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Sorin POPA

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Sorin POPA*)

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*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, ROMANIA.

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The first operator algebra analogue of the rigidity phenomena in representation of groups and ergodic theory were obtained by Connes ([2]). He used discrete groups G with the property T of Kazhdan to construct type II_1 factors M=L(G) with discrete automorphism group Aut M/Int M. As a consequence these factors were shown to have countable fundamental group $\mathcal{F}(M)$ (as defined in [9]), a fact that may be viewed as a typical and specific rigidity phenomena in operator algebra. Then in [3] Connes defined the property T for arbitrary type II_1 factors, in a way that makes equiválent the property T of a group G and of its von Neumann algebra L(G).

We present in this note some new rigidity results concernin finite von Neumann algebras with proprety T.

Our first theorem shows that in some sense the set of subfactors with property T of an arbitrary II_1 factor is poor. To state it we recall that two von Neumann subalgebras B, B_o of a type II_1 factor M are iner conjugate if there is a unitary $u \in M$ such that $uBu^* = B_o$. And that two factors M and N are called stable equivalent if either there exists a projection $e \in M$ so tha N is isomorphic to eMe or there exists a projection $f \in N$ so that M is isomorphic to fNf.

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THEOREM 1. Let M be a separable type II_1 factor. The set of classes of inner conjugate subfactors with property T of M having finite dimensional relative commutant in M is countable. Moreover the set of classes of stable equivalent subfactors with property T of M (not necessarily with the same unit as M) is also countable.

This theorem generalizes the results in [2], [10]. Another extension of those results is given below. On the other hand the preceding theorem is related to a well known próblem in operator algebra asking whether there exists a separable type II_1 factor containing copies of any other separable finite factor (cf. [12] problem 4.4.29). Indeed if in this problem we require in addition the subfactors to have small relative commutant then by Thoerem 1 the answer to the problem is no. If in turn there would exist such a universal separable II_1 factor then, again by the theorem, it would follow that, up to stable equivalence, there would be only countably many rigid factors. That this is true or not seems to be a problem of independent interest.

The next result shows that for a separable II_1 factor M to have a countable fundamental group $\mathfrak{F}(M)$ or, more generally, a countable set of indices of subfactors $\mathcal{J}(M)$ ([7]) it is sufficient to contain a subfactor with property T with small relative commutant.

THEOREM 2. Let M be a separable type II_1 factor. If M contains a subfactor with property T, $N\subset M$, so that $N'\cap M$ has a nontrivial atomic part then $\mathfrak{F}(M)$ and $\mathcal{J}(M)$ are countable.

The last theorem we discuss contains a new type of rigidity result, not considered until now in operator algebra or in ergodic theory. Namely, we compare the different restrictions of a measured equivalence relation and show that, once it contains an ergodic action of a rigid group, most of them are nonisomorphic (in the sense of orbit equivalence [6]).

THEOREM 3. Let $\mathcal{R}$ be an ergodic countable measured equivalence relation on a nonatomic probability space with $\mathcal{R}$-invariant measure $(X,\mathcal{X},\mu)$. Suppose $\mathcal{R}$ contains the free ergodic action of a group with property T. Then there is a countable set $S_0 \subset [0,1]$ such that whenever $F \in \mathcal{X}$ and $\mu(F) \notin S_0$ the restriction of $\mathcal{R}$ to F is non-isomorphic to $\mathcal{R}$. In other words, if M is the type II_1 factor with normalized trace τ constructed from $\mathcal{R}$ as in [6] and if $A=L^{\infty}(X,\mathcal{X},\mu) \subset M$ is the corresponding Cartan subalgebra then for any projection $e \in A$ with $\tau(e) \notin S_0$ there are no isomorphisms of M onto eMe carying A onto Ae.

Since the group measure algebra constructed in [5] satisfies the above conditions and is so that the corresponding II_1 factor is isomorphic to its tensor product by the hyperfinite II_1 factor R, we get by Theorem 3:

COROLLARY. There exist separable II_1 factors with uncountable many nonconjugate Cartan subalgebras. More precisely there exists a separable type II_1 factor M with a Cartan subalgebra $A \subset M$ so that, for a certain countable set $S_0 \subset [0,1]$, given any projection $e \in A$ with $\tau(e) \notin S_0$ there are no isomorphisms of M onto eMe carying A onto Ae, but so that $M \simeq M \otimes R$ (and thus $M \simeq eMe$ for all e).

Although with a statement of ergodic theory flavor, Theorem 3 has a purely operator algebra proof. It is in fact the immediate consequence of the following more general:

THEOREM 4. Let M be a separable type II_1 factor and $B\subset M$ a von Neumann subalgebra. Assume there exist type II_1 subfactors $N_0\subset N\subset M$ so that $B\subset N$, $N_0'\cap N=\mathbb{C}$, $N'\cap M=\mathbb{C}$ and so that N_0 and the inclusion $B\subset N$ have the property T (see [11]). Then there exists a countable set $S_0\subset[0,1]$ such that for every projection $e\in B$ with $\tau(e)\notin S_0$ there are no isomorphisms of M onto eMe sending B onto eBe.

Theorems 1, 2 and 4 have a common line of proof:

1° One begins by contradiction and one uses an appropriate separability argument (like in [10]) to get from it the existence of pointwise arbitrary close subfactors with property T (or morphisms from the same subfactor with property T);

2° One uses this to get completely positive maps, of the same property T subfactor, close in certain points to the identity of that subfactor (like in [10]); 3° then one uses the trick of [4] to show that these completely positive maps follow uniformly close to the identity; 4° then one interprets this to get that the initial algebras (or morphisms) are uniformly close (in the sense of [1]); 5° one uses Christensen's results ([1]) to obtain that the algebras (or morphisms) are inner conjugate thus getting a contradiction which ends the proof.

Details of all these as well as an exposition of the necessary technical background can be found in [11].

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Department of Mathematics, INCREST, Bd.Pacii 220, 79622
Bucharest, Romania.