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STIINTIFICA SI TEHNICA

ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No.59/1986

BUCURESTI

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November 1986

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Cohomology with compact supports for Stein spaces

§o. Introduction

It is known [9],[4] that if X is a Stein space of dimension n then $H_i(X;\mathbb{Z})=o$ for $i>n$ and $H_n(X;\mathbb{Z})$ is free. If X has no singularities Poicaré duality shows that the cohomology groups with compact supports $H^i_c(X;\mathbb{Z})$ vanish for $i<n$ and $H^n_c(X;\mathbb{Z})$ is free. When X is singular a similar statement does not hold. However if X is locally a set-theoretic complete intersection we prove that $H^i_c(X;\mathbb{Z})=o$ for $i<n$ and $H^n_c(X;\mathbb{Z})$ is free (Theorem 1). When X has isolated singularities this result is already known [11].

Our proof is essentially based on a result of Stehlé on the coverings of Stein spaces which enables us to glue together some local topological properties to get a global one. We need also some simple facts about the homology of q-complete open subsets of euclidian spaces which are usually deduced [17] from the classical Morse theory [13].

The vanishing of the cohomology with compact supports has a direct consequence the Lefschetz theorem on hyperplane sections for singular spaces (the homology statement) which was proved by Hamm [9] using Morse theory on singular spaces and regular Whitney stratifications. A stronger version of Lefschetz theorem is proved by Hamm in [1o] (for homotopy groups).

§1. Preliminaries

All complex spaces are assumed to be reduced and with countable topology.

Let X be a complex space of pure dimension k. X is called a locally set-theoretic complete intersection if for any $x \in X$ there exist an open neighbourhood U of x and an embedding $\tau : U \hookrightarrow D \subset \mathbb{C}^N$, where D is an open subset of $\mathbb{C}^N$, such that $V = \tau(U)$ is a set-theoretic complete intersection in D, i.e. V can be given by $(N-k)$ equations in D.

Our main result is the following :

Theorem 1. Let X be a Stein space of pure dimension k. If X is locally a set-theoretic complete intersection then : $H^i_c(X;\mathbb{Z})=0$ for $i<k$ and $H^k_c(X;\mathbb{Z})$ is free.

The principal ingredient for the proof of this theorem is the following result due to Stehlé [19] :

Lemma of Stehlé. Let $\mathcal{U}=(U_\alpha)$ be an open covering of $\mathbb{R}^N$. Then there exists a locally finite open covering $\mathcal{B}=(B_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^N$ which is a refinement of $\mathcal{U}$ with the following properties:

a) the sets B_i and $C_j=\bigcup_{i\leq j} B_i$ are convex and relatively compact for any $i,j\in\mathbb{N}$.

b) $\overline{C}_j$ and ∂C_j are semi-analytic for any $j\in\mathbb{N}$.

Remark 1

The set C_{j+1} is obtained from C_j in the following way :

one takes a point a_{j+1} sufficiently close to C_j and defines $C_{j+1}=\overbrace{\mathrm{conv}(C_j,a_{j+1})}^{\circ}$. B_{j+1} is the intersection of C_{j+1} with a small ball with center a_{j+1}.

In the proof of Theorem 1 we shall use also some elementary facts about q-complete open subsets of the euclidian space $\mathbb{C}^N$. If $U\subset\mathbb{C}^N$ is an open subset a C^∞ function $\varphi:U\to\mathbb{R}$ is called strongly q-convex if the Levi form $\mathcal{L}(\varphi)_x$ has at least $(N-q)$ positive (>0) eigenvalues at any point $x\in U$.
U is said to be q-complete if there exists a strongly q-convex function $\varphi:U\to\mathbb{R}$ which is an exhaustion function, i.e. $\{\varphi<c\}\subset\subset U$ for any $c\in\mathbb{R}$.

<u>Example 1.</u> Let $V\subset\mathbb{C}^N$ be a Stein open subset, $f_0,\dots,f_q\in\mathcal{O}(V)$ and put $A=\{f_0=\dots=f_q=0\}$. Then $U=V\setminus A$ is q-complete.
Indeed, if τ is a strongly plurisubharmonic exhaustion function on V then an easy computation shows that $\varphi=\tau+1/(\sum_{i=0}^{q}|f_i|^2)^{q+1}$ is a strongly q-convex exhaustion function on U (see [18]).

If U is a q-complete domain in $\mathbb{C}^N$ the strongly q-convex exhaustion function $\varphi:U\to\mathbb{R}$ can be chosen with nondegenerate critical points (by the density of Morse functions) hence the classical Morse theory shows that [17] :

<u>Lemma 1.</u> If $U\subset\mathbb{C}^N$ is a q-complete domain then $H_i(U;G)=0$ for $i>N+q$ and any abelian group G.

In fact for the proof of Theorem 1 we shall need also a stronger statement, namely :

<u>Lemma 2.</u> Let $U\subset\mathbb{C}^N$ be a q-complete domain, ($F\subset\mathbb{C}^N$ a Stein open subset containing U) and $D\subset F$ a Runge open subset. Then $H_i(U,U\cap D;G)=o$ for $i>N+q$ and any abelian group G.

Proof

First we make the following remark : if τ is a strongly q-convex exhaustion function on U and $V=\{\tau<o\}$ then $H_i(U,V;G)=o$ for $i>N+q$. This follows easily by an exhaustion argument and the approximation of τ with Morse functions. To prove Lemma 2 it suffices to show that for any compact $K\subset U\cap D$ there is an open subset V such that $K\subset V\subset U\cap D$ and $H_i(U,V;G)=o$ for $i>N+q$. Since D is Runge in F there is a strongly plurisubharmonic function ψ on F such that $K\subset\{\psi<o\}\subset\subset D$. Let $\varphi>o$ be a strongly q-convex exhaustion function on U and choose $\varepsilon_o>o$ sufficiently small such that $\tau=\varepsilon_o\varphi+\psi<o$ on K. Then τ is a strongly q-convex exhaustion function on U and if we set $V=\{\tau<o\}$ it follows that $K\subset V\subset U\cap D$ and $H_i(U,V;G)=o$ for $i>N+q$ which proves the lemma.

<u>Lemma 3.</u> Let $D\subset\mathbb{C}^N$ be an open subset, $A\subset D$ a closed analytic subset of pure dimension k which is a set-theoretic complete intersection and G any abelian group. Assume that A is Stein. If $A'\subset A$ is an open subset of A which is Runge in A then $H^i_c(A';G)=H^i_c(A;G)=o$ for $i<k$ and $H^k_c(A';G)\to H^k_c(A;G)$ is injective.

Proof

We may assume $k<N$ since for $k=N$ the result follows from the previous lemma. In view of a result of Siu [16] we may also suppose that D is Stein. Using an exhaustion argument

we see that it is enough to consider the case when $A'=D'\cap A$ with D' a Runge domain in D (since A' can be exhausted with such intersections).

By Poincaré duality $H^i_c(A;G)\cong H_{2N-i}(D,D\setminus A;G)$ and $H^i_c(A';G)\cong H_{2N-i}(D',D'\setminus A';G)$. In the exact sequence :

$$\ldots\to H_{2N-i}(D;G)\to H_{2N-i}(D,D\setminus A;G)\to H_{2N-i-1}(D\setminus A;G)\to\ldots$$

we have $H_{2N-i}(D;G)=H_{2N-i-1}(D\setminus A;G)=o$ for $i<k$ since D is Stein and $D\setminus A$ is $(N-k-1)$ complete (from Example 1). It follows that $H^i_c(A;G)=o$ for $i<k$. A similar argument shows that $H^i_c(A';G)=o$ for $i<k$. To see that $H^k_c(A';G)\to H^k_c(A;G)$ is injective, or equivalently $H_{2N-k}(D',D'\setminus A';G)\xrightarrow{\alpha} H_{2N-k}(D,D\setminus A;G)$ is injective, we consider the commutative diagram with exact lines :

$$\begin{array}{ccccccc}
\ldots\to H_{2N-k}(D;G) & \to & H_{2N-k}(D,D\setminus A;G) & \to & H_{2N-k-1}(D\setminus A;G) & \to\ldots \\
\uparrow & & \alpha\uparrow & & \beta\uparrow & \\
\ldots\to H_{2N-k}(D';G) & \to & H_{2N-k}(D',D'\setminus A';G) & \to & H_{2N-k-1}(D'\setminus A';G) & \to\ldots
\end{array}$$

We have $H_{2N-k}(D;G)=H_{2N-k}(D';G)=o$ since $k<N$ and D,D' are Stein open subsets of $\mathbb{C}^N$. From Lemma 2 the map β is injective, hence α is also injective, as desired.

§2. Proof of Theorem 1

Let $\tau:X\to\mathbb{C}^N$ be a proper, injective, holomorphic map (which exists by [14]). Hence $X'=\tau(X)$ is an analytic subset of $\mathbb{C}^N$ and $\tau:X\to X'$ is an analytic homeomorphism. Let $\mathcal{U}=(U_\alpha)$ be an open covering of X such that each U_α is isomorphic to a set-theoretic complete intersection.

From the Lemma of Stehlé there exists a locally finite open covering $(B_i)_{i\in\mathbb{N}}$ of $\mathbb{C}^N$ with properties a), b) and such that $T_i=\tau^{-1}(B_i)$ defines an open covering $\mathcal{T}=(T_i)_{i\in\mathbb{N}}$ which is a refinement of $\mathcal{U}$. If we set $V_i=\tau^{-1}(C_i)$ then $T_i, V_i, T_{i+1}\cap V_i$ are Stein open sets and $(T_{i+1}\cap V_i, T_{i+1})$ is a Runge pair (since the sets in the Lemma of Stehlé are convex, in particular they are Runge). From Lemma 3 $H_c^j(T_{i+1};G)=H_c^j(T_{i+1}\cap V_i;G)=o$ for $j<k$ and $H_c^k(T_{i+1}\cap V_i;G)\to H_c^k(T_{i+1};G)$ is injective for any abelian group G.

If we consider the Mayer-Vietoris exact sequence :

$$\dots\to H_c^j(T_{i+1}\cap V_i;G)\to H_c^j(V_i;G)\oplus H_c^j(T_{i+1};G)\to H_c^j(V_{i+1};G)\to H_c^{j+1}(T_{i+1}\cap V_i;G)\to\dots$$

it follows that the maps $H_c^j(V_i;G)\to H_c^j(V_{i+1};G)$ are bijective for $j<k$ and injective for $j=k$. Hence, if $j<k$, $H_c^j(V_i;G)\approx H_c^j(V_1;G)= H_c^j(T_1;G)=o$. Taking inductive limit we obtain $H_c^j(X;G)=o$ for $j<k$ and any abelian group G. From the universal coefficient formula for the cohomology with compact supports :

$$H_c^j(X;G)\approx H_c^j(X;\mathbb{Z})\otimes G\oplus \mathrm{Tor}(H_c^{j+1}(X;\mathbb{Z}),G)$$

it follows that $H_c^j(X;\mathbb{Z})=o$ for $j<k$ and $H_c^k(X;\mathbb{Z})$ is torsion free. To prove the theorem it remains to show that $H_c^k(X;\mathbb{Z})$ is free. But $H_c^k(X;\mathbb{Z})=\varinjlim H_c^k(V_j;\mathbb{Z})$ and the maps $H_c^k(V_j;\mathbb{Z})\to H_c^k(V_{j+1};\mathbb{Z})$ are injective. So, to prove the theorem it suffices to show that $H_c^k(V_j;\mathbb{Z})$ is free of finite type for any j and $H_c^k(V_j;\mathbb{Z})$ is a direct summand in $H_c^k(V_{j+1};\mathbb{Z})$.

First we remark that for any s $H_c^s(V_j;\mathbb{Z})\approx H_c^s(C_j\cap X';\mathbb{Z})= H_c^s((\overline{C}_j\cap X')\setminus(\partial C_j\cap X');\mathbb{Z})\approx H^s(\overline{C}_j\cap X', \partial C_j\cap X';\mathbb{Z})$ which is finitely generated because $\overline{C}_j\cap X', \partial C_j\cap X'$ are semi-analytic compact sets. On the other hand we know that $H_c^k(V_i;\mathbb{Z})$ is torsion free (since $H_c^{k-1}(V_j;G)=o$ for any abelian group G) hence being finitely generated it is free. Hence it remains to verify that $H_c^k(V_j;\mathbb{Z})$ is a direct summand in $H_c^k(V_{j+1};\mathbb{Z})$. To prove this assertion it suffices to show that $H_c^k(V_j;\mathbb{Z})\to H_c^k(V_{j+1};\mathbb{Z})$ has torsion free cokernel. But $H_c^k(V_j;\mathbb{Z})\approx H_c^k(X'\cap C_j;\mathbb{Z})\approx H_{2N-k}(C_j,C_j\setminus X';\mathbb{Z})\approx$ $H_{2N-k-1}(C_j\setminus X';\mathbb{Z})$. From the exact sequence :

$$H_{2N-k-1}(C_j\setminus X';\mathbb{Z})\rightarrow H_{2N-k-1}(C_{j+1}\setminus X';\mathbb{Z})\rightarrow H_{2N-k-1}(C_{j+1}\setminus X',C_j\setminus X';\mathbb{Z})$$

and the universal coefficient formula we have only to check that $H_{2N-k}(C_{j+1}\setminus X',C_j\setminus X';G)=o$ for any abelian group G. Property a) in the Lemma of Stehlé and the excision theorem show that $H_{2N-k}(C_{j+1}\setminus X',C_j\setminus X';G)\approx H_{2N-k}(B_{j+1}\setminus X',B_{j+1}\cap C_j\setminus X';G)$. In the exact sequence :

$$\ldots\rightarrow H_{2N-k}(B_{j+1}\setminus X';G)\rightarrow H_{2N-k}(B_{j+1}\setminus X',B_{j+1}\cap C_j\setminus X';G)\rightarrow H_{2N-k-1}(B_{j+1}\cap C_j\setminus X';G)\rightarrow H_{2N-k-1}(B_{j+1}\setminus X';G)\rightarrow\ldots$$

we have :

$H_{2N-k}(B_{j+1}\setminus X';G)\approx H_c^{k-1}(B_{j+1}\cap X';G)\approx H_c^{k-1}(T_{j+1};G)=o$, $H_{2N-k-1}(B_{j+1}\cap C_j\setminus X';G)\approx H_c^k(T_{j+1}\cap V_j;G)$ and $H_{2N-k-1}(B_{j+1}\setminus X';G)\approx H_c^k(T_{j+1};G)$. Since $H_c^k(T_{j+1}\cap V_j;G)\rightarrow H_c^k(T_{j+1};G)$ is injective it follows that $H_{2N-k}(B_{j+1}\setminus X',B_{j+1}\cap C_j\setminus X';G)=o$ and the proof of *Theorem* 1 is complete.

Remark 2. Let X be a complex space and $x\in X$. We put :

$$s(x)=\min\left\{d \;\middle|\; X_x \text{ is isomorphic to a germ of analytic set } Y_o \text{ in } \mathbb{C}^p \text{ which can be given by } p-\dim X_x+d \text{ equations}\right\}$$

and we define $s(X)=\sup_{x\in X} s(x)$.

Clearly $s(X)=o$ is equivalent to X is locally a set-theoretic complete intersection. Theorem 1 can be strengthened (with the same proof) as follows :

Theorem 2. Let X be a Stein space of pure dimension k. Then : $H_c^i(X;\mathbb{Z})=o$ for $i<k-s(X)$ and $H_c^i(X;\mathbb{Z})$ is free for $i=k-s(X)$.

Remark 3. If in Theorem 1 the assumption "X is locally a set-theoretic complete intersection" is replaced by the stronger hypothesis "X is locally a complete intersection" (and if we assume also that X has bounded Zariski dimension) then the proof can be simplified in view of the following result of Ballico [3] : If $X\subset\mathbb{C}^N$ is an analytic subset of pure dimension k and X is locally a complete intersection then $\mathbb{C}^N\setminus X$ is (N-k-1) complete.

We give now some immediate consequences of Theorem 1 :

Corollary 1. Let X be a compact complex space of pure dimension k and $A\subset X$ a closed subset which is ANR. Assume that $X\setminus A$ is Stein and is locally a set-theoretic complete intersection. Then $H_i(X,A;\mathbb{Z})=0$ for $i<k$.

Proof

The homology groups $H_i(X,A;\mathbb{Z})$ being finitely generated the assertion of Corollary 1 is equivalent ([7] ,p.136) to the condition : $H^i(X,A;\mathbb{Z})=0$ for $i<k$ and $H^k(X,A;\mathbb{Z})$ is free. But $H^i(X,A;\mathbb{Z})\approx H^i_c(X\setminus A;\mathbb{Z})$ and the desired conclusion follows from Theorem 1.

From Corollary 1 we deduce the Lefschetz theorem on hyperplane sections for singular varieties proved by Hamm in [9] :

Corollary 2. Let $A\subset\mathbb{P}_N$ be an analytic subset of pure dimension k and $H\subset\mathbb{P}_N$ a hyperplane. Assume that $A\setminus A\cap H$ is locally a set-theoretic complete intersection in $\mathbb{P}_N$. Then $H_i(A,A\cap H;\mathbb{Z})=0$ for $i<k$.

Remark 4. If in Theorem 1 we assume that X is a closed analytic subset of a Stein manifold Y and X is locally a set-theoretic complete intersection in Y (these conditions are satisfied for example in Corollary 2) then it is not necessary to invoke Siu's result (which was needed in Lemma 3). In fact for the proof of Lemma 3 it suffices the semi-local version of Siu's theorem.

Remark 5. Let X be a complex space of pure dimension k, $x\in X$ a singular point and $N(x)=\dim T_xX$. Choose a local embedding of a neighbourhood V of x in a Stein open subset of $\mathbb{C}^{N(x)}$. From the recent results of M.Peternell [15], Diederich and Fornaess [5] on q-complete manifolds with corners it follows that $D\setminus V$ is $N(x)-[N(x)/N(x)-k]$ complete, where $[a]$ denotes the largest integer $\leq a$. If X has bounded Zariski dimension we set $\lambda(X)=\sup_{x\in X}(N(x)-k)$ and if X is singular we put $v(X)=[k/\lambda(X)]$. The arguments given in the proof of Theorem 1 can be repeated word by word and one obtains :

Theorem 3. Let X be a pure dimensional singular Stein space of bounded Zariski dimension. Then $H^i_c(X;\mathbb{Z})=0$ for $i<v(X)$ and $H^i_c(X;\mathbb{Z})$ is free for $i=v(X)$.

Of course this result implies a Lefschetz-type theorem for singular varieties which depends on Zariski dimension. The precise statement is left to the reader.

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