\varepsilon I$, the map u is well-defined and continuous, and we have $\Phi=\mathrm{Ad}\, u \circ (\varphi^* \otimes \Psi)$.

<u>4.3.</u> Now consider a system:

$$C(X_1)\otimes A_1 \xrightarrow{\Phi_1} C(X_2)\otimes A_2 \xrightarrow{\Phi_2} \cdots$$

where for each k, X_k is a compact space, A_k is a finite-dimensional C^*-algebra, Φ_k is an isometric *-homomorphism such that:

$$\Phi_k(g\otimes 1_{A_k}) = g \circ \varphi_k \otimes 1_{A_{k+1}}, \quad g\in C(X_k)$$

with $\varphi_k : X_{k+1} \to X_k$ a surjective continuous map.
Let $X := \varprojlim (X_k, \varphi_k)$

Assume that for any $k\geq 2$, X_k is connected and $H^1(X_k, U((\Phi_{k-1}(C(X_{k-1})\otimes A_{k-1})(x))^c)_c)$ is reduced to the distinguished element for some $x\in X_k$ (and hence for all $x\in X_k$). Here $(\Phi_{k-1}(C(X_{k-1})\otimes A_{k-1})(x))^c$ is the relative commutant of $\Phi_{k-1}(C(X_{k-1})\otimes A_{k-1})(x)$ in A_k.

Then, by Proposition 4.2., for any $k\geq 1$ there exists $\Psi_k \in \mathrm{Hom}_i(A_k, A_{k+1})$ (unique, up to inner equivalence) such that $\Phi_k \sim \varphi_k^* \otimes \Psi_k$. Let $A := \varinjlim (A_k, \Psi_k)$

We thus obtain the following:

<u>Theorem</u>. The C^*-algebra $\varinjlim (C(X_k)\otimes A_k, \Phi_k)$ is *-isomorphic to the (spatial) C^*-tensor product $C(X)\otimes A$.

<u>4.4.</u> The isomorphism problem for the above considered inductive limits can be settled in certain cases by using the following result. We give a proof for the sake of the completeness.

<u>Proposition</u>. Let X, Y be compact spaces and A, B unital C^*-algebras with trivial centers. Then $C(X)\otimes A \simeq C(Y)\otimes B$ if and only if X and Y are homeomorphic and $A\simeq B$.

Proof: Suppose that $\Phi: C(X)\otimes A\to C(Y)\otimes B$ is a $*$-isomorphism. Since Φ maps $Z(C(X)\otimes A)$ onto $Z(C(Y)\otimes B)$, $C(X)\simeq C(Y)$, i.e. X and Y are homeomorphic.

Let m be a maximal ideal in C(X) and let χ be the corresponding character of C(X). We consider the surjective $*$-homomorphism $\chi\otimes id_A: C(X)\otimes A\to\mathbb{C}\otimes A$. Since $\ker(\chi\otimes id_A)=m\otimes A$, we have $A\simeq\mathbb{C}\otimes A\simeq C(X)\otimes A/m\otimes A$. But $\Phi(m\otimes 1_A)=m'\otimes 1_B$ with m'a maximal ideal in C(Y), since Φmaps $C(X)\otimes 1_A$ $(=Z(C(X)\otimes A))$ onto $C(Y)\otimes 1_B(=Z(C(Y)\otimes B))$. We have $A\simeq C(X)\otimes A/m\otimes A\simeq\Phi$ $(C(X)\otimes A)/$ $/\Phi(m\otimes A)=C(Y)\otimes B/m'\otimes B\simeq B$, which completes the proof.

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