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ON THE EFFECTIVE COEFFICIENTS OF FIBER - REINFORCED PERIODIC MATERIALS

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Abstract. In the case $a_{ij}=a_{ij}(x_1)$ we obtain the effective coefficient a^o_{ij} by choosing suitable test functions in the local problem. We give an example for which a^o_{12} may be positive or negative, even if the corresponding initial coefficients are equal to zero.

Introduction

We consider a composite material formed by a matrix of low resistence in which fibers (or inclusions) of high resistence are periodically inserted. When the dimension of this periods is very small compared with that of the entire medium, the characterization of the material is obtained by passing to the limit in a family of problems, indexed by the ratio between the two dimensions. In this way, the initial medium with spatial periodic coefficients is approximated by a medium with constant coefficients; they are obtained by solving a problem in a single period - so called local standard problem ([1] , [2]). The material obtained in this way(in general non- isotropic) have a better resistence properties in certain directions.

We consider here the standard problem (1) which characterize the thermal conductivity of a composite medium with two components, the periodicity cell being the unit square (fig. 1); in this case, the effective conductivity is given by (2):

$$(1)\quad \begin{cases} \partial(a_{ij}\partial P^k/\partial y_j)/\partial y_i = 0 \quad \text{in} \quad \mathring{Y}_1 \cup \mathring{Y}_2 \\ n_i\left[a_{ij}\partial P^k/\partial y_j + a_{ip}\delta_{pk}\right]/_{\Sigma} = 0 \\ a_{ij} = Y \text{ periodic, elliptic; } P^k = Y \text{ periodic} \end{cases}$$

Fig. 1

$$(2)\quad a^o_{ij} = \langle a_{ij}\rangle + \langle a_{ik}\partial P^j/\partial y_k\rangle ,$$

where $\langle\cdot\rangle = \int \cdot \, dy/|Y|$ denote the mean value over the period Y.

In general , the problem (1) have not an exact solution.

We consider $y_1 = x$, $y_2 = y$. For instance, if the coefficients a_{ij} depend only on x, it is possible to obtain the effective coefficients using the formulas (3) given in [3]. These formulas were obtained using the regularity of the solution and the weak limits of certain combinations of the initial coefficients.

$$(3)\quad \begin{cases} a_{11}^{o} = \langle 1/a_{11}\rangle^{-1}, \quad a_{i1}^{o}/a_{11}^{o} = \langle a_{i1}/a_{11}\rangle \text{ if } i \neq 1 \\ a_{22}^{o} - a_{21}^{o}\, a_{12}^{o}/a_{11}^{o} = \langle a_{22} - a_{21}\, a_{12}/a_{11}\rangle . \end{cases}$$

In the first part of the paper, we obtain the effective coefficients by choosing suitable test functions in the variational formulation of the problem (1). With this method we can obtain some characterizations of the solutions, even in the symmetric case , for a_{ij} depending on x and y (cf. [4]).

In the last part we consider an example of a periodic medium for which it is possible to obtain the exact formulas for the effective coefficient a_{12}^{o}. This value may be positive or negative for different " orientation " of the inclusion, even if $a_{12} = a_{21} = 0$.

1. The case $a_{ij} = a_{ij}(x)$

First, we consider the inclusion to be symmetrical with respect to the symmetry axis of the unit square Y (fig. 2)

Let $a_{ij} = \mu\delta_{ij}$, where μ takes the values α and β in Y_2 (fiber) and $Y - Y_2$ (matrix of low resistence). We use the test function ϕ defined by (4), where $x_1 + x_2 = 1$, x_i being the co-ordonates which define the boundaries of the inclusion.

$$(4)\quad \phi(x) = \begin{cases} x & , x \in (0, x_2) \\ -x_2(x-1)/x_1 & , x \in (x_2, 1) \end{cases}$$

Fig. 2

In this case , we have the following symmetry properties (cf. [4]):

$$\begin{cases} P^1 = 0 \quad \text{on} \quad x=0 \quad \text{and} \quad x=1 \\ P^1 \quad \text{odd in the direction } Ox \\ P^1 \quad \text{even in the direction } Oy. \end{cases}$$

The function ϕ being Y periodic, the variational formulation of problem (1) is

$$(5) \qquad \int_Y \mu \, \partial P^1/\partial x \cdot \partial \phi/\partial x = -\int_Y \mu \, \partial \phi/\partial x .$$

In the following we denote by $a = x_2 - x_1$ and $\Gamma_i = \{(x,y), x = x_i\}$.

Using the above symmetry properties, we have

$$(6) \qquad \int_{\Gamma_2} P^1 = -(\alpha - \beta) \cdot a \cdot x_1 / (2x_1\alpha + a\beta)$$

The computation of a^o_{11} follows :

$$(7) \qquad a^o_{11} = \langle a_{11} \rangle + \int_Y a_{1k} \partial P^1/\partial y_k = \langle a_{11} \rangle + (\alpha - \beta) \int_\Sigma P^1 n_1 =$$

$$= a_{11} + 2(\alpha - \beta) \int_{\Gamma_2} P^1 = \alpha\beta \, / (\alpha(1-a) + \beta a).$$

In order to compare with the exact formula (3) , we compute

$$\langle 1/a_{11} \rangle^{-1} = \{ a/\alpha + (1-a)/\beta \}^{-1} = \alpha\beta / (\alpha(1-a) + \beta a) .$$

Then the relation (3) for a^o_{11} it is obtained .

For the effective coefficients a^o_{12} and a^o_{21} we have :

$$a^o_{12} = \langle a_{12} \rangle + \int_Y a_{11} \partial P^2/\partial x ,$$

$$a^o_{21} = \langle a_{21} \rangle + \int_Y a_{22} \partial P^1/\partial y ,$$

which give us $a^o_{12} = a^o_{21} = 0$, because $P^2 = 0$(P^2 verifying an equation with the right hand side equal to zero) and P^1 is Y periodic, a_{22} depending only on x. The last relation in (3) it is also verified , because

$$a^o_{22} = \langle a_{22} \rangle + \int_Y a_{22} \partial P^2/\partial y = \langle a_{22} \rangle$$

Let us remark that because $a_{12} = a_{21} = 0$, the symmetry condition give us $a^o_{12} = a^o_{21} = 0$.

Now, we consider $a_{12} \neq 0$. We denote by

$$a_{11} = \begin{cases} \alpha \text{ in } Y_2 \\ \beta \text{ in } Y_1 \end{cases} \qquad a_{21} = \begin{cases} \gamma \text{ in } Y_2 \\ \delta \text{ in } Y_1 \end{cases}$$

In this case, the form of a^o_{11} is the same as above, because the equation for P^1 is

$$\int_Y a_{11} \partial P^1/\partial x \cdot \partial \phi/\partial x + \int_Y a_{12} \partial P^1/\partial y \, \partial \phi/\partial x = - \int_Y a_{11} \cdot \partial \phi/\partial x$$

and $\partial P^1/\partial y$ is odd in the direction Oy; therefore we obtain (5), but with a_{11} instead of μ.

For a^o_{21} we have

$$(8) \qquad a^o_{21} = \langle a_{21} \rangle + \int_Y a_{21} \partial P^1/\partial x = \gamma a + \delta(1-a) - 2(\gamma - \delta) \frac{(\alpha-\beta) a x_1}{\alpha(1-a)+\beta a}$$
$$= \{ a\gamma\beta + (1-a)\delta\alpha \} / \{ \alpha(1-a) + \beta a \},$$

obtained using the expression of $\int_{\Gamma_2} P^1$ given by (6), but with a_{11} instead of μ.

The formula (3) give us :

$$\langle a_{21}/a_{11} \rangle = a\gamma/\alpha + (1-a)\delta/\beta = \{ a\gamma\beta + (1-a)\delta\alpha \} / \alpha\beta,$$

$$a^o_{21} = \langle a_{21}/a_{11} \rangle a^o_{11} = \{ a\gamma\beta + (1-a)\delta\alpha \} / \{ \alpha(1-a) + \beta a \},$$

therefore we obtain the same result for a^o_{21}.

To obtain a^o_{12}, we have (because $P^2 = 0$ in this case)

$$(9) \qquad a^o_{12} = \langle a_{12} \rangle + \int_Y a_{11} \partial P^2/\partial x = a\gamma + (1-a)\delta - 2(\alpha-\beta) \frac{(\gamma-\delta) a x_1}{\alpha(1-a) + \beta a}.$$

The above formula was obtained using the fact that P^2 verify an equation of the same type with those of P^1, but in the right hand side a_{11} is replaced by a_{12}. That is why P^2 will have the same symmetry property as P^1. We note that the expression given by (8) and (9) are the same.

The computation of a^o_{22} is made using the relation :

$$a^o_{22} = \langle a_{22}\rangle + \int_Y a_{21}\partial P^2/\partial x = \langle a_{22}\rangle - 2(\gamma-\delta)\frac{(\gamma-\delta)ax_1}{\alpha(1-a)+\beta a},$$

and we need to verify that

$$a^o_{21}\, a^o_{12}/a^o_{11} - \langle a_{21}\, a_{12}/a_{11}\rangle = -2\frac{(\gamma-\delta)^2 ax_1}{\alpha(1-a)+\beta a}$$

Using (7) and (8) we obtain :

$$a^o_{21}\, a^o_{12}/a^o_{11} - \langle a_{21}\, a_{12}/a_{11}\rangle = \frac{\{a\gamma\beta+(1-a)\delta\alpha\}^2}{\alpha\beta[(1-a)\alpha+a\beta]} - \left[a\frac{\gamma^2}{\alpha} + (1-a)\frac{\delta^2}{\beta}\right] =$$

$$= -(\gamma-\delta)^2 a(1-a)/\{\alpha(1-a)+\beta a\}.$$

Therefore in this case also we obtain the last of the relations (3). We emphasize that in this case P^1 and P^2 have necessary the same symmetry properties ,unlike in the two dimensional case where P^2 is even in the direction Ox and odd in the direction Oy.

Finally we consider $a_{ij} = a_{ij}(x)$ but the inclusion is not symmetric with respect to the symmetry axis of the period Y. We have

$$a^o_{11} = \langle a_{11}\rangle + (\alpha-\beta)\left\{\int_{\Gamma_2} P^1 - \int_{\Gamma_1} P^1\right\} \tag{10}$$

$$a^o_{12} = \langle a_{12}\rangle + (\gamma-\delta)\left\{\int_{\Gamma_2} P^1 - \int_{\Gamma_1} P^1\right\} \tag{11}$$

$$a^o_{22} = \langle a_{22}\rangle + (\gamma-\delta)\left\{\int_{\Gamma_2} P^2 - \int_{\Gamma_1} P^2\right\} \tag{12}$$

We consider the following test function :

$$\psi(x) = \begin{cases} x & , x\in(0,x_1) \\ x(1-x_2-x_1)+x_1(2x_2-1)\;/(x_2-x_1), & x\in(x_1,x_2) \\ 1-x & , x\in(x_2,1) \end{cases} \tag{13}$$

The equations of P^1 and P^2 give us

$$\int_{\Gamma_2} P^1 - \int_{\Gamma_1} P^1 = -\ a(1-a)(\alpha-\beta)/\{\alpha(1-a)+\beta a\} \tag{14}$$

$$\int_{\Gamma_2} P^2 - \int_{\Gamma_1} P^2 = -\ a(1-a)(\gamma-\delta)/\{\alpha(1-a)+\beta a\}, \tag{15}$$

and the expression of a^o_{ij} are the same with those obtained above.

The difference between the case of symmetric and respectively non-symmetric inclusion consist of the different forms of the test functions. The test function ψ, having the same derivatives for $x=0$ and $x=1$, the values of P^i on this part of Y are canceled. In the symmetric case we have $P^1 = 0$ on $x = 0$ and $x = 1$.

An interesting property is the following: if we consider a_{21} = constant(with the same value in the fiber and in the matrix), then using (15) we obtain $a^o_{22} = \langle a_{22} \rangle$.

2. The case $a_{ij} = a_{ij}(x,y)$

We consider symmetric inclusion(fig.3), and $a_{ij} = \mu\delta_{ij}$, where μ takes the values β and α in Y_1, respectively Y_2.

The test function given by (4) have a jump across of Γ_2.
For this reason, following the above method, we have two unknowns :

$\int_{BC} P^1$ and $\int_{\Gamma_2-BC} P^1 = R$ which satisfy the relation

$$\left\{ 2\alpha + \beta a/x_1 \right\} \int_{BC} P^1 + \frac{\beta}{x_1} \int_{\Gamma_2-BC} P^1 = -(\alpha - \beta)a^2$$

The last relation gives us

$$(16) \quad \int_{BC} P^1 = \frac{(\beta - \alpha)\, a^2 x_1}{(1-a)\alpha + a\beta} - \frac{R}{\alpha(1-a) + \beta a} \cdot \beta ,$$

and we obtain for a^o_{11}

$$(17) \quad a^o_{11} = \alpha a^2 + (1-a^2)\beta + 2(\alpha - \beta) \int_{BC} P^1 =$$

$$= \beta \cdot \left\{ \alpha(1-a-a^2) + \beta a(1-a) - 2(\alpha - \beta)R \right\} / \left\{ \alpha(1-a) + \beta a \right\}.$$

We want now to compute, using the method described in [6], the lower limit of the coefficient a^o_{11}; we compute the harmonical mean of a_{11} and, after that, the harmonical mean of the previous value.

$$MH_x\left(\frac{1}{a_{11}}\right) = \begin{cases} (1-a)/\beta + a/\alpha = \left\{ \alpha(1-a) + \beta a \right\} / \alpha\beta & , x \in (x_1, x_2) \\ 1/\beta & , x \notin (x_1, x_2) \end{cases}$$

Therefore , the lower limit is given by .

(18) $AINF = MH_y(1/MH_x) = \beta\{\alpha(1-a-a^2)+\beta a(1-a)\}/\{\alpha(1-a)+\beta a\}$.

The relation $a^0_{11} \geqslant AINF$ gives us

$$-2(\alpha-\beta)R/\{\alpha(1-a)+\beta a\} \geqslant 0,$$

and we obtain , if $\alpha > \beta$ (in general the fiber is more resistent than the matrix), then $R \leqq 0$. This result agrees with the numerical results obtained in [5] , because in the right part of the unit square Y we have obtained for P^1 only negative values.

Now , we consider a test function $\phi \in C(Y)$ which is not periodic. The integral form of problem (1) is :

$$(19) \quad 0 = -\int_{\Sigma}[\mu]\phi n_1 + \beta\int_{Y}\partial P^1/\partial y_i \phi n_i - \int_{\Sigma} P^1[\mu\partial\phi/\partial x_i]n_i -$$
$$-\beta\int_{\partial Y} P^1 \partial\phi/\partial y_i \cdot n_i + \int_{Y}\mu P^1 \Delta\phi$$

where $\vec{n}$ is the outward normal to Y_2, $\Sigma = Y_1 Y_2$ and $[\ ]$ denotes the jump . It is possible to solve the problem - to compute $\int_{\Sigma} P^1$ - if we have a test function with the following properties :

$$\begin{cases} \phi(0,y) = \phi(1,y) \ ; \ \partial\phi/\partial y(x,0) = \partial\phi/\partial y(x,1); \\ \Delta\phi = \text{const., even in } x \text{ or odd in } y. \end{cases}$$

It is well known that if f is even then f' is odd; but the converse is not true. For instance if $g(x) = x(x-1)$, then $g(x) = x^3/3 - x^2/2 + A$; we have $g(1) = -1/6 = 0 \neq -g(0)$.

In the particular case of test function ϕ wich depends only on x , we have

$$0 = -\int_{\Sigma}[\mu]\phi n_1 + \beta\int_{x=1}\partial P^1/\partial x\{\phi(1)-\phi(0)\} -$$
$$-\int_{BC} P^1[\mu]\{\phi/_{BC} + \phi/_{AD}\} + \int_{Y}\mu P^1 \Delta\phi$$

If we consider $\phi = x$, we obtain :

$$(20) \quad 2[\mu]\int_{BC} P^1 = \beta\int_{x=1}\partial P^1/\partial x - [\mu]\,|Y_2|.$$

It is interesting to see that the relation (20) is obtained for all test functions which verify $\phi'' =$ even (in order to cancel the mean value of P^1):

Proposition.

$$\Phi \in C^{\infty}(Y),\ \Phi'' \text{ even } \Longrightarrow \Phi'(x_1) + \Phi'(x_2) = 2\{\Phi(1) - \Phi(0)\} .$$

Proof.

First, we consider the following developpements for derivatives:

$$\Phi'(x_1) = \Phi'(0) + \Phi''(0)x_1 + \Phi'''(0)x_1^2/2 + \Phi^{IV}(0)x_1^3/6 + \dots$$
$$\Phi'(x_2) = \Phi'(1) - \Phi''(1)x_1 + \Phi'''(1)x_1^2/2 - \Phi^{IV}(1)x_1^3/6 + \dots$$

We have $\Phi^{(2k+1)}(x)$ odd and $\Phi^{(2k)}(x)$ even for $k \geq 1$, therefore $\Phi'(x_1) + \Phi'(x_2) = \Phi'(0) + \Phi'(1)$.

Considering the same developpements for Φ, we have :

$$\Phi(1) = \Phi(0) + \Phi'(0) + \Phi''(0)/2 + \Phi'''(0)/6 + \Phi^{IV}(0)/24 + \dots$$
$$\Phi(0) = \Phi(1) - \Phi'(1) + \Phi''(1)/2 - \Phi'''(1)/6 + \Phi^{IV}(1)/24 + \dots ,$$

then we obtain $\quad 2\{\Phi(1) - \Phi(0)\} = \Phi'(0) + \Phi'(1)$ ∎.

It follows that is not possible to obtain a system for the two unknowns.

In some particular cases it is possible to compute some elements of the solution. In the case $Y_2 = \{ (x,y) ,\ 1/4 \leq x,y \leq 3/4 \}$, we consider $\Phi(x) = (x-1/2)(x-1)$, and (20) gives us

$$\int_{Y_d} \mu P^I = -[\mu]/64 ,$$

where $Y_d = \{ (x,y) ,\ x \geq 1/2 \}$. This result agrees with the inequality $R \leq 0$ obtained above.

3. A particular configuration

We consider the configuration given in fig. 4 , where the unit square contains two oblique strips in which the fiber is introduced.

Fig. 4 Fig. 5

We denote by $\vec{e_i}$ the versors of the cartesian axis and by $\vec{f_i}$ the versors of the oblique axis. We have the relations :

$$\vec{f_k} = \sum \theta_{pk} \vec{e_p} \quad , \quad \vec{e_p} = \sum \eta_{sp} \vec{f_s}$$

$$\theta = \begin{pmatrix} \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 1 \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{pmatrix}$$

Let $a_{ij} = \mu \delta_{ij}$; A_{pq} represents the elastic coefficients in the oblique co-ordonates : $A_{11} = A_{22} = 2\mu$, $A_{12} = A_{21} = -\sqrt{2}\mu$. In the oblique co-ordonates the periodicity cell is represented by a parallelogram ,in which a symmetric strip of fiber is inserted, with the thikness 1/3; A_{pq} depend only on x_2' (' being the oblique co-ordonates). If μ takes the values α in the fiber and 1 in the matrix, we obtain,via (3):

$$A^o_{11} = \{2(\alpha + 2)(2\alpha + 1) + 9\alpha\} / 3(2\alpha + 1)\sqrt{2}$$

$$A^o_{12} = A^o_{21} = -3\alpha/(2\alpha + 1) \quad , \quad A^o_{22} = 6\alpha/(2\alpha + 1)\sqrt{2}$$

Using these expressions and passing to the cartesian co-ordonates, we receive:

$$a^o_{12} = \theta_{1p}\theta_{2q} A^o_{pq} = \sqrt{2}(4\alpha^2 + \alpha + 4)/12(2\alpha + 1)$$

It is possible to see that $(4\alpha^2 + \alpha + 4) > 0$, $(\mp)\alpha$; therefore in this case $a^o_{12} > 0$, $(\mp)\alpha$.

If we consider a configuration of the same type,but with the fiber orientated downwards (fig. 5), we receive:

$$\vec{f_k} = \sum \theta_{pk} \vec{e_p} \quad , \quad \vec{e_p} = \sum \eta_{sp} \vec{f_s}$$

$$\theta = \begin{pmatrix} \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & 1 \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} \sqrt{2} & 0 \\ 1 & 1 \end{pmatrix}$$

The elastic coefficients in the oblique co-ordonates are: $A_{11} = A_{22} = 2\mu$, $A_{12} = A_{21} = \mu\sqrt{2}$. Using (3) we have

$$A^o_{11} = \{2(2\alpha + 1)(\alpha + 2) + 9\alpha\} / 3(2\alpha + 1)\sqrt{2}$$

$$A^o_{12} = A^o_{21} = 3\alpha/(2\alpha + 1) \quad , \quad A^o_{22} = 6\alpha/(2\alpha + 1)\sqrt{2}$$

and it yields :

$$a^{o}_{12} = -\sqrt{2}(4\alpha^{2} + \alpha + 4)/12(2\alpha + 1).$$

Therefore in this case $a^{o}_{12} < 0$, $(\forall)\ \alpha$.

A consequence of the above result is the following: it is not possible to obtain estimations for a^{o}_{12} with respect to the mean value of a_{12}. If we consider $a^{o}_{12} \leq \langle a_{12}\rangle$, it follows that, for $a_{12} = 0$ we have $a^{o}_{12} \leq 0$, which is not true in general. In the sense of the definition given in [6] for the inequality between two symmetric matrix, we can obtain $a^{o}_{ii} \leq \langle a_{ii}\rangle$.

R E F E R E N C E S

[1] A.Benssoussan, J.L.Lions, G.Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978

[2] E.Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory, Lect, Notes inPhysics No.127, Springer Verlag, 1980

[3] W.H.Mc. Connel, On the Approximation of Elliptic Operators with Discontinous Coefficients, Ann.Sc.Norm.Pisa, Ser.IV, No.1, 1976, pp. 121- 137

[4] J.F.Bourgat, Numerical experiments of the homogenization method for operators with periodic coefficients, Raport IRIA No. 277, 1978, Roquencourt, France

[5] H.I.Ene, G.I.Paşa, Metoda omogenizării, Editura Acad.RSR, in press

[6] L.Tartar, Estimation de coefficients homogenises, Lect. Notes in Math., No.704, Springer Verlag, 1977, pp.364

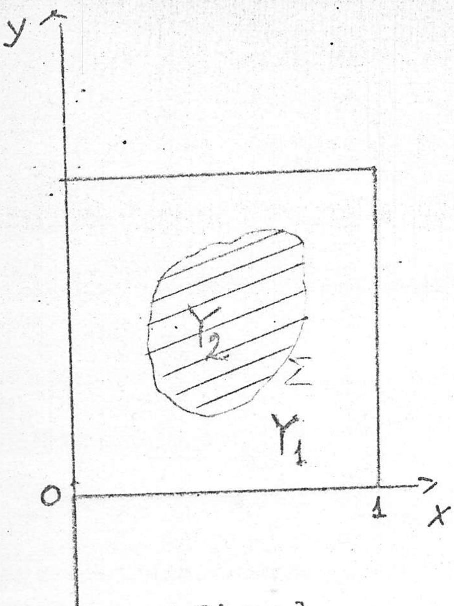

Fig. 1

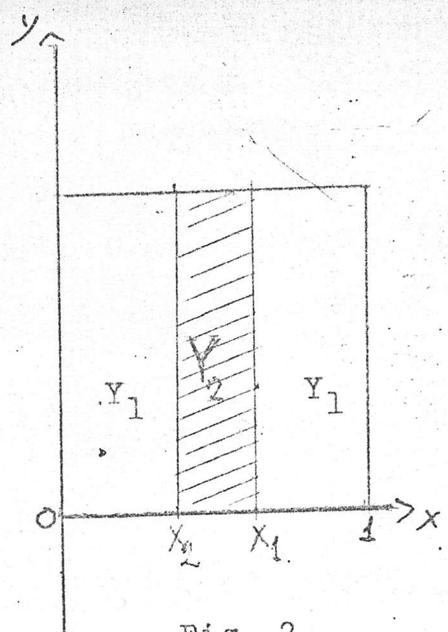

Fig. 2

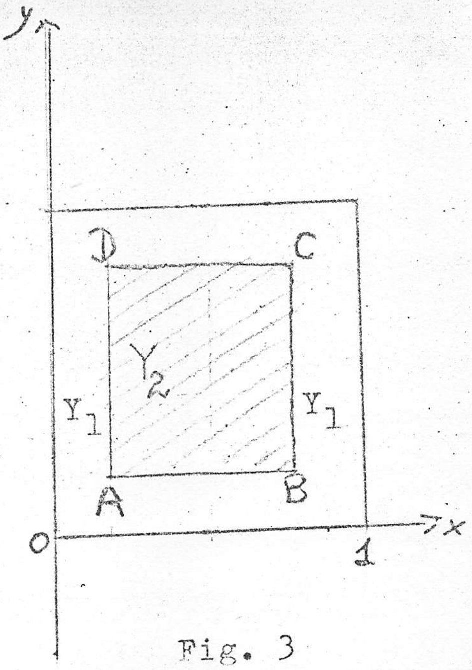

Fig. 3

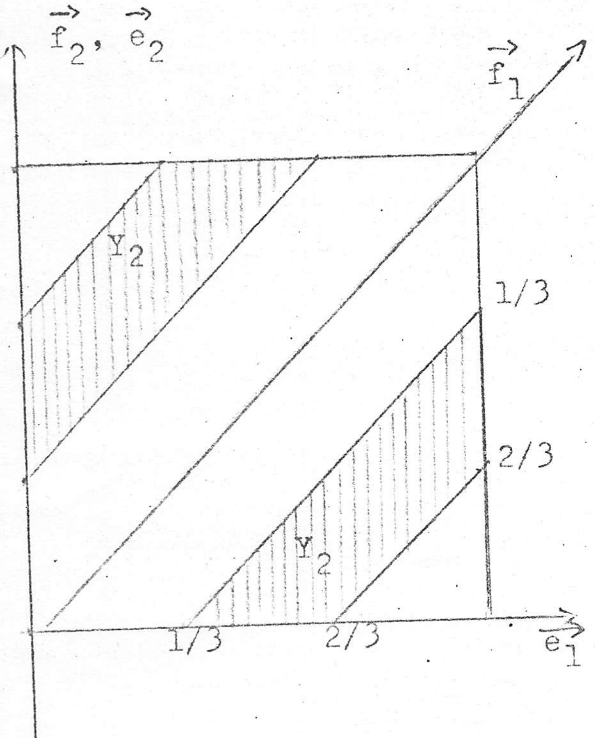

Fig. 4

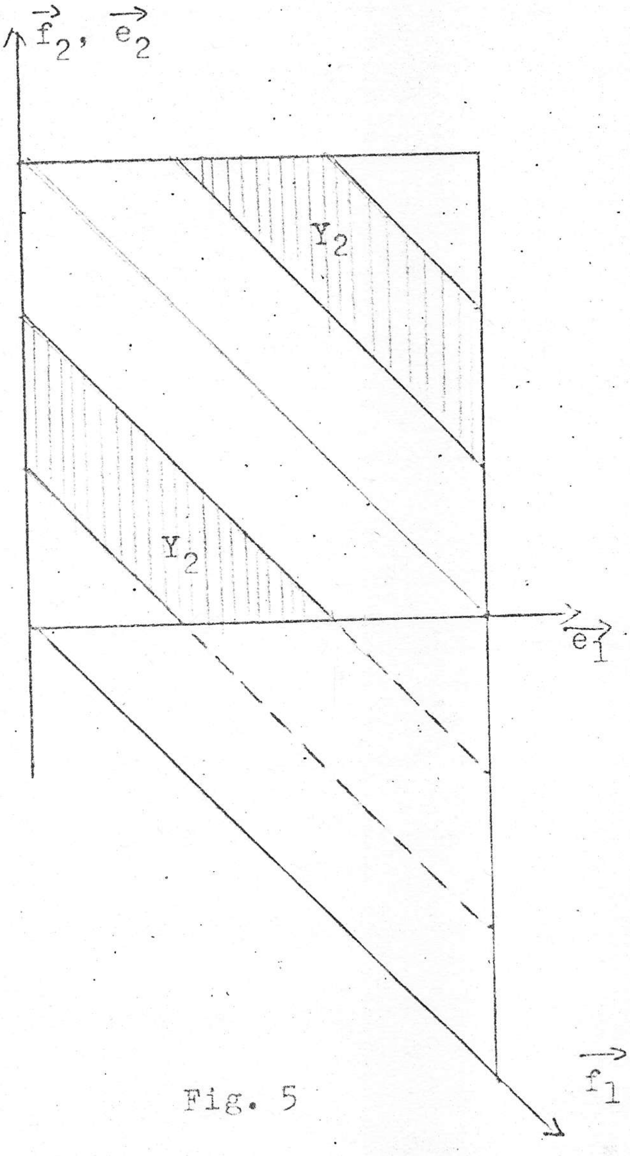

Fig. 5