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INTRODUCTION

Let E be a smooth finite dimensional real vector bundle with the typical fibre V. Assume that F is a finite group and let τ be a projective representation of F in V. Let $G(\tau)$ be the subgroup of GL(V) defined by

$$G(\tau)=\{a\in GL(V):a\tau(x)a^{-1}=\tau(x);\ x\in F\}.$$

Following the usual terminology in fibre bundles theory [4] we shall say that the bundle E has a $G(\tau)$-structure if there exists a smooth atlas $\{(h_i,V_i):i\in I\}$ for E such that the transition functions $\{g_{ij}:i,j\in I\}$ of this atlas have their values in $G(\tau)$.

Our main objective in the present paper is to give a unified tratment of all these structures. The simplest examples arise in connection with the projective representations of the cyclic group of order two. More precisely (see Section 1 below) in this case one obtains product or complex structures.

Althoughout different objects have to be studied in the different concrete cases, our hope is that the present approach will make it evident that some general results follow from the same algebraic manipulations.

The presentation is somewhat expository and the paper is organized as follows.

The first section begins by recalling the notion of a two-cocycle for a group and continues with the construction of the convolution algebra associated with a two-cocycle. A number of examples with a geometrical significance is also discussed.

The second part analyses the derivations in a unital and associative algebra over a convolution algebra. This section containes the main technical results of the paper.

The third and final section deals with a geometrical question. To be more specific, let us suppose that E is a smooth vector bundle endowed with a $G(\tau)$-structure. The problem is that of an explicit description of all linear connections on E which preserve in an appropriate sense the given $G(\tau)$-structure.

1. CONVOLUTION ALGEBRAS

We begin by recalling a few basic facts concerning two--cocycles and convolution algebras. The reader who wants to get a historical perspective and a deeper insight into these topics should refer for example to [3], [5] or [9].

1.1. Suppose that F is a fixed finite group, noted multiplicatively, and let R^* be the multiplicative group of all nonzero real numbers.

By a two-cocylce for F with values in R^* we shall mean a map m of FxF into R^* such that

(1.1) $\quad m(x,e)=m(e,x)=1 \; ; \quad x\varepsilon F,$

(1.2) $\quad m(x,y)m(xy,z)=m(x,yz)m(y,z) \; ; \quad x,y,z\varepsilon F \; ,$

where e denotes the identity of F.

Under pointwise multiplication the set $Z^2(F)$ of all R^*-valued two-cocycles for F becomes an abelian group. Its identity is the two-cocycle m_o defined by $m_o(x,y)=1$ $(x,y\varepsilon F)$.

Given m and m' in $Z^2(F)$, they are said to be cohomologous, $m\cong m'$ in symbols, if there exists a map k of F into R* such that

$$(1.3) \qquad m'(x,y)=m(x,y)k(x)k(xy)^{-1}k(y) \; ; \qquad x,y\varepsilon F.$$

1.2. Assume that m is a fixed two-cocycle for F. Let C(F) be the real vector space of all real valued functions on F. For f and g in C(F) one defines an element $f*g$ of C(F) by

$$(1.4) \qquad f*g(x)=\sum_{y\varepsilon F} f(xy)g(y^{-1})m(xy,y^{-1}) \; ; \qquad x\varepsilon F.$$

From (1.1) and (1.2) it follows easily that, under the operation (1.4) as multiplication, C(F) becomes a real unital and associative algebra, denoted by C(F,m) and referred to as the convolution algebra associated with m.

Let us mention that $C(F,m_o)$ is the usual real group algebra corresponding to F. It is also easy to prove that if m and m' are cohomologous, then the algebras C(F,m) and C(F,m') are isomorphic.

For any x in F one defines the element e_x in C(F,m) by

$$(1.5) \qquad e_x(x)=1 \; ; \quad e_x(y)=0 \qquad (y\varepsilon F,\ y\neq x).$$

The set $\{e_x : x\varepsilon F\}$ is a linear basis of C(F,m) and the equation (1.4) leads to the relations

$$(1.6) \qquad e_x*e_y=m(x,y)e_{xy} \; ; \qquad x,y\varepsilon F.$$

The basis $\{e_x : x\varepsilon F\}$ will be called the canonical basis of $C(F,m)$.

1.3. We might mention that the algebra $C(F,m)$ together with the map

$$\tau_o : F \to C(F,m), \qquad \tau_o(x) = e_x \qquad (x\varepsilon F)$$

has the following universal property: if A is an associative real algebra with unit 1 and τ is a map of F into A such that

(i) $\tau(e) = 1$,

(ii) $\tau(x)\tau(y) = m(x,y)\tau(xy)$; $\quad x,y\varepsilon F$,

then there exists a morphism of real algebras $\lambda : C(F,m) \to A$ and one only, such that $\tau = \lambda\tau_o$.

We make now a brief digression. Throughout in what follows, by a $C(F,m)$-algebra we shall mean a real unital and associative algebra A together with a morphism of real algebras $\lambda : C(F,m) \to A$. The map λ is referred to as the structural morphism of A. The next two examples are basic for our purposes.

a) Suppose that V is a finite dimensional real vector space, m is a fixed two-cocycle in $Z^2(F)$, and let $End(V)$ be the algebra of all endomorphisms of V. We recall that by a projective representation of F in V, with the multiplier m, one means a map τ of F into $End(V)$ such that

(i) $\tau(e) = 1_V$,

(ii) $\tau(x)\tau(y) = m(x,y)\tau(xy)$; $\quad x,y\varepsilon F$,

where 1_V is the identity map of V.

From the universal property of $C(F,m)$ described above it is clear that there is a natural correspondence between the projective representations of F in V, with the multiplier m, and the C(F,m)-algebra structures on End(V).

b) Assume now that E is a smooth finite dimensional real vector bundle with the typical fibre V, and let τ be a projective representation of F in V. We denote by $G(\tau)$ the subgroup of GL(V) defined by

$$G(\tau)=\{a\in GL(V): a\tau(x)a^{-1}=\tau(x); \quad x\in F\}. \tag{1.7}$$

An elementary and standard argument shows that E has a $G(\tau)$--structure if and only if the algebra End(E) of all smooth vector bundle endomorphisms of E is a C(F,m)-algebra, where m is the multiplier of τ (compare for instance with the proof of Proposition 1.1 in [6]).

More about smooth vector bundles endowed with such a structure will be discussed in the last section of the paper.

1.4. We continue by presenting some examples of convolution algebras. The following result will be used several times in the sequel. The proof needs only minor calculations and is omitted.

PROPOSITION 1. Let C be a finite dimensional associative real algebra with unit 1. Assume that there exist:

(i) a finite group F with the order equal to the dimension of C ;

(ii) a linear basis $\{C_x : x\in F\}$ of C with $C_e=1$;

(iii) a map m of FxF into R*, such that

$$C_x * C_y = m(x,y)C_{xy} ; \quad x,y\in F. \tag{1.8}$$

Then m is a two-cocycle for F and the algebra C is isomorphic to $C(F,m)$.

This simple result will enable us to show that certain algebras related to a number of interesting geometric structures are in fact convolution algebras. The two-cocycle m introduced by (1.8) is called the cocycle associated with the basis $\{C_x : x \in F\}$.

a) The simplest examples appear when $F=\{e,x\}$ is the cyclic group of order two. Let m_o be the identity of $Z^2(F)$ and let m_1 be the two-cocycle defined by

$$m_1(e,e)=m_1(e,x)=m_1(x,e)=1, \qquad m_1(x,x)=-1.$$

If m is an arbitrary two-cocycle for F, then one has $m \cong m_o$ or $m \cong m_1$. Thus, there exist in this case only two non-isomorphic convolution algebras, $C(F,m_o)$ and $C(F,m_1)$.

The algebra $C(F,m_o)$ is isomorphic to $R \oplus R$, an algebra related to the so-called product structures (see Example 2.10 in [6]), and the algebra $C(F,m_1)$ is isomorphic to C, the algebra of complex numbers.

b) A generalization of the algebras $R \oplus R$ and C is given by the class of algebras $P_{n,\varepsilon}$, where $n \geq 2$ is an integer and $\varepsilon = \pm 1$. We explicitly set

$$(1.9) \qquad P_{n,\varepsilon} = R[t]/(t^n - \varepsilon),$$

where $R[t]$ is the algebra of real polynomials of the variable t, and $(t^n-\varepsilon)$ denotes the ideal generated by $t^n-\varepsilon$.

Taking the images of the polynomials $1, t, \ldots, t^{n-1}$ in $P_{n,\varepsilon}$, one obtains a linear basis in $P_{n,\varepsilon}$. Applying Proposition 1 to our case we find a two-cocycle for the group Z/nZ. It turns out that $P_{n,\varepsilon}$ is isomorphic to a convolution algebra.

We should remark that the algebras $P_{n,\varepsilon}$ appear in connection with the so-called polynomial structures.

c) Another example is that of the algebra H of quaternions. The usual basis of H over R leads to a two-cocycle for the group of Klein and, hence, H is also isomorphic to convolution algebra.

d) Our last class of examples is given by the Clifford algebras. We shall now outline the concrete description of these algebras. The reader who is interested in a more detailed discussion is referred to [1], [4] or [9].

Let p and q be two positive integers, with $p+q=n\geqq 1$. The Clifford algebra $C^{p,q}$ is the real unital and associative algebra generated by a set $\{e_1,e_2,\dots,e_n\}$ with relations

(1.10) $e_i^2=-1; \quad 1\leqq i\leqq p,$

(1.11) $e_i^2=1; \quad p+1\leqq i\leqq p+q,$

(1.12) $e_ie_j+e_je_i=0; \quad 1\leqq i,j\leqq n, \quad i\neq j.$

Let F be the set of all subsets of $\{1,2,\dots,n\}$. For I and J in F one defines

(1.13) $I\wedge J=(I\cup J)-(I\cap J).$

Under this operation F becomes an abelian group of order 2^n.

Next, for any I in F one defines the element e_I of $C^{p,q}$ as follows:

(i) $e_\emptyset=1$;

(ii) if $I=\{i(1),\, i(2),\dots,i(s)\}$ with $1\leqq i(1)<i(2)<\dots$ $\dots<i(s)\leqq n$, then

(1.14) $e_I=e_{i(1)}e_{i(2)}\cdots e_{i(s)}$.

The elements $\{e_I : I\in F\}$ form a basis of $C^{p,q}$ and for any I and J in F there is a number m(I,J), which is +1 or -1, such that

(1.15) $$e_I e_J = m(I,J) e_{I\wedge J} .$$

Proposition 1 implies that m is a two-cocycle for F and the Clifford algebra $C^{p,q}$ is isomorphic to a convolution algebra. Note that $C^{0,1}$, $C^{1,0}$ and $C^{2,0}$ are isomorphic, respectively, to $R \oplus R$, C and H.

2. DERIVATIONS IN C(F,m)-ALGEBRAS

Throughout this section A will denote a fixed C(F,m)-algebra with the structural morphism $\lambda : C(F,m) \to A$. By $\{e_x : x\in F\}$ we denote the canonical basis of C(F,m) and let $\lambda(e_x) = b(x)$ $(x\in F)$.

Our aim in what follows is to describe the set of all derivations ∂ in A which satisfy a system of equations of the form

$$\partial(b(x)) = a(x); \quad x\in F, \quad a(x)\in A.$$

2.1. A derivation in the algebra A is by definition a real linear map ∂ of A into A with the property

(2.1) $$\partial(aa') = (\partial a)a' + a(\partial a'); \quad a,a'\in A.$$

The space of all derivations in A will be denoted by Der(A).

For any a and a' in A one defines their commutator $[a,a']$ by

(2.2) $$[a,a'] = aa' - a'a .$$

For a fixed a, the map $\theta(a) : A \to A$, $\theta(a)a' = [a,a']$ $(a'\in A)$ is a derivation in A, called the inner derivation corresponding to a. The function θ of A into Der(A) is real linear. In certain cases all derivations are inner. For example if A=End(V), where V is a real finite dimensional vector space.

2.2. In order to state the main results of this section we need a few preliminary constructions.

Consider first the tensor product of real vector spaces $A \otimes \dot{C}(F,m)$. An element A in this tensor product is of the form

$$(2.3) \qquad \mathrm{A} = \sum_{x \varepsilon F} a(x) \otimes e_x ; \qquad a(x) \varepsilon A .$$

Let $L(A,\lambda)$ be the subspace of all elements A of the form (2.3) which satisfy the equations

$$(2.4) \qquad a(x)b(y)+b(x)a(y)=m(x,y)a(xy); \qquad x,y \varepsilon F.$$

Recall that $b(x)=\lambda(e_x)$ $(x\varepsilon F)$, hence

$$(2.5) \qquad b(x)b(y)=m(x,y)b(xy); \qquad x,y\varepsilon F.$$

Next we introduce a linear map ϕ:Der(A) $\rightarrow L(A,\lambda)$ by the formula

$$(2.6) \qquad \phi(\partial) = \sum_{x\varepsilon F} \partial(b(x)) \otimes e_x ; \qquad \partial \varepsilon \mathrm{Der}(A).$$

An obvious consequence of (2.5) and (2.1) is that $\phi(\partial)$ really belongs to $L(A,\lambda)$ for each derivation ∂.

Our aim is to solve equations of the form $\phi(\partial)$=A, A$\varepsilon L(A,\lambda)$. The essential step in this is accomplished by the construction of a linear map $\psi: L(A,\lambda) \rightarrow A$ with the property that $\phi(\theta\psi(\mathrm{A}))$=A, for all A in $L(A,\lambda)$.

PROPOSITION 2. For each $A=\sum_{x\varepsilon F} a(x) \otimes e_x$ in $L(A,\lambda)$ we define

$$\psi(A)=(\text{ord } F)^{-1} \sum_{x\varepsilon F} m(x,x^{-1})^{-1} a(x)b(x^{-1}) . \tag{2.7}$$

Then the inner derivation $\partial=\theta(\psi(A))$ satisfies the equation $\phi(\partial)=A$.

PROOF. We have to prove that

$$\psi(A)b(y)-b(y)\psi(A)=a(y); \quad y\varepsilon F. \tag{2.8}$$

Suppose that y in F is fixed and note that

$$\psi(A)b(y)=(\text{ord } F)^{-1} \sum_{x\varepsilon F} m(x,x^{-1})^{-1} a(x)b(x^{-1})b(y).$$

From (2.5) one has

$$b(x^{-1})b(y)=m(x^{-1},y)b(x^{-1}y),$$

and the equation (2.4) leads to

$$a(x)b(x^{-1}y)+b(x)a(x^{-1}y)=m(x,x^{-1}y)a(y).$$

Consequently one obtains

$$\psi(A)b(y)=c_1-c_2 \tag{2.9}$$

where

$$c_1=(\text{ord } F)^{-1} \sum_{x\varepsilon F} m(x,x^{-1})^{-1} m(x^{-1},y)m(x,x^{-1}y)a(y), \tag{2.10}$$

and

$$c_2=(\text{ord } F)^{-1} \sum_{x\varepsilon F} m(x,x^{-1})^{-1} m(x^{-1},y)b(x)a(x^{-1}y). \tag{2.11}$$

The relation (2.8) will follow from (2.9) if it is shown that

(2.12) $c_1=a(y)$,

(2.13) $c_2=-b(y)\psi(A)$.

To this end, note first that (1.1) and (1.2) imply

(2.14) $m(x,x^{-1}y)m(x^{-1},y)=m(x,x^{-1})$; $x\varepsilon F$.

The last identity and the formula (2.10) clearly give (2.12).

It remains to prove (2.13). From (2.14) one obtains also

$$c_2=(\text{ord } F)^{-1}\sum_{x\varepsilon F} m(x,x^{-1}y)^{-1}b(x)a(x^{-1}y),$$

thus, after the transformation $x\to yx$, one finds

(2.15) $$c_2=(\text{ord } F)^{-1}\sum_{x\varepsilon F} m(yx,x^{-1})^{-1}b(yx)a(x^{-1}).$$

On the other hand one has

(2.16) $$b(y)\psi(A)=(\text{ord } F)^{-1}\sum_{x\varepsilon F} m(x,x^{-1})^{-1}b(y)a(x)b(x^{-1}).$$

An obvious consequence of (2.4) is that $a(e)=0$. Hence, by (2.4) again, one obtains

$$a(x)b(x^{-1})=-b(x)a(x^{-1}),$$

thus

$$b(y)a(x)b(x^{-1})=-m(y,x)b(yx)a(x^{-1}).$$

From (2.16) it follows that

(2.17) $$b(y)\psi(A)=-(\text{ord } F)^{-1}\sum_{x\varepsilon F} m(x,x^{-1})^{-1}m(y,x)b(yx)a(x^{-1}).$$

Finally, note that (1.1) and (1.2) yield

(2.18) $m(y,x)m(yx,x^{-1})=m(x,x^{-1})$; $x\varepsilon F$.

Now, (2.13) follows easily from (2.18), (2.15) and (2.17). The proof of Proposition 2 is complete.

2.3. Proposition 2 shows at once that the map ϕ defined by (2.6) is surjective. Actually, an alternative formulation of Proposition 2 can be given as follows.

Let $Der(A)^{\lambda}$ be the subspace of $Der(A)$ defined by

$$Der(A)^{\lambda}=\{\partial \varepsilon Der(A): \partial(b(x))=0;\ x\varepsilon F\}, \tag{2.19}$$

and let $\iota: Der(A)^{\lambda} \to Der(A)$ be the inclusion map. Then we have:

THEOREM 1. The sequence of vector spaces

$$0 \to Der(A)^{\lambda} \xrightarrow{\iota} Der(A) \xrightarrow{\lambda} L(A,\lambda) \to 0 \tag{2.20}$$

is exact; moreover, the map $\theta\psi: L(A,\lambda) \to Der(A)$ provides a splitting for the sequence (2.20).

2.4. Theorem 1 enables us to represent the space $Der(A)$ as a direct sum of $Der(A)^{\lambda}$ and $L(A,\lambda)$, in a specific way. An immediate consequence of this representation can be stated in the next form.

THEOREM 2. Let $\chi: Der(A) \to Der(A)$ be the map defined by $\chi = id-\theta\psi\phi$, where id is the identity map of $Der(A)$. Then we have:

i) χ is a projection from $Der(A)$ onto $Der(A)^{\lambda}$, that is, $\chi^2=\chi$ and image $(\chi)=Der(A)^{\lambda}$;

ii) for a fixed A in $L(A,\lambda)$ the equation $\phi(\partial)=A$ has the general solution

$$\partial=\chi(\partial')+\theta\psi(A); \qquad \partial'\varepsilon Der(A). \tag{2.21}$$

2.5. We conclude this section with a few remarks. It is implicit in Theorem 2 above that if ∂ is an inner derivation, then $\chi(\partial)$ is also an inner derivation. More precisely, let $\chi_o : A \to A$ be the map defined by

$$(2.22) \qquad \chi_o(a) = (\mathrm{ord}\ F)^{-1} \sum_{x \in F} m(x,x^{-1})^{-1} b(x) a b(x^{-1}); \qquad a \in A .$$

A direct computation shows that $\chi(\theta(a)) = \theta(\chi_o(a))$ for all a in A. Actually the map χ_o is a projection of A onto the commutant of the set $\{b(x) : x \in F\}$.

In addition to Theorem 1 we find that the system of commutator equations

$$(2.23) \qquad [a, b(x)] = a(x); \qquad x \in F,$$

where $A = \sum_{x \in F} a(x) \times e_x$ is an element of $L(A,\lambda)$, has the general

$$(2.24) \qquad a = \chi_o(a') + \psi(A); \qquad a' \in A .$$

3. LINEAR CONNECTIONS ON SMOOTH C(F,m)-BUNDLES

By a smooth C(F,m)-bundle we shall mean a smooth vector bundle E, together with a morphism of real algebras $\lambda : C(F,m) \to \mathrm{End}(E)$. We recall that End(E) is the algebra of all smooth vector bundle endomorphisms of E. Our aim in this section is to present a geometrical application of Theorem 2. An outline only will be given and the detailed proofs will be omitted.

We have to note that Theorem 3 below, the main result of this section, gives the unified form of certain well-known results concerning product, complex or quaternionic structures. We refer the reader to [2], [7], [8] and to the papers quoted there.

3.1. We begin by recalling some notations and definitions.

Assume that E is a smooth vector bundle over the smooth manifold M. Let $C^{\infty}(M)$ be the algebra of real valued smooth functions on M and let $\Gamma(E)$ be the $C^{\infty}(M)$-module of all smooth sections of E. According to an usual isomorphism we shall identify throughout in what follows the algebra End(E) with the $C^{\infty}(M)$-algebra of all $C^{\infty}(M)$-linear maps from $\Gamma(E)$ into $\Gamma(E)$.

The space $Der(C^{\infty}(M))$ of derivations in the algebra $C^{\infty}(M)$ is exactly the space $\mathcal{X}(M)$ of all smooth vector fields on E. The space Der(End(E)) will be denoted simply by $\mathcal{X}(M,E)$. Note that $\mathcal{X}(M,E)$ has a natural structure of $C^{\infty}(M)$-module. In order to define natural relationships between $\mathcal{X}(M)$ and $\mathcal{X}(M,E)$ we have to recall the notion of linear connections on E.

A linear connection on E is by definition a real bilinear map ∇ of $\mathcal{X}(M)\times\Gamma(E)$ into $\Gamma(E)$ such that

(3.1) $\qquad \nabla(\alpha X,\sigma)=\alpha\nabla(X,\sigma),$

(3.2) $\qquad \nabla(X,\alpha\sigma)=X(\alpha)\sigma+\alpha\nabla(X,\sigma),$

for all X in $\mathcal{X}(M)$, σ in $\Gamma(E)$ and α in $C^{\infty}(M)$. Suppose that ∇ is a fixed linear connection on E and let us define for any X in $\mathcal{X}(M)$ a map $\nabla(X)$ of $\Gamma(E)$ into $\Gamma(E)$ by

(3.3) $\qquad \nabla(X)\sigma=\nabla(X,\sigma)\ ;\ .\ \sigma\varepsilon\Gamma(E)$

In additions one introduces a function $\hat{\nabla}(X)$ which associates with any T in End(E) the map $\hat{\nabla}(X)T$ of $\Gamma(E)$ into $\Gamma(E)$ given by

(3.4) $\qquad \hat{\nabla}(X)T=\nabla(X)\circ T-T\circ\nabla(X).$

An immediate computation shows that $\hat{\nabla}(X)T$ is $C^{\infty}(M)$-linear, therefore $\hat{\nabla}(X)T$ is an element of End(E). Actually $\hat{\nabla}(X)$ is a devitation in the algebra End(E) and the map $\hat{\nabla}: \mathcal{X}(M) \to \mathcal{X}(M,E)$ is $C^{\infty}(M)$-linear.

3.2. Suppose now that F is a finite group, m is a two--cocycle for F and the smooth vector bundle E is a C(F,m)--bundle. Denote by $\lambda: C(F,m) \to End(E)$ the structural morphism and let $\lambda(e_x) = S(x)$ $(x \varepsilon F)$. The set $\sum = \{S(x) : x \varepsilon F\}$ is referred to as a C(F,m)-structure on E.

Given a linear connection ∇ on E we shall say that ∇ preserves the C(F,m)-structure $\sum$ if

$$(3.5) \qquad \hat{\nabla}(X)S(x) = 0; \; X \varepsilon \mathcal{X}(M), \; x \varepsilon F.$$

Following an already introduced notation, one has that the linear connection ∇ preserves the structure $\sum$ if and only if the derivations $\hat{\nabla}(X)$ belong to $\mathcal{X}(M,E)^{\lambda}$ for all X in $\mathcal{X}(M)$.

The main result of this section gives a concrete description of all linear connections on E which preserve the structure $\sum$. This description appears in fact as an application of Theorem 2.

More precisely, if ∇ is a linear connection on E and X is a vector field in $\mathcal{X}(M)$ we define first the map $\chi\nabla(X): \Gamma(E) \to \Gamma(E)$ by

$$(3.6) \qquad \chi\nabla(C) = (\mathrm{ord}\ F)^{-1} \sum_{x \varepsilon F} m(x,x^{-1}) S(x) \circ \nabla(X) \circ S(x^{-1}) .$$

Next let us denote by $\chi\nabla$ the map of $\mathcal{X}(M) \times \Gamma(E)$ into $\Gamma(E)$ obtained as follows:

$$(3.7) \qquad \chi\nabla(X,\sigma) = \chi\nabla(X)\sigma; \quad X \varepsilon \mathcal{X}(M), \quad \sigma \varepsilon \Gamma(E)$$

By a straightforward computation one finds that $\chi\nabla$ is a linear connection on E and moreover we have:

PROPOSITION 3. Let ∇ be a linear connection on E. Then

i) $\chi\nabla$ preserves the structure Σ, and

ii) ∇ preserves the structure Σ if and only if $\nabla=\chi\nabla$.

3.3. More generally, Theorem 2 furnishes a description of the set of all linear connections ∇ on E which satisfy a system of equations of the form

$$(3.8)\qquad \hat{\nabla}(X)S(x)=T(x)(X);\qquad X\varepsilon\mathcal{X}(M),\quad x\varepsilon F,$$

where $\mathcal{T}=\{T(x):x\varepsilon F\}$ is a collection of $C^{\infty}(M)$-linear maps from $\mathcal{X}(M)$ into End(E). According to the results of Section 2, the next analogue of the second part of Theorem 2 follows.

THEOREM 3. The system of equations (3.8) has solutions if and only if

$$(3.9)\qquad T(x)(X)S(y)+S(x)T(y)(X)=m(x,y)T(xy)(X)$$

for all X in $\mathcal{X}(M)$ and x,y in F.

If the collection satisfies the conditions (3.9) then the system (3.8) has the general solution

$$(3.10)\qquad \nabla=\chi\nabla'+\psi(\mathcal{T})$$

where ∇' is a linear connection on E, and $\psi(\mathcal{T})$ is the map of $\mathcal{X}(M)\times\Gamma(E)$ into $\Gamma(E)$ defined by

$$(3.11)\qquad \psi(\mathcal{T})(X,\sigma)=(\text{ord } F)^{-1}\sum_{x\varepsilon F} m(x,x^{-1})^{-1}T(x)(X)S(x^{-1})\sigma;$$

$$X\varepsilon\mathcal{X}(M),\quad \sigma\varepsilon\Gamma(E).$$

REFERENCES

1. ATIYAH, M.F.; BOTT, R.; SHAPIRO, A.: Clifford modules. Topology 3, 3-38 (1964).
2. CRUCEANU, V.: Sur les connections compatibles avec certaines structures sur un fibre vectoriel banachique. Czech. Math. J. 24, 126-142 (1974).
3. CURTIS, C.W.; REINER, I.: Representation Theory of Finite Groups and Associative Algebras. New York, N.Y.: Interscience Publishers 1962.
4. HUSEMOLLER, D.: Fibre bundles. New York: McGraw-Hill 1966.
5. KIRILLOV, A.A.: Elements of the Theory of Representations (in Russian). Moscow: Nauka 1972.
6. KOBAYASHI, S.: Transformation Groups in Differential Geometry. Berlin: Springer 1972.
7. MARTIN, M.: Geometric structures and derivations. Rev. Roum. Math. Pures Appl. 29(3), 239-248 (1984).
8. MARTIN, M.: Geometric structures on vector bundles (in Romanian), Stud. Cerc. Mat. 36, I(2), 113-142 (1984); II(3), 171-192(1984); III(4), 269-292(1984).
9. VARADARAJAN, V.S.: Geometry of Quantum Theory, Berlin: Springer 1985.