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Gelu I. PAŞA*)

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*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, ROMANIA.

BOUNDS FOR EFFECTIVE COEFFICIENTS OF PERIODIC FIBER - REINFORCED MATERIALS

GELU I. PAŞA
INCREST , Dept. of Mathematics
Bdul Păcii 220, 79622 Bucharest
ROMANIA

Abstract

A generalization of the estimates obtained by L.Tartar [6] for the case of linear elasticity, without the symmetry condition $A_{ijkh} = A_{khij}$, is present. This case is possible when the non-orthogonal co-ordinates are used. A comparison with the estimates obtained by Z.Hashin and W.Rosen [3] is given.

Key words : bounds for effective coefficients
compensated compactness

Introduction

We consider an elastic medium, whose coefficients a_{ijkh} are Y periodic, Y being a parallelepipedon in R^n. The composite materials may be such media, where in a matrix with low resistance are periodically inserted fibers (or inclusions) of high resistance. The material obtained in this way have a superior resistance than that of the matrix.

In this paper we study composite materials for which the dimension of the periodicity cell Y is very small compared with that of the entire medium. Using the fact that the coefficients have a limit in a suitable topology (being periodic and bounded) we try to replace the initial problem with another one with constant coefficients; they are computed by solving an equation for a single cell.

Because even that problem for a single cell is complicated, it is very useful to obtain a priori estimates for the constant coefficients (named homogenized or effective coefficients). In this direction, a great number of methods were proposed. We reffer here to the method proposed by Hashin and Rosen [3] which is based on the minimum principle for the potential energy ; they need an exact solution for the studied configuration (in their case , the hexagonal periodicity).

In 1976, Tartar [6] obtain more general estimations, using the compensated compactness, introduced by himself and F.Murat (see Murat [4]).

In this paper we give a generalization of the results obtained by Tartar [6] for the linear elasticity, without the symmetry condition $a_{ijkh} = a_{khij}$. This situation may occur , for example, when we try to apply the Tartar's method to a configuration whose directions of periodicity are not orthogonal (as in the case studied in Hashin ,Rosen [3]). In this case, the coefficients are not symmetric with respect to the axis parallel to the directions of periodicity.

For some particular values of the coefficients we compare the results given by these two methods.

§ 1. Bounds for homogenized coefficients

In the following, using the method of Tartar [6] and the idea of Mc. Connel [2], we give the estimates of the homogenized coefficients for the system of linear elasticity.

We begin with some definitions:

Definition 1.

Let $A_{ijkh} = A_{ijhk} = A_{jikh}$ Y periodic, strongly elliptic,

$$A^{\varepsilon}_{ijkh}(x) = A_{ijkh}(x/\varepsilon).$$

Let the equation :

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(A^{\varepsilon}_{ijkh}\dfrac{\partial u^{\varepsilon}_k}{\partial x_h}\right) = F_j \text{ in } \Omega \subset R^n \\ \underline{u}^{\varepsilon} / \partial\Omega = 0. \end{cases}$$

Using the results of the homogenization theory. (Sanchez-Palencia [5] ; Benssoussan, Lions, Papanicolaou [1]) it is possible to prove the existence of A^{o}_{ijkh} and $\underline{u}^{o}$ such that $\underline{u}^{\varepsilon} \rightarrow \underline{u}^{o}$ in the weak topology of Sobolev space $H^1_o(\Omega)$ and

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(A^{o}_{ijkh}\dfrac{\partial u^{o}_k}{\partial x_h}\right) = F_j \\ \underline{u}^{o} / \partial\Omega = 0. \end{cases}$$

In this case, A^{ε}_{ijkh} are H-convergent to A^{o}_{ijkh} :

$$A^{\varepsilon}_{ijkh} \xrightarrow{H} A^{o}_{ijkh}.$$

Definition 2.

Let M, N two "matrix" such that $M_{ijkh} = M_{jikh} = M_{ijhk}$.

Then

$$M \leq N \iff (M\,c, c) \leq (N\,c, c) \ (\forall)\ c_{ij} = c_{ji} \in R^3 \times R^3,$$

(.,.) being the scalar product in R^9.

Remarque 1.

If $A^{\varepsilon}_{ijkh} \xrightarrow{H} A^{o}_{ijkh}$ then we have $A^{\varepsilon}_{ijkh}\dfrac{\partial u^{\varepsilon}_k}{\partial x_h} \xrightarrow{H} A^{o}_{ijkh}\dfrac{\partial u^{o}_k}{\partial x_h}$,

which is a consequence of the convergence theorem, where in the local problem we use the transposed matrix of A^{ε}_{ijkh} .

Proposition 1.

Let $p^{\varepsilon}_{ij} \in H^{1}(\Omega)$ such that $p^{\varepsilon}_{ij} \rightharpoonup p^{*}_{ij}$ weakly in $L_2(\Omega)$ and $\| \partial/\partial x_i (p^{\varepsilon}_{ij}) \|_{L_2(\Omega)} \leq$ ct. Let $\underline{u}^{\varepsilon} \in H^1(\Omega)$ such that $\underline{u}^{\varepsilon} \rightharpoonup \underline{u}^{*}$ weakly in $H^1(\Omega)$. Then :

$$\int_{\Omega} p^{\varepsilon}_{ij} \partial u^{\varepsilon}_i / \partial x_j \, \phi \longrightarrow \int_{\Omega} p^{*}_{ij} \partial u^{*}_i / \partial x_j \, \phi \, , \quad (\forall) \quad \phi \in C^{\infty}_0(\Omega) .$$

Proof.

We put $\underline{h}^{\varepsilon i} = (p^{\varepsilon}_{i1}, p^{\varepsilon}_{i2}, \ldots, p^{\varepsilon}_{in})$, $\underline{v}^{\varepsilon i} = (\partial u^{\varepsilon}_i / \partial x_1, \partial u^{\varepsilon}_i / \partial x_2, \ldots, \partial u^{\varepsilon}_i / \partial x_n)$.

Then we have $\| \operatorname{div} \underline{h}^i \|_{L_2(\Omega)} \leq$ ct,

$$(\operatorname{rot} \underline{v}^i)_{sq} = \partial^2 u^{\varepsilon}_i / \partial x_s \partial x_q - \partial^2 u^{\varepsilon}_i / \partial x_q \partial x_s = 0 ,$$

and we may apply the compensated compactness for vectors, cf. Bensoussan, Lions, Papanicolaou [1] (see also Murat [4]) ∎

In the following we consider the elastic coefficients with the property:

$(\exists)$ a coefficient with $ij = kh$ (for example A_{1313}) such that

$$(*) \begin{cases} 1) \; A_{1312} = A_{1213} \, , \; A_{13qs} = 0 \quad \text{for } sq \neq 12, 13 \, , \\ 2) \; (A)^{-1}_{13sq} = (A)^{-1}_{sq13} \, , \; (A)^{-1}_{12sq} = (A)^{-1}_{sq12} \, . \end{cases}$$

.. First we prove a result which does not use the compensated compactness :

Theorem 1.

Let $A^{\varepsilon} \xrightarrow{H} Q$ and $A^{\varepsilon} \rightharpoonup A$, $(A^{\varepsilon})^{-1} \rightharpoonup (B)^{-1}$ weakly in $L_2(\Omega)$.

Then :

a) $Q \leq A$

b) $Q_{1313} \geq B_{1313}$

Proof.

a) We consider $c_{ij} = c_{ji} \in R^3 \times R^3$ and start with

(1) $\quad (A^{\varepsilon}(v^{\varepsilon} - c), v^{\varepsilon} - c) \geq 0$;

where v^ε is defined by

$$(2)\quad \begin{cases} \partial(A^\varepsilon_{ijkh}\,\partial u^\varepsilon_k/\partial x_h)/\partial x_i = 0\,, & v^\varepsilon_{kh} = \partial u^\varepsilon_k/\partial x_h \\ u^\varepsilon_k - c_{kh}\cdot x_h \in H^1_0(\Omega)\,. & \end{cases}$$

Passing to the limit with $\varepsilon \longrightarrow 0$ in (1) and using that $v^\varepsilon \rightharpoonup c$ we have

$$(\,Q\,c,c\,) - 2.(Q\,c,c\,) + (\,A\,c,c\,) \geqslant 0\,,$$

and it is clear that $Q \leqslant A$. Therefore, for the first part of the theorem the condition (✳) was not necesary.

b) For the lower bound we consider

$$(3)\qquad ((A^\varepsilon)^{-1}(A^\varepsilon v^\varepsilon + d\,), A^\varepsilon v^\varepsilon + d\,) \geqslant 0$$

where v^ε is given by (2) for $c_{13} \neq 0$ and $c_{sq} = 0$, $sq \neq 13$, and $d_{ij} = -B_{ijkh}c_{hk}$; then $d_{ij} = 0$ for $ij \neq 12\,,13$. We have:

$$(4)\qquad (A^\varepsilon)^{-1}_{pqst}A^\varepsilon_{stij}\frac{\partial u^\varepsilon_i}{\partial x_j}A^\varepsilon_{pqmn}\frac{\partial u^\varepsilon_m}{\partial x_n} = \delta_{pi}\delta_{qj}A^\varepsilon_{pqmn}\frac{\partial u^\varepsilon_i}{\partial x_j}\frac{\partial u^\varepsilon_m}{\partial x_n} \rightharpoonup (Q\,c,c)$$

$$(5)\qquad (A^\varepsilon)^{-1}_{pqst}A^\varepsilon_{stij}\partial u^\varepsilon_i/\partial x_j\cdot d_{pq} = \delta_{pi}\delta_{qj}\,\partial u^\varepsilon_i/\partial x_j d_{pq} \rightharpoonup (d\,,c\,)$$

$$(6)\qquad (A^\varepsilon)^{-1}_{pqst}\,d_{pq}\,d_{st} \rightharpoonup (B^{-1}d,d\,),$$

the convergences being in the weak topology of $L_2(\Omega)$. We have to remark that only the symmetric part of $(\partial u^\varepsilon_i/\partial x_j)$ appeared and that we have used the inversion formula $A_{ijkh}\,(A)^{-1}_{khpq} = \delta_{ip}\,\delta_{jq}$.

It is possible to see that in those three terms defined above the symmetry of A^ε was not necessary.

The term $((A^\varepsilon)^{-1}d\,,\,A^\varepsilon v^\varepsilon)$ may be studied using the condition (✳)

$$(7)\qquad (A^\varepsilon)^{-1}_{pqst}d_{st}A^\varepsilon_{pqij}\partial u^\varepsilon_i/\partial x_j = \left[(A^\varepsilon)^{-1}_{pq12}d_{12} + (A^\varepsilon)^{-1}_{pq13}d_{13}\right]\cdot$$

$$\cdot\, A^\varepsilon_{pqij}\partial u^\varepsilon_i/\partial x_j \rightharpoonup d_{13}\,c_{13}\,.$$

Using (3)-(7) we obtain :

$$(8)\qquad Q_{1313}c^2_{13} + 2\,d_{13}c_{13} + (B)^{-1}_{pqst}d_{pq}d_{st} \geqslant 0\,.$$

But $$(B)^{-1}_{pqst} d_{pq} d_{st} = (B)^{-1}_{pqst} d_{st} (-B_{pqmn} c_{mn}) =$$

$$= -(B)^{-1}_{pqst} B_{pq13} c_{13} d_{st} = - d_{13} c_{13} .$$

We have to remark that if A^ε have some symmetry properties , B and A will have also these properties, because they are obtained by averaging the coefficients of A^ε . In the example presented in § 2 we will prove such symmetry properties of A and B .

The last relation and relation (8) implies :

$$Q_{1313} c_{13}^2 + d_{13} c_{13} \geqslant 0 \quad , \quad Q_{1313} c_{13}^2 - B_{1313} c_{13}^2 \geqslant 0 .$$

Therefore, the condition (*) was used only in the last part of the theorem ☐

In Tartar [6] was noticed that the bounds obtained using the above theorem are not sufficiently precise for the case when A^ε is not continous and the values of A^ε in the fiber tends to zero or to infinity.

To obtain better limits, we define the following convergence which generalises that introduced in Tartar [6] :

Definition 3.

$A^\varepsilon \xrightarrow{13} A$ iff $\quad 1/A^\varepsilon_{1313} \longrightarrow 1/A_{1313} \; , A^\varepsilon_{13sq}/A^\varepsilon_{1313} \longrightarrow A_{13sq}/A_{1313},$

$$A^\varepsilon_{sq13}/A^\varepsilon_{1313} \longrightarrow A_{sq13}/A_{1313},$$

$$A^\varepsilon_{sqkh} - A^\varepsilon_{sq13} \cdot A^\varepsilon_{13kh}/A^\varepsilon_{1313} \longrightarrow A_{sqkh} - A_{sq13} \cdot A_{13kh}/A_{1313},$$

where all convergences are in the weak topology of $L_2(\Omega)$ and (sq), $(kh) \neq (13)$.

We prove first a preliminary result:

Proposition 2.

Let $A^\varepsilon \xrightarrow{13} A$ and $c_{kh} = c_{hk} \in R^3 \times R^3$ . .

Let $A^{\varepsilon}_{13sq} = A^{\varepsilon}_{sq13}$. Then $(\exists)$ $w^{\varepsilon}_{kh} \in L_2(\Omega)$ such that:

1) $w^{\varepsilon}_{kh} \longrightarrow c_{kh}$ weakly in $L_2(\Omega)$,

2) $A^{\varepsilon}_{ijkh} w^{\varepsilon}_{ij} w^{\varepsilon}_{kh} \longrightarrow A_{ijkh} c_{ij} c_{kh}$ weakly in $L_2(\Omega)$.

Proof.

We put $p^{\varepsilon}_{ij} = A^{\varepsilon}_{ijkh} w^{\varepsilon}_{kh}$ for $(ij) \neq (13)$,

$p^{\varepsilon}_{13} = K$ (which will be defined latter) ,

w^{ε}_{13} = variable , $w^{\varepsilon}_{sq} = c_{sq}$ for $(sq) \neq (13)$.

For $(sq) \neq (13)$ we have :

$$K = p^{\varepsilon}_{13} = A^{\varepsilon}_{1313} w^{\varepsilon}_{13} + A^{\varepsilon}_{13sq} c_{sq} \quad ;$$

$$w^{\varepsilon}_{13} = K/A^{\varepsilon}_{1313} - A^{\varepsilon}_{13sq} c_{sq} / A^{\varepsilon}_{1313} \longrightarrow K/A_{1313} - A_{13sq} c_{sq} / A_{1313} .$$

For $(ij) \neq (13)$ we have, replacing w^{ε}_{13} :

$$p^{\varepsilon}_{ij} \longrightarrow c_{sq}\left(A_{ijsq} - A_{ij13}\, A_{13sq}/A_{1313}\right) + K.A_{ij13}/A_{1313} .$$

Using the last relation we can compute the weak limit of the product $p^{\varepsilon}_{rt} w^{\varepsilon}_{rt}$, because p^{ε}_{13} and w^{ε}_{sq} are constant for $(sq) \neq (13)$:

$$p^{\varepsilon}_{rt} w^{\varepsilon}_{rt} \longrightarrow K(K/A_{1313} - A_{13sq} c_{sq}/A_{1313}) +$$

$$+ c_{ij}\left\{A_{ij13} \cdot K/A_{1313} + c_{sq}\left(A_{ijsq} - A_{ij13} \cdot A_{13sq}/A_{1313}\right)\right\} =$$

$$= K^2/A_{1313} - K \cdot A_{13sq} c_{sq}/A_{1313} + K \cdot A_{ij13} c_{ij}/A_{1313} +$$

$$+ c_{sq} A_{ijsq} c_{ij} - c_{ij} c_{sq} A_{ij13} A_{13sq}/A_{1313} .$$

From the condition $A_{13sq} = A_{sq13}$ and $K = c_{13} A_{1313} + A_{13sq} c_{sq}$ it follows :

$$p^{\varepsilon}_{rt} w^{\varepsilon}_{rt} \longrightarrow A_{1313} c^2_{13} + 2\, c_{13} A_{13sq} c_{sq} + c_{sq} c_{ij} A_{sqij}$$

Taking in account the choice of K we obtain $w^{\varepsilon}_{13} \longrightarrow c_{13}$ □

We define now :

MA3 = arithmetical mean with respect to $x_1, x_2, x_4, \dots, x_n$,

MA2 = arithmetical mean with respect to $x_1, x_3, x_4, \dots, x_n$.

Theorem 2.

Let $MA3(A^{\varepsilon}) \xrightarrow{13} B$, $A^{\varepsilon} \xrightarrow{H} Q$, $A_{13sq} = A_{sq13}$.

Then $Q \leq B$.

Proof.

We consider $c_{ij} = c_{ji} \in R^3 \times R^3$, w^{ε}_{ij} the sequence defined in Proposition 2 for $MA3(A^{\varepsilon})$ and c_{ij} , v^{ε}_{kh} given by (2). Then we have the following convergences in the weak sense of $L_2(\Omega)$:

$$v^{\varepsilon}_{kh} \longrightarrow c_{kh} , \quad w^{\varepsilon}_{ij} \longrightarrow c_{ij} ,$$

$$MA3(A^{\varepsilon} v^{\varepsilon}, v^{\varepsilon}) \longrightarrow MA3(Q c , c) = (Q c , c)$$

$$MA3(A^{\varepsilon} v^{\varepsilon}, w^{\varepsilon}) \longrightarrow MA3(Q c , c) = (Q c , c)$$

$$MA3(A^{\varepsilon} w^{\varepsilon}, w^{\varepsilon}) = (MA3((A^{\varepsilon}) w^{\varepsilon}, w^{\varepsilon}) \longrightarrow (B c , c) ,$$

where we used the fact that w^{ε}_{ij} for $(ij) \neq (13)$ are constant and w^{ε}_{13} depends only on x_3 ; therefore :

$$\mathrm{rot}(0, 0, w^{\varepsilon}_{13}, 0, \dots, 0) = 0$$

$$\| \mathrm{div}(p^{\varepsilon}_{11}, p^{\varepsilon}_{12}, p^{\varepsilon}_{13}, \dots, p^{\varepsilon}_{1n}) \|_{L_2(\Omega)} \leq ct,$$

with $p^{\varepsilon}_{ij} = A^{\varepsilon}_{ijkh} \partial u^{\varepsilon}_k / \partial x_h$ and we may use the compensated compactness for $MA3(A^{\varepsilon} v^{\varepsilon}, w^{\varepsilon})$. We start with

(10) $$MA3(A^{\varepsilon}(v^{\varepsilon} - w^{\varepsilon}), v^{\varepsilon} - w^{\varepsilon}) \geq 0 ,$$

and we obtain ,using the above convergences :

$$(Q c , c) - 2 (Q c , c) + (B c , c) \geq 0 \quad \square$$

Theorem 3.

Let A^{ε} with the condition (*), $MA2[(A^{\varepsilon})^{-1}] \xrightarrow{13} (E)^{-1}$, $A^{\varepsilon} \xrightarrow{H} Q$.

Then $Q_{1313} \geqslant E_{1313}$.

Proof.

We consider v^{ε}_{kh} given by (2) with $c_{13} \neq 0$, $c_{sq}=0$ for $(sq) \neq (13)$ and $d_{ij} = - \dot{E}_{ijkh} c_{kh}$, and w^{ε}_{kh} the sequence given in Proposition 2 for $MA2[(A^{\varepsilon})^{-1}]$ and the constants d_{ij}.

Considering :

(11) $\quad MA2((A^{\varepsilon})^{-1}(A^{\varepsilon} v^{\varepsilon} \pm w^{\varepsilon}), A^{\varepsilon} v^{\varepsilon} + w^{\varepsilon}) \geqslant 0$,

we notice that w^{ε}_{13} may depend only by x_2 and consequently :

$$\mathrm{div}(w^{\varepsilon}_{13},0,..,0) = 0 , \quad \mathrm{rot}(\partial u^{\varepsilon}_3/\partial x_1, \partial u^{\varepsilon}_3/\partial x_2, \ldots, \partial u^{\varepsilon}_3/\partial x_n) = 0,$$

$$\mathrm{div}(0,0,w^{\varepsilon}_{13},0,..0)=0, \quad \mathrm{rot}(\partial u^{\varepsilon}_1/\partial x_1, \partial u^{\varepsilon}_1/\partial x_2, \ldots, \partial u^{\varepsilon}_1/\partial x_n) = 0.$$

Using the compensated compactness for $MA2(v^{\varepsilon}, w^{\varepsilon})$ we obtain :

$$\int_{\Omega} (1/2)(\partial u^{\varepsilon}_1/\partial x_3 + \partial u^{\varepsilon}_3/\partial x_1) w^{\varepsilon}_{13} \phi \longrightarrow \int_{\Omega} c_{13} d_{13} \phi, (\forall)\phi \in C^{\infty}_{0}(\Omega).$$

Considering the Remark 1, above observation, and condition (*), we have the following convergences in the weak sense of $L_2(\Omega)$:

$$MA2((A^{\varepsilon})^{-1}(A^{\varepsilon} v^{\varepsilon}), A^{\varepsilon} v^{\varepsilon}) \longrightarrow (Q c , c)$$

$$MA2((A^{\varepsilon})^{-1}(A^{\varepsilon} v^{\varepsilon}), w^{\varepsilon}) \longrightarrow d_{13} c_{13}$$

$$MA2((A^{\varepsilon})^{-1}(w^{\varepsilon}), w^{\varepsilon}) = (MA2(A^{\varepsilon})^{-1} w^{\varepsilon}, w^{\varepsilon}) \longrightarrow (E^{-1} d , d).$$

For the term $MA2((A^{\varepsilon})^{-1} w^{\varepsilon}, A^{\varepsilon} v^{\varepsilon})$, in a manner similar to that of the second part of Theorem 1, taking also in account the condition (*), we have $MA2((A^{\varepsilon})^{-1} w^{\varepsilon}, A^{\varepsilon} v^{\varepsilon}) \longrightarrow d_{13} c_{13}$. The above convergences and the relation (11) give us, as in Theorem 1 :

$Q_{1313} - E_{1313} \geqslant 0$ □

§ 2. A Particular configuration

In this section we consider an example in order to illustrate the above theoretical results. In Hashin, Rosen [3] it is studied a composite material formed by cylindrical fibres of radii r, disposed in the vertexes of a hexagonal array, surrounded by a mâtrix. The method is based on an exact solution in cylindrical co-ordinates, and for this reason the interior of each hexagon is partially filled by the matrix, wich is formed of cylinders of radii 1/2 if the edge lenghts of these hexagons are equal to 1. We consider the medium formed by periodic parallelograms(fig. 1):

Fig. 1

Fig. 2

The stress-strain relation is written in the reduced form :

$$\sigma_{11} = C_{11}e_{11} + C_{12}e_{22} + C_{12}e_{33}$$

$$\sigma_{22} = C_{12}e_{11} + C_{22}e_{22} + C_{23}e_{33}$$

$$\sigma_{33} = C_{12}e_{11} + C_{23}e_{22} + C_{22}e_{33}$$

$$\sigma_{12} = 2\, C_{44}e_{12}$$

$$\sigma_{13} = 2\, C_{44}e_{13}$$

$$\sigma_{23} = (\, C_{22} - C_{23}\,)e_{23} \quad ,$$

where the usual six-by-six matrix notations has been used : C_{11} stands for a_{1111} , C_{12} stands for a_{1122}, and so on. The coefficients have the usual symmetry properties and depend only on x_2 and x[illegible] ; a[illegible] = a[illegible] ;

For the corresponding homogeneous medium it is considered the same type of isotropy. In Tartar [6] the estimates are obtained in a co-ordinate system parallel to the directions of periodicity; we have to transform the cartezian form of the equation

$$(12) \qquad \partial(a^{\varepsilon}_{ijkh}\, \partial u^{\varepsilon}_{k}/\partial x_h)/\partial x_j = F_i$$

in the new system which have an angle of 60^{o} between the axis. If $\underline{g}_k$ and $\underline{f}_p$ are the unit vectors of the cartezian and oblique co-ordinates, we have:

$$\underline{g}_k = \alpha_{pk}\, \underline{f}_p \qquad\qquad \underline{f}_p = \beta_{kp}\, \underline{g}_k ,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/\sqrt{3} \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & \sqrt{3}/2 \end{bmatrix}$$

Then from (12) we obtain

$$\partial(A^{\varepsilon}_{pqst}\, \partial u'_s/\partial x'_t)/\partial x'_q = F'_p$$

where $'$ denotes the components in the new system and

$$(13) \qquad 2\, A_{pqst} = \alpha_{pi}\, \beta_{qj}\, (\beta_{hs}\, \alpha_{tk} + \beta_{ks}\, \alpha_{th})\, a_{ijkh} .$$

The stress-strain relations have the form :

$$(14) \quad \begin{cases} \sigma'_{11} = a_{1111} e'_{11} + a_{1122} e'_{22} + a_{1133} e'_{33} \\ \sigma'_{22} = (4a_{2211}/3) e'_{11} + (5a_{2222} - a_{3322})/3\ e'_{22} + \\ \qquad\qquad + (5a_{2233} - a_{2222})/3\ e'_{33} - (4a_{2323}/3) e'_{23} \\ \sigma'_{33} = (4a_{3311}/3) e'_{11} + (4a_{3322}/3) e'_{22} + (4a_{3333}/3) e'_{33} \\ \sigma'_{12} = (4a_{1313}/3) e'_{12} - (2a_{1313}/3) e'_{13} \\ \sigma'_{13} = -(2a_{1313}/3) e'_{12} + (4a_{1313}/3) e'_{13} \\ \sigma'_{23} = -(2a_{3311}/3) e'_{11} - (2a_{2222}/3) e'_{22} - \\ \qquad\qquad - (2a_{2233}/3) e'_{33} + (4a_{2323}/3) e'_{23} , \end{cases}$$

therefore

$$A_{1313} = A_{1212} = 4a_{1313}/3 .$$

For the matrix A^{ε}_{pqst} we have the first part of the condition (*) : $A^{\varepsilon}_{13sq} = A^{\varepsilon}_{sq13}$. It remains to prove only the symmetry of the columns and rows 12sq,13sq,sq12,sq13 for the matrix $(A^{\varepsilon})^{-1}$, using the reduced form of (A^{ε}). Each minor of the row A^{ε}_{13sq} have two proportional columns(for sq $\neq$ 12 and 13) : after the elimination of the row 13 ,the columns 12 and 13.

In a similar manner , the minors of the column A_{sq13} are zero(sq $\neq$ 13 and 12) ; after the elimination of the column 13 the rows 12 and 13 being proportional; therefore:

$$(A^{\varepsilon})^{-1}_{sq13} = 0, \quad (A^{\varepsilon})^{-1}_{13sq} = 0 \ , \quad sq \neq 12 \text{ and } 13 \ .$$

The minors of the elements A^{ε}_{1312} and A^{ε}_{1213} are both equal to

$$-(2 \cdot a^{\varepsilon}_{1313}/3)\begin{vmatrix} A^{\varepsilon}_{1111} & A^{\varepsilon}_{1122} & A^{\varepsilon}_{1133} & 0 \\ A^{\varepsilon}_{2211} & A^{\varepsilon}_{2222} & A^{\varepsilon}_{2233} & A^{\varepsilon}_{2323} \\ A^{\varepsilon}_{2311} & A^{\varepsilon}_{2322} & A^{\varepsilon}_{2323} & 0 \end{vmatrix}$$

and consequently we have $(A^{\varepsilon})^{-1}_{13sq} = (A^{\varepsilon})^{-1}_{sq13}$, $(A^{\varepsilon})^{-1}_{12sq} = (A^{\varepsilon})^{-1}_{sq12}$, the condition (*) .

The element $(A^{\varepsilon})^{-1}_{1313}$,obtained using the developpement of $\det(A^{\varepsilon})$ with respect to the row A^{ε}_{12st} , is given by

$$(15) \qquad (A^{\varepsilon})^{-1}_{1313} = 1/a^{\varepsilon}_{1313} \ .$$

We note that $(A^{\varepsilon})^{-1}_{1313}$ is used to construct the sequence w^{ε} defined in Theorem 3 ; it is important to see that the homogenized coefficient corresponding to a^{ε}_{1313} is estimated in Hashin,Rosen [3] using only the values of this coefficient ; in the method presented here, the estimates may generally depend on the other coefficients.

We must compute, following Definition 3 , the 13-limit of arithmetical mean with respect to x'_2 and the 13-limit of harmonical mean with respect to x'_3 of the coefficient A_{1313} ,for which is possible to obtain upper and lower bounds :

$$\lim_{\varepsilon \to 0} MA2(A^{\varepsilon}_{1313}) = B_{1313} \;, \quad \lim_{\varepsilon \to 0} MA3\left[(A^{\varepsilon}_{1313})^{-1}\right] = E^{-1}_{1313} .$$

We obtain finally :

$$E_{1313} \leq Q_{1313} \leq B_{1313} \qquad (16)$$

Using the method given here ,it is possible to obtain bounds for the effective coefficients in the case when the space between the periodical fibers is completely filled by the matrix of low resistence. We consider in the following this situation .

Now we analyse the geometry of the periodicity cell (fig. 1). In the vertex of the romb of edge lenghts 1 and 60^o in the origin, are situated sectors of radii r and 60^o(resp. 120^o), filled by the fibre . The rest of the romb is filled by the matrix of low resistence. We can see that AB = CD . For simplicity we put $x_3'=t, x_2'=s$. We have (see fig.2):

$$t = OM \;, \quad OZ = t\sqrt{3}/2$$

$$ZH = \left(\sqrt{4r^2 - 3t^2}\right)/2 \;, \quad UH = 2\cdot ZH = \sqrt{4r^2 - 3t^2}$$

The maximum value of t for which a parallel line to Os intersects the circle of radii r is $ON = 2r/\sqrt{3}$. We consider the case when such a parallel may intersect only two circles (centers whose are situated on Os), then we consider $r < \sqrt{3}/4$, which is in concordance with the above hypothesis.

We put $A_{1313} = h$ in the fiber, $A_{1313} = 1$ in the matrix.

For the upper bound we have

$$MA2(A_{1313}) = \begin{cases} h\sqrt{4r^2 - 3t^2} + 1 - \sqrt{4r^2 - 3t^2} \;, & t \in (0, 2r/\sqrt{3}) \\ 1 \;, & t \in (2r/\sqrt{3}, 1/2) \end{cases}$$

and using that the weak limit in $L_2(\Omega)$ of a periodic function of (x/ε), for $\varepsilon \to 0$, is the mean value on a cell (cf. Sanches-Palencia [5]) , we obtain :

(17) $$A^{+}_{1313} = \left\{ 2 \cdot \int_0^{2r/\sqrt{3}} \left[(h-1)\sqrt{4r^2 - 3t^2} + 1 \right]^{-1} dt + 1 - 4r/\sqrt{3} \right\}^{-1}$$

The lower bound is given by:

(18) $$A^{-}_{1313} = 2 \cdot \int_0^{2r/\sqrt{3}} \left[\left(\frac{1}{h} - 1\right)\sqrt{4r^2 - 3t^2} + 1 \right]^{-1} dt + 1 - 4r/\sqrt{3}$$

In Hashin, Rosen [3] are given the following bounds:

(19) $$G^{+} = (m_g v_1 + v_2)$$

(20) $$G^{-} = (v_1 / m_g + v_2)^{-1}$$

where

$$m_g = \left[h(1+b^2) + 1 - b^2 \right] / \left[h(1-b^2) + 1 + b^2 \right] , \quad b = 2r ,$$

$$v_1 = 0.918 , \quad v_2 = 0.082 = 1 - v_1 .$$

In order to compare these two methods, we compute the expressions (17),(18),(19),(20) for different values of h and r. Considering, for example, r=0.2 and r= 0.3 , we obtain the following results:

Table 1: r= 0.2

h	A^{-}_{1313}	G^{-}	G^{+}	A^{+}_{1313}
2	0.819	1.103	1.112	0.840
3	0.846	1.157	1.159	0.907
6	0.879	1.232	1.236	1.037
10	0.889	1.275	1.282	1.140
20	0.908	1.302	1.310	1.236
30	0.909	1.314	1.323	1.271
50	0.912	1.323	1.333	1.315
100	0.914	1.331	1.341	1.351
⋮	⋮	⋮	⋮	⋮
∞	0.917	1.339	1.349	1.393

Table 2 : r = 0.3

h	A^{-}_{1313}	G^{-}	G^{+}	A^{+}_{1313}
5	1.086	1.551	1.579	1.341
15	1.198	1.785	1.844	1.846
35	1.236	1.872	1.945	2.116
65	1.250	1.905	1.984	2.249
115	1.258	1.922	2.005	2.326
155	1.260	1.928	2.012	2.354
195	1.262	1.931	2.016	2.371
⋮	⋮	⋮	⋮	⋮
∞	1.268	1.945	2.032	2.441

One can see that in general the estimates given in Hashin , Rosen [3] are more precise that those given by the method presented here.

We want to emphasize that only for A^{ε}_{1313} it is possible to obtain both the upper and the lower limits using the above theoretical results. In general, because A^{ε} verifies the first part of the condition (⁎) , in the sense of Definition 2 only the upper limit may be computed.

In Hashin,Rosen [3] there are computed also the bounds for following constants :

$$K_{23} = (a_{2222} + a_{2233}) \quad , \quad G_{23} = (a_{2222} - a_{2233})$$

$$E_1 = a_{1111} - 2a_{1122} /(a_{2222} + a_{2233}) .$$

If we consider $\lambda_{ij} = 0, \lambda_{22} = \lambda_{23} = \lambda$ then (14) yields

$$A_{ijkh}\lambda_{ij}\lambda_{kh} = 8 \cdot \lambda^2 (a_{2222} + a_{2233})/3 .$$

For $\lambda_{ij} = 0, \lambda_{23} = \lambda_{32} = \lambda$ we have

$$A_{ijkh}\lambda_{ij}\lambda_{kh} = 8 \cdot \lambda^2 (a_{2222} - a_{2233})/3.$$

Therefore we can apply our method to obtain upper bounds for K_{23} and G_{23} . Because in general the coefficients with $ij \neq kh$ can not be directly estimated ,it seems that for E_1 (which represents the longitudinal Young's modulus) the method presented here is not applicable .

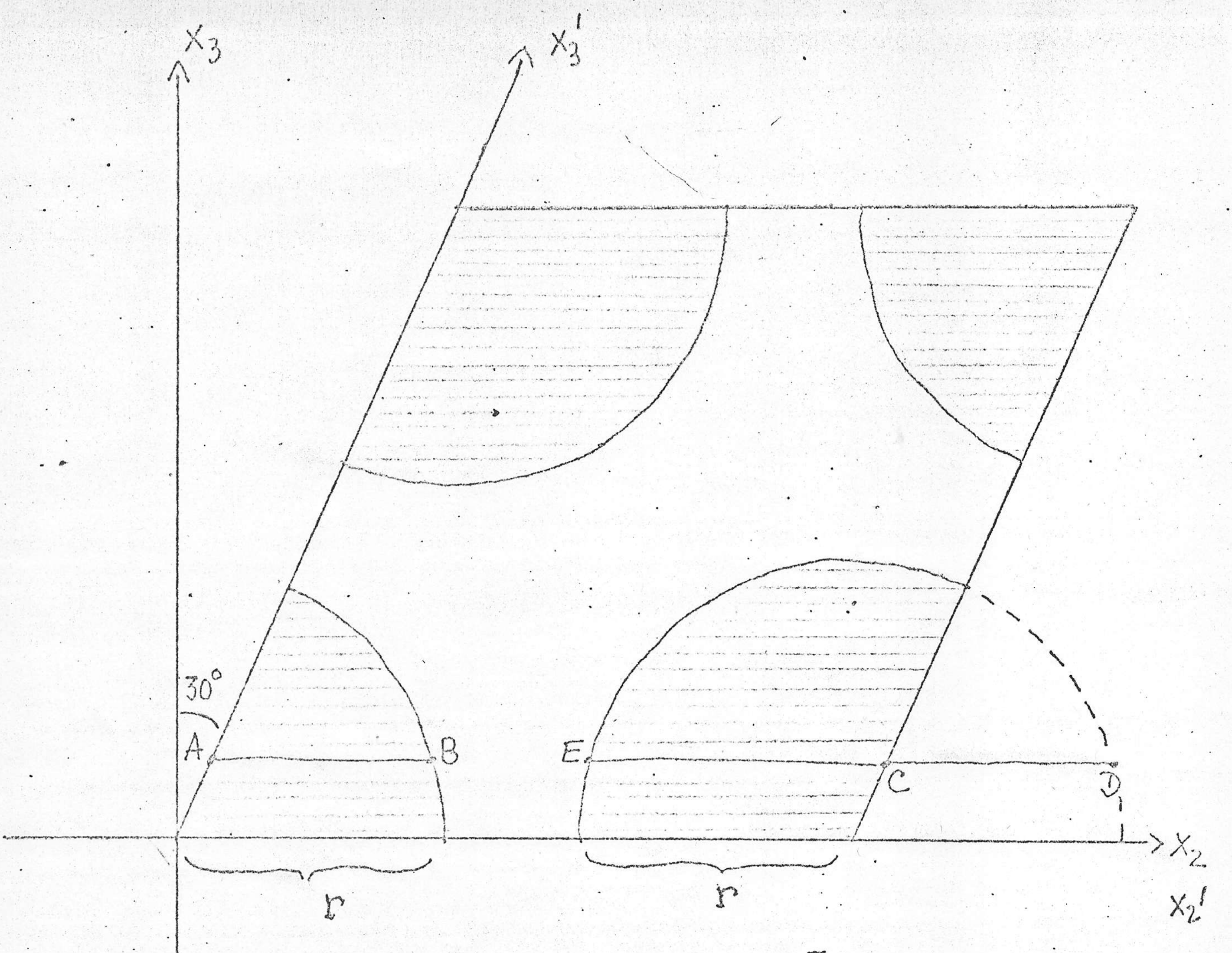

Fig. 1

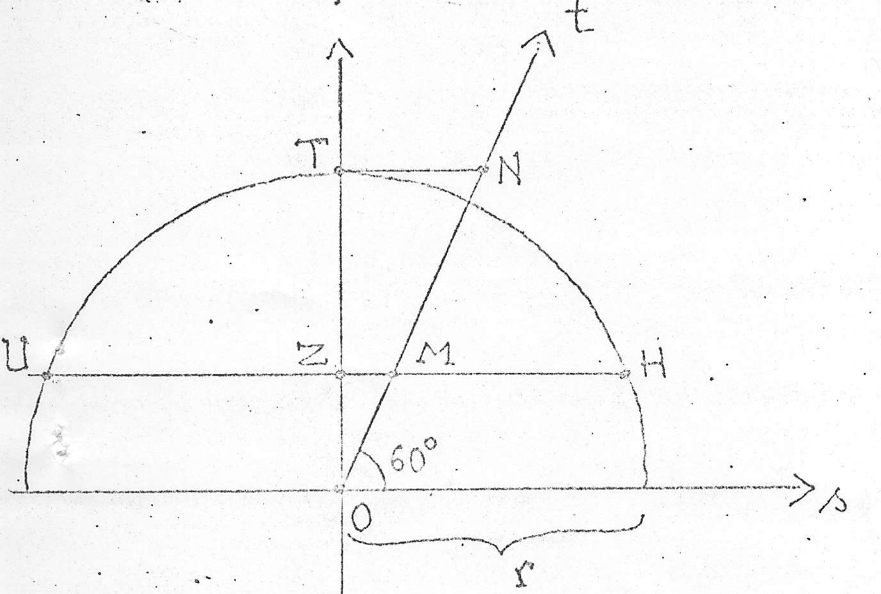

Fig. 2

R E F E R E N C E S

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