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CONTINUOUS FIELDS OF AF - ALGEBRAS

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INTRODUCTION

In [5] E.G. Effros posed the problem of studying inductive limits of C^* -algebras of the form $C(X) \otimes A$, with A finite-dimensional, as a generalization of the AF-algebras.

Let X be a connected compact space. In this paper we give some classification results concerning inductive limits $\lim_{\rightarrow} C(X) \otimes A_i$, with A_i finite-dimensional, where the bonding homomorphisms are unital, injective and $C(X)$ -linear. The problem here is to measure and to store the possible twistings over X of the embeddings of A_i into A_{i+1} . The $C(X)$ -linear $*$ -homomorphisms $C(X) \otimes A_i \rightarrow C(X) \otimes A_{i+1}$ correspond to homomorphisms $A_i \rightarrow C(X) \otimes A_{i+1}$ which are classified, modulo inner equivalence, by matrices of complex vector bundles over X . (see Corollary 2.2). Each inductive limit $L = \lim_{\rightarrow} C(X) \otimes A_i$, with $C(X)$ -linear connecting $*$ -homomorphisms, is isomorphic to the C^* -algebra defined by a continuous field E_L of AF-algebras canonically associated with L (see Proposition 3.1). This field is not always trivial as it is shown in Proposition 5.1. Moreover, we are able to classify the inductive limits L in the case when the fibres of E_L are simple, using the

semiring of the homotopy classes of projections in $\bigcup_{n=1}^{\infty} M_n \otimes L$ (see Theorem 4.4). If the canonical map $\text{Vect}(X) \rightarrow K^0(X)$ is injective (in particular, this occurs provided that X is a connected finite CW-complex of dimension ≤ 3) this result may be given using the pointed ordered group $(K_0(L), K_0(L)_+, [1_L])$ (see Theorem 4.6). Also we classify the C^* -algebras L as $C(X)$ -modules (see Theorems 4.3 and 4.5).

1. PRELIMINARIES

If A, B are unital C^* -algebras we shall denote by $\text{Hom}(A, B)$ the space of all unital $*$ -homomorphisms from A to B endowed with the topology of pointwise convergence. Two homomorphisms $\phi_1, \phi_2 \in \text{Hom}(A, B)$ are said to be inner equivalent if there is a unitary $u \in B$ such that $\phi_2 = u\phi_1 u^*$. Let $\text{Hom}(A, B)/\sim$ be the set of classes of inner equivalent homomorphisms from A to B . If A and B are $C(X)$ -modules, we shall denote by $\text{Hom}_{C(X)}(A, B)$ the subspace of $\text{Hom}(A, B)$ consisting of all $C(X)$ -linear homomorphisms.

We shall use $\text{Vect}(X)$ to denote the set of isomorphism classes of complex vector bundles on X , and $\text{Vect}_k(X)$ to denote the subset of $\text{Vect}(X)$ given by bundles of dimension k . $\text{Vect}(X)$ is a semiring under the operations \oplus and \otimes . In $\text{Vect}_k(X)$ we have one naturally distinguished element $[k]$ —the class of the trivial bundle of dimension k .

As usual we denote by $G(n, k)$ the Grassmann manifold of all subspaces of \mathbb{C}^n of dimension k and by $U(n)$ the Lie group of all unitaries of M_n . Any continuous map $F: X \rightarrow G(n, k)$

defines a vector bundle $E_F = \{(x, F(x)\eta) : x \in X, \eta \in \mathbb{C}^n\} \subset X \times \mathbb{C}^n$. Let $H^1(X, U(k)_\mathbb{C})$ denote the cohomology set associated with the sheaf of germs of continuous functions $X \rightarrow U(k)$. We have a bijection $\text{Vect}_k(X) \rightarrow H^1(X, U(k)_\mathbb{C})$ which takes classes of vector bundles to classes of cocycles ([8]).

We describe below the cocycle of E_F . The fibration:

$$U(k) \times U(n-k) \rightarrow U(n) \rightarrow G(n, k)$$

induces the exact sequence of pointed cohomology sets:

$$C(X, U(n)) \rightarrow C(X, G(n, k)) \xrightarrow{\delta} H^1(X, U(k)_\mathbb{C}) \times H^1(X, U(n-k)_\mathbb{C}) \rightarrow H^1(X, U(n)_\mathbb{C})$$

(for details see [2]). Denote $\delta(F) = (\delta_1(F), \delta_2(F))$.

1.1. LEMMA. The vector bundle E_F is given by the cocycle $\delta_1(F)$.

Proof. Choose an open covering (U_i) of X and continuous maps $u_i: U_i \rightarrow U(n)$ such that:

$$F(x) = u_i(x) \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix} u_i(x)^* \quad \text{on } U_i.$$

Then:

$$u_i(x) * u_j(x) = \begin{bmatrix} u_{ij}(x) & 0 \\ 0 & u'_{ij}(x) \end{bmatrix} \quad \text{on } U_i \cap U_j$$

and $\delta(F) = ((U_i, u_{ij}), (U_i, u'_{ij}))$, by definition. Consider the local trivializations for E_F :

$$U_i \times \mathbb{C}^k \xrightarrow{\phi_i} E_F|_{U_i} = \{ (x, u_i(x) \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix} u_i(x) * \eta) : x \in U_i, \eta \in \mathbb{C}^n \}$$

$$\text{given by } \phi_i(x, \xi) = (x, u_i(x) \begin{bmatrix} \xi \\ 0 \end{bmatrix}), \quad x \in U_i, \quad \xi \in \mathbb{C}^k.$$

The cocycle (U_i, b_{ij}) of E_F can be computed using the local trivializations:

$$\begin{aligned} b_{ij}(x) \xi &= (\phi_i^{-1})_x (\phi_j)_x \xi = (\phi_i^{-1})_x u_j(x) \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \\ &= (\phi_i^{-1})_x u_i(x) u_i(x) * u_j(x) \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \\ &= (\phi_i^{-1})_x u_i(x) \begin{bmatrix} u_{ij}(x) & 0 \\ 0 & u'_{ij}(x) \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \\ &= (\phi_i^{-1})_x u_i(x) \begin{bmatrix} u_{ij}(x) \xi \\ 0 \end{bmatrix} = u_{ij}(x) \xi, \quad x \in U_i \cap U_j, \quad \xi \in \mathbb{C}^k. \end{aligned}$$

1.2. COROLLARY. Let $F: X \rightarrow G(n, q)$ be a continuous map and define a continuous map $\tilde{F}: X \rightarrow G(nk+p, qk)$

$$\tilde{F}(x) = v_i(x) \begin{bmatrix} F(x) \otimes 1_k & 0 \\ 0 & 0_p \end{bmatrix} v_i(x)^*, \quad x \in U_i$$

where (U_i) is an open covering of X and $v_i: U_i \rightarrow U(nk+p)$ are continuous maps satisfying:

$$v_i(x) * v_j(x) = \begin{bmatrix} 1_n \otimes a_{ij}(x) & 0 \\ 0 & a'_{ij}(x) \end{bmatrix}, \quad x \in U_i \cap U_j$$

for some continuous maps $a_{ij}: U_i \cap U_j \rightarrow U(k)$ and $a'_{ij}: U_i \cap U_j \rightarrow U(p)$.
 Let H be the vector bundle corresponding to the cocycle (U_i, a_{ij}) .
 Then $E_{\tilde{F}}$ is isomorphic to $E_F \otimes H$.

Proof. We may assume that $F(x) = u_i(x) \begin{bmatrix} 1_q & 0 \\ 0 & 0 \end{bmatrix} u_i(x)^*$ on U_i ,

where $u_i: U_i \rightarrow U(n)$ are continuous and:

$$u_i(x) * u_j(x) = \begin{bmatrix} u_{ij}(x) & 0 \\ 0 & u'_{ij}(x) \end{bmatrix} \quad \text{on } U_i \cap U_j.$$

We get the following formula for \tilde{F} on U_i :

$$\tilde{F}(x) = v_i(x) \begin{bmatrix} u_i(x) \otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1_q & 0 \\ 0 & 0 \end{bmatrix} \otimes 1_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_i(x)^* \otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} v_i(x)^*$$

so that we can compute $\delta(\tilde{F})$. Indeed, for $x \in U_i \cap U_j$ we have:

$$\begin{aligned} & \begin{bmatrix} u_i(x)^* \otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} v_i(x) * v_j(x) \begin{bmatrix} u_j(x) \otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} = \\ & = \begin{bmatrix} u_i(x)^* \otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} \begin{bmatrix} 1_n \otimes a_{ij}(x) & 0 \\ 0 & a'_{ij}(x) \end{bmatrix} \begin{bmatrix} u_j(x) \otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} = \\ & = \begin{bmatrix} u_{ij}(x) \otimes a_{ij}(x) & 0 & 0 \\ 0 & u'_{ij}(x) \otimes a_{ij}(x) & 0 \\ 0 & 0 & a'_{ij}(x) \end{bmatrix} \end{aligned}$$

Hence E_F is given by the cocycle $(U_i, u_{ij} \otimes a_{ij})$.

2. HOMOMORPHISMS OF $C(X)$ -MODULES.

In this section we classify the homomorphisms in $\text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)$ within inner equivalence, where $A = M_{n_1} \oplus \dots \oplus M_{n_r}$, $B = M_{m_1} \oplus \dots \oplus M_{m_s}$ and X is compact and connected.

Any homomorphism $\phi \in \text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)$ is uniquely determined by its restriction to A . This allows us to identify $\text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)$ with $\text{Hom}(A, C(X) \otimes B)$ as topological spaces, identification which preserves the inner equivalence classes. By Proposition 1 in [3] it follows that there is a bijection

$$\delta: \text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B) / \sim \rightarrow \{ \underline{E} = (E_{pq}) \in M_{s \times r}(\text{Vect}(X)) : E[\underline{n}] = [\underline{m}] \} \quad (1)$$

where $[\underline{n}] := ([n_1], \dots, [n_r])$, $[\underline{m}] := ([m_1], \dots, [m_s])$. Explicitly, $E[\underline{n}] = [\underline{m}]$ means:

$$(E_{p1} \otimes [n_1]) \oplus \dots \oplus (E_{pr} \otimes [n_r]) = [m_p], \quad p=1, 2, \dots, s.$$

The description of δ can be obtained using the local structure of homomorphisms $A \rightarrow C(X) \otimes B$ given in [10] or by Proposition 1 in [3]. For simplicity, suppose that $B = M_m$. Thus, for a homomorphism $\phi \in \text{Hom}(A, C(X) \otimes B)$ there are: an open covering (U_i) of X , continuous maps $v_i: U_i \rightarrow U(m)$ and positive integers k_{11}, \dots, k_{1r} such that:

$$\phi(a)(x) = v_i(x) (a_1 \otimes 1_{k_{11}} \oplus \dots \oplus a_r \otimes 1_{k_{1r}}) v_i(x)^* \quad (2)$$

where $x \in U_i$, $a = a_1 \oplus \dots \oplus a_r \in A$ and:

$$v_i(x) * v_j(x) = \begin{bmatrix} 1_{n_1} \otimes a_{ij}^1(x) & & 0 \\ & \ddots & \\ 0 & & 1_{n_r} \otimes a_{ij}^r(x) \end{bmatrix} \text{ on } U_i \cap U_j$$

If $\delta(\phi) = (E_{1q})$ then each vector bundle E_{1q} is given by the cocycle (U_i, a_{ij}^q) . Note that $\text{rank } E_{1q} = k_{1q}$.

If C is a unital C^* -algebra we shall denote by $D(C)$ the set of homotopy classes of selfadjoint projections in $\bigcup_{n=1}^{\infty} M_n \otimes C$. Recall that $D(C)$ is a semigroup under the operation induced by the direct sum of projections and $D(\cdot)$ is a covariant functor.

Let $C = C(X) \otimes A$. It is known that there is an isomorphism of semigroups $D(C(X) \otimes A) \rightarrow \text{Vect}(X)^r$ which maps the class of a projection $F \in C(X) \otimes A \otimes M_n$, having the decomposition

$$F = F_1 \oplus \dots \oplus F_r \in \bigoplus_{k=1}^r C(X) \otimes M_{n_k} \otimes M_n$$

to $(E_{F_1}, \dots, E_{F_r}) \in \text{Vect}(X)^r$. Any homomorphism

$\phi \in \text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)$ induces a map $\phi_*: D(C(X) \otimes A) \rightarrow D(C(X) \otimes B)$ or equivalently a map $\phi_*: \text{Vect}(X)^r \rightarrow \text{Vect}(X)^s$.

$\text{Vect}(X)^r$ is a free module over the unital semiring $\text{Vect}(X)$.

Let e_1, \dots, e_r be its canonical basis, $e_i = (0, \dots, [1], \dots, 0)$ with $[1]$ on the i -th position. We denote by

$\text{Hom}_{\text{Vect}(X)}(\text{Vect}(X)^r, \text{Vect}(X)^s)$ the set of all homomorphisms of $\text{Vect}(X)$ -modules $\text{Vect}(V)^r \rightarrow \text{Vect}(X)^s$. As usual any element of $\text{Hom}_{\text{Vect}(X)}(\text{Vect}(X)^r, \text{Vect}(X)^s)$ is given by a unique matrix in $M_{s \times r}(\text{Vect}(X))$ with respect to the canonical bases.

2.1. PROPOSITION. The map ϕ_* is $\text{Vect}(X)$ -linear and its matrix it is equal to $\delta(\phi) = (E_{pq})$.

Proof. We may assume that $B = M_m$. Using (2) and the canonical bijection $\text{Hom}(A, C(X) \otimes B) \rightarrow \text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)$ we get the following description for ϕ :

$$\phi(G)(x) = v_i(x) (G_1(x) \otimes 1_{k_{11}} \oplus \dots \oplus G_r(x) \otimes 1_{k_{1r}}) v_i(x)^*$$

$x \in U_i$, $G = \bigoplus_{i=1}^r G_i \in \bigoplus_{i=1}^r C(X) \otimes M_{n_i}$, where k_{11}, \dots, k_{1r} are positive

integers ($n_1 k_{11} + \dots + n_r k_{1r} = m$), (U_i) is an open covering of X and $v_i: U_i \rightarrow U(m)$ are continuous maps satisfying:

$$v_i(x) * v_j(x) = \begin{bmatrix} 1_{n_1} \otimes a_{ij}^1(x) & & 0 \\ & \ddots & \\ 0 & & 1_{n_r} \otimes a_{ij}^r(x) \end{bmatrix}, x \in U_i \cap U_j.$$

Let $\phi_n: C(X) \otimes A \otimes M_n \rightarrow C(X) \otimes M_m \otimes M_n$, $\phi_n := \phi \otimes \text{id}_{M_n}$,

$n \geq 1$. Since ϕ_* is a homomorphism of semigroups it is enough to describe the homotopy class of $\phi_n(F)$ for a projection $F \in C(X) \otimes M_{n_1} \otimes M_n \subset C(X) \otimes A \otimes M_n$. One can easily obtain the following formula:

$$\phi_n(F)(x) = v_i(x) \otimes 1_n \begin{bmatrix} F(x) \otimes 1_{k_{11}} & 0 \\ 0 & 0_p \end{bmatrix} \cdot v_i(x)^* \otimes 1_n, \quad x \in U_i$$

where $p = mn - k_{11}n_1n$. Since:

$$(v_i(x) \otimes 1_n)^* (v_j(x) \otimes 1_n) = \begin{bmatrix} 1_{nn_1} \otimes a_{ij}^1(x) & 0 \\ 0 & a_{ij}'(x) \end{bmatrix}, \quad x \in U_i \cap U_j$$

where $a_{ij}'(x) := \bigoplus_{q=2}^r 1_{nn_q} \otimes a_{ij}^q(x)$, it follows by Corollary 1.2 that

$\phi_n(F)$ gives a vector bundle isomorphic to $E_F \otimes E_{11}$, where E_{11} is the vector bundle corresponding to the cocycle (U_i, a_{ij}^1) .

2.2. COROLLARY. The map $\phi \rightarrow \phi_*$ induces a bijection

$$\text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B) / \sim \rightarrow \{E \in \text{Hom}_{\text{Vect}(X)}(\text{Vect}(X)^r, \text{Vect}(X)^s) : E[\underline{n}] = [\underline{m}]\}$$

Proof. Use (1) and Proposition 2.1.

Let $K_0(C(X) \otimes A)$ be the Grothendieck group for the abelian semigroup $D(C(X) \otimes A)$. Let $K_0(C(X) \otimes A)_+$ be the image of $D(C(X) \otimes A)$ in $K_0(C(X) \otimes A)$. $(K_0(C(X) \otimes A), K_0(C(X) \otimes A)_+)$ is an ordered group. The isomorphism $D(C(X) \otimes A) \rightarrow \text{Vect}(X)^r$ induces an isomorphism of ordered groups $(K_0(C(X) \otimes A), K_0(C(X) \otimes A)_+) \rightarrow (K^0(X)^r, K^0(X)_+^r)$ where $K^0(X)_+$ is the image of $\text{Vect}(X)$ in $K^0(X)$. Recall that $K^0(X)$ has a

natural structure of ring. In $K_0(C(X) \otimes A)$ we distinguish the class of the unity $[1_{C(X) \otimes A}] = [\underline{n}]$. We shall denote by $\text{Hom}_{K^0(X)}((K^0(X)^r, K^0(X)^r_+, [\underline{n}]), (K^0(X)^s, K^0(X)^s_+, [\underline{m}]))$ the set of all pointed ordered group homomorphisms which are $K^0(X)$ -linear.

2.3. COROLLARY. Assume that the canonical map $\text{Vect}(X) \rightarrow K^0(X)$ is injective. Then the map $\Phi \rightarrow K_0(\Phi)$ induces a bijection:

$$\text{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B) / \sim \longrightarrow \text{Hom}_{K^0(X)}((K^0(X)^r, K^0(X)^r_+, [\underline{n}]), (K^0(X)^s, K^0(X)^s_+, [\underline{m}]))$$

3. CONTINUOUS FIELDS OF AF-ALGEBRAS

Let X be a compact space and let $(A_i)_{i=1}^\infty$ be a sequence of finite-dimensional C^* -algebras. We consider a system:

$$\dots \rightarrow C(X) \otimes A_i \xrightarrow{\Phi_i} C(X) \otimes A_{i+1} \rightarrow \dots \quad (3)$$

where each $*$ -homomorphism Φ_i is unital, injective and $C(X)$ -linear. We show that the corresponding C^* -inductive limit $L = \varinjlim (C(X) \otimes A_i, \Phi_i)$ is $*$ -isomorphic, by a $C(X)$ -module isomorphism, to the C^* -algebra of the sections of some continuous field of AF-algebras.

Since we can canonically identify $\text{Hom}(A_i, C(X) \otimes A_{i+1})$ with $C(X, \text{Hom}(A_i, A_{i+1}))$, each Φ_i defines a continuous map $X \ni x \rightarrow \Phi_i(x) \in \text{Hom}(A_i, A_{i+1})$. Note that each $\Phi_i(x)$ is injective.

For any $x \in X$ define the AF-algebra $A(x) = \varinjlim (A_i, \phi_i(x))$. We want to define a continuous field of AF-algebras $E_L = ((A(x))_{x \in X}, \Gamma)$. Let L_0 be the algebraic inductive limit of the system (3). Then define $\eta: L_0 \rightarrow \prod_{x \in X} A(x)$ by $\eta([F])(x) = [F(x)]$, $x \in X$, $F \in L_0$. ($[a]$ denotes the image of a in the corresponding inductive limit).

Define Γ to be the closure of $\eta(L_0) \subset \prod_{x \in X} A(x)$ with respect to the norm $\|a\| = \sup_{x \in X} \|a(x)\|$. It is easily seen that E_L is a continuous field of AF-algebras. Moreover, η extends to a $C(X)$ -linear $*$ -isomorphism from L onto Γ . Thus, we have the following:

3.1. PROPOSITION. The inductive limit L is $*$ -isomorphic to Γ by a $C(X)$ -module isomorphism.

3.2. REMARK. If each A_i is a factor or if the space X is connected, then $A(x) \cong A(y)$, $x, y \in X$. If X is locally contractible, then the field E_L is locally trivial.

3.3. PROPOSITION. Let L, L' be inductive limits of the above type such that the fibres $A(x), A'(x)$ ($x \in X$) of $E_L, E_{L'}$ are simple. Then, for any $*$ -isomorphism $\phi: L \rightarrow L'$ there is a homeomorphism $\psi: X \rightarrow X$ such that:

$$\phi(f \cdot a) = f \circ \psi \cdot \phi(a), \quad f \in C(X), \quad a \in L.$$

Proof. Let $\eta: L \rightarrow \Gamma$ and $\eta': L' \rightarrow \Gamma'$ be the $*$ -isomorphisms

constructed in the proof of Proposition 3.1. Let ψ be the *-isomorphism which makes the diagram:

$$\begin{array}{ccc} L & \xrightarrow{\eta} & \Gamma \\ \Phi \downarrow & & \downarrow \psi \\ L' & \xrightarrow{\eta'} & \Gamma' \end{array}$$

commutative. Since η and η' are $C(X)$ -linear, it is enough to prove that $\psi(fa) = f \circ \phi \cdot \psi(a)$, $f \in C(X)$, $a \in \Gamma$.

Since each $\Lambda(x)$ is simple, the maximal ideals of Γ are of the form $I_x := \{a \in \Gamma : a(x) = 0\}$, $x \in X$. Since ψ is an isomorphism, it induces a homeomorphism $\phi: X \rightarrow X$ such that $\psi(I_x) = I_{\phi^{-1}(x)} := \{a' \in \Gamma' : a'(\phi^{-1}(x)) = 0\}$. For $f \in C(X)$ and $a \in \Gamma$ we have $(f - f(x))a \in I_x$ hence $\psi((f - f(x))a)(\phi^{-1}(x)) = 0$, that is $\psi(fa)(\phi^{-1}(x)) = f(x)\psi(a)(\phi^{-1}(x))$. The proof is complete.

3.4. REMARK. Assume that all the AF-algebras $\Lambda(x)$ and $\Lambda'(x)$ are simple. Using Proposition 3.1 and a similar argument with that given in the proof of Proposition 3.3 one can see that $L \cong L'$ if and only if the field E_L is isomorphic to the pullback $\phi^*E_{L'}$ for some homeomorphism $\phi: X \rightarrow X$.

4. CLASSIFICATION RESULTS

Let X be a compact connected space. In this section we shall consider inductive limits $L = \varinjlim (C(X) \otimes A_i, \phi_i)$, where $(A_i)_{i=1}^{\infty}$ is a sequence of finite-dimensional C^* -algebras and

each $\phi_i \in \text{Hom}_{C(X)}(C(X) \otimes A_i, C(X) \otimes A_{i+1})$ is injective. Note that L inherits a natural structure of $C(X)$ -module. Consider $D(L)$, the semigroup of homotopy classes of selfadjoint projections in $\bigcup_{n=1}^{\infty} M_n \otimes L$ (see section 2). Since $D(L) = \varinjlim D(C(X) \otimes A_i)$, $D(L)$ inherits a natural structure of module over the semiring $\text{Vect}(X)$. Our classification of the inductive limits L will be given in terms of $D(L)$ and $K_0(L)$. Consider two inductive limits $L = \varinjlim (C(X) \otimes A_i, \phi_i)$ and $L' = \varinjlim (C(X) \otimes A'_i, \phi'_i)$ of the above type. Set $L_i := C(X) \otimes A_i$ and $L'_i := C(X) \otimes A'_i$.

4.1. LEMMA. Let $\phi: L \rightarrow L'$ be a $*$ -isomorphism such that $\phi(fa) = f \circ \phi \cdot \phi(a)$, $f \in C(X)$, $a \in L$, for some homeomorphism $\phi: X \rightarrow X$. Then there is a commutative diagram of $*$ -homomorphisms:

$$\begin{array}{ccccccc}
 L_i(1) & \longrightarrow & L_i(2) & \longrightarrow & L_i(3) & \longrightarrow & \dots \\
 \searrow \alpha_1 & & \nearrow \beta_1 & & \searrow \alpha_2 & & \nearrow \beta_2 \\
 & & L'_j(1) & \longrightarrow & L'_j(2) & \longrightarrow & \dots
 \end{array}$$

such that $\alpha_i(f) = f \circ \phi$ and $\beta_i(f) = f \circ \phi^{-1}$, $f \in C(X)$. The converse is also true.

Proof. We prove only the nontrivial implication. Using Glimm's Lemma [7, Lemma 1.8] as in the proof of Lemma 2.6 in [1], we can get suitable unitaries $u_i \in L'$, $v_i \in L$ such that the homomorphisms $\alpha_i = u_i \phi u_i^*$ and $\beta_i = v_i \phi^{-1} v_i^*$ to have the desired properties.

Let S be a unital semiring. Consider two inductive limits $T = \varinjlim (S_i^{r_i}, \theta_i)$ and $T' = \varinjlim (S_i'^{r_i'}, \theta_i')$ where θ_i and θ_i' are homomorphisms (not necessarily injective) of S -modules. Note that T and T' inherit a natural structure of S -modules. Set $S_i = S_i^{r_i}$ and $S_i' = S_i'^{r_i'}$. We shall distinguish an element s_i (resp. s_i') in S_i (resp. S_i') such that $\theta_i(s_i) = s_{i+1}$ (resp. $\theta_i'(s_i') = s_{i+1}'$). Then T and T' will be pointed in the obvious way, by $t = [s_i]$ and $t' = [s_i']$. Let $J: S \rightarrow S$ be an isomorphism of semirings.

4.2. LEMMA. Let $\Lambda: (T, t) \rightarrow (T', t')$ be an isomorphism of pointed semigroups such that $\Lambda(sa) = J(s)\Lambda(a)$, $s \in S$, $a \in T$. Then there is a commutative diagram of homomorphisms of pointed semigroups:

$$\begin{array}{ccccccc}
 (S_i(1), s_i(1)) & \longrightarrow & (S_i(2), s_i(2)) & \longrightarrow & (S_i(3), s_i(3)) & \longrightarrow & \dots \\
 \searrow \gamma_1 & & \nearrow \delta_1 & & \searrow \gamma_2 & & \nearrow \delta_2 \\
 & (S_j'(1), s_j'(1)) & \longrightarrow & (S_j'(2), s_j'(2)) & \longrightarrow & \dots
 \end{array}$$

such that $\gamma_k(sa) = J(s)\gamma_k(a)$, $\delta_k(sb) = J^{-1}(s)\delta_k(b)$, $s \in S$, $a \in S_i(k)$, $b \in S_j'(k)$. The converse is also true.

Proof. The proof uses the fact that S_k and S_k' are finitely generated as S -modules.

4.3. THEOREM. Let $L = \varinjlim (C(X) \otimes A_i, \phi_i)$, $L' = \varinjlim (C(X) \otimes A_i', \phi_i')$. Then L and L' are $*$ -isomorphic by a $C(X)$ -linear isomorphism if and only if $D(L)$ and $D(L')$ are isomorphic as semigroups, by a $\text{Vect}(X)$ -linear isomorphism

which takes the class of 1_L to the class of $1_{L'}$.

Proof. The proof uses Corollary 2.2, Lemma 4.1, (with $\phi = \text{id}_X$) and Lemma 4.2 (with $S = \text{Vect}(X)$ and $J = \text{id}_S$).

4.4. THEOREM. Assume that the fibres of the continuous fields E_L and $E_{L'}$ (see section 3) are simple. Then L and L' are $*$ -isomorphic if and only if there is an isomorphism of semigroups $\Lambda: D(L) \rightarrow D(L')$ which takes the class of 1_L to the class of $1_{L'}$ and such that:

$$\Lambda(sa) = J(s)\Lambda(a), \quad s \in \text{Vect}(X), \quad a \in D(L)$$

where $J: \text{Vect}(X) \rightarrow \text{Vect}(X)$ is an isomorphism of semirings induced by some homeomorphism $X \rightarrow X$.

Proof. The proof uses Corollary 2.2, Proposition 3.3, Lemma 4.1, Lemma 4.2, and the following remarks:

a) Let A, B be finite dimensional C^* -algebras and let $\phi \in \text{Hom}(C(X) \otimes A, C(X) \otimes B)$ be a $*$ -homomorphism satisfying $\phi(fa) = f \circ \phi \cdot \phi(a)$, $f \in C(X)$, $a \in C(X) \otimes A$. Then we have a factorization $\phi = \phi_1 \phi^*$

$$C(X) \otimes A \xrightarrow{\phi^*} C(X) \otimes A \xrightarrow{\phi_1} C(X) \otimes B$$

where $\phi^*(F) = F \circ \phi$ and ϕ_1 is a $C(X)$ -linear $*$ -homomorphism.

b) If $\gamma: \text{Vect}(X)^r \rightarrow \text{Vect}(X)^t$ is a homomorphism of semigroups satisfying $\gamma(sa) = J(s)\gamma(a)$, $s \in \text{Vect}(X)$, $a \in \text{Vect}(X)^r$,

then we have the factorization $\gamma = \alpha J^{(r)}$

$$\text{Vect}(X)^r \xrightarrow{J^{(r)}} \text{Vect}(X)^r \xrightarrow{\alpha} \text{Vect}(X)^t$$

where $J^{(r)}(s_1, \dots, s_r) = (J(s_1), \dots, J(s_r))$ and α is $\text{Vect}(X)$ -linear.

We denote by $K_O(L)_+$ the image of $D(L)$ into $K_O(L)$. Since $K_O(L) = \varinjlim K_O(L_i)$ and $K_O(L)_+ = \varinjlim K_O(L_i)_+$ it follows that $K_O(L)$ inherits a natural structure of $K^O(X)$ -module and the triplet $(K_O(L), K_O(L)_+, [1_L])$ is a pointed ordered group. When the canonical map $\text{Vect}(X) \rightarrow K^O(X)$ is injective the above two Theorems can be formulated in terms of K_O -groups in the following way:

4.5. THEOREM. L and L' are $*$ -isomorphic by a $C(X)$ -linear isomorphism if and only if $(K_O(L), K_O(L)_+, [1_L])$ and $(K_O(L'), K_O(L')_+, [1_{L'}])$ are isomorphic as pointed ordered groups by a $K^O(X)$ -linear isomorphism.

4.6. THEOREM. Assume that the fibres of the continuous fields E_L and $E_{L'}$ are simple. Then L and L' are $*$ -isomorphic if and only if there is an isomorphism of pointed ordered groups

$$\Lambda: (K_O(L), K_O(L)_+, [1_L]) \rightarrow (K_O(L'), K_O(L')_+, [1_{L'}])$$

such that: $\Lambda(sa) = J(s)\Lambda(a)$, $s \in K^O(X)$, $a \in K_O(L)$ where

$J: K^O(X) \rightarrow K^O(X)$ is a ring isomorphism induced by some homeomorphism $X \rightarrow X$.

5. APPLICATIONS

Assume that X is a finite connected CW-complex of dimension ≤ 3 . Then there is an isomorphism of rings $\chi: K^0(X) \rightarrow (\mathbb{Z} \times H^2(X, \mathbb{Z}), +, \cdot)$ given by $\chi[E] = (\text{rank}(E), c_1(E))$, $E \in \text{Vect}(X)$, where $c_1(E)$ is the first Chern class of E . The ring structure on $\mathbb{Z} \times H^2(X, \mathbb{Z})$ is given by:

$$(k, \alpha) + (l, \beta) = (k+l, \alpha+\beta)$$

$$(k, \alpha) \cdot (l, \beta) = (kl, l\alpha + k\beta) \quad \text{where } \alpha, \beta \in H^2(X, \mathbb{Z}), \quad k, l \in \mathbb{Z}.$$

Also, in this case the map $\text{Vect}(X) \rightarrow K^0(X)$ is injective. These facts follow from the properties of stability of vector bundles (see [9]). When $X = S^2$ we obtain that:

$$K^0(S^2) = \{s+tx: s, t \in \mathbb{Z}, x^2=0\} = \mathbb{Z}[x]/(x^2) \quad \text{and}$$

$$K^0(S^2)_+ = \{s+tx: (s, t) \in \mathbb{N}^* \times \mathbb{Z} \cup \{(0, 0)\}\}$$

Let $3 < p_1 < p_2 < \dots$ be a sequence of prime integers, $a = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$, $\underline{n}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and define $\underline{n}_i = \begin{bmatrix} n_i' \\ n_i'' \end{bmatrix}$ by $\underline{n}_{i+1} = a_i \underline{n}_i$ where $a_i = p_i a$, $i \geq 1$.

Let $A_i = M_{n_i'} \oplus M_{n_i''}$ and consider a simple AF-algebra A given by the Bratteli system:

$$A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} \dots$$

We shall consider a C^* -algebra $L = \varinjlim (C(S^2) \otimes A_i, \phi_i)$ whose pointed ordered K_0 -group is given by the inductive limit

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corresponding to the following system of $K^0(S^2)$ -linear homomorphisms:

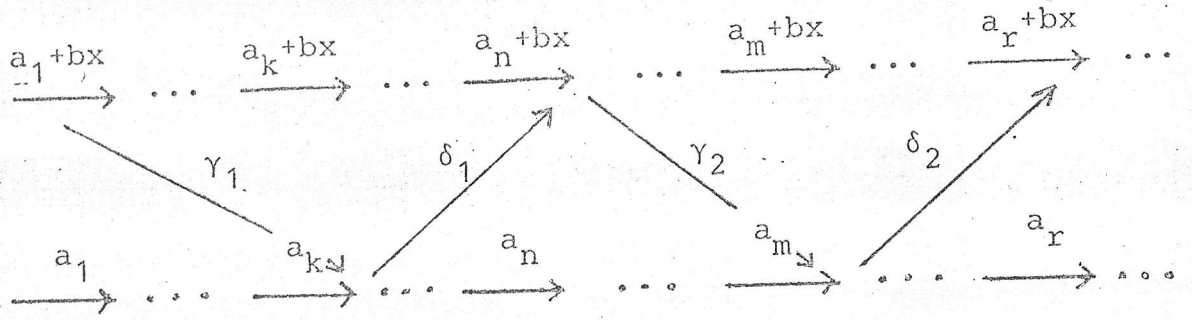
$$K^0(S^2)^2 \xrightarrow{a_1+bx} K^0(S^2)^2 \xrightarrow{a_2+bx} K^0(S^2)^2 \xrightarrow{a_3+bx} \dots$$

where $b = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$. Note that ϕ_i is such that $K_0(\phi_i) = a_i + bx$ and ϕ_i is injective.

The following proposition shows that the C^* -algebras studied in this paper do not reduce to the C^* -algebras given by trivial fields of AF-algebras.

5.1. PROPOSITION. The inductive limit $L = \varinjlim (C(S^2) \otimes A_i, \phi_i)$ is not $*$ -isomorphic to any C^* -algebra of the form $C(S^2) \otimes B$, with B an AF-algebra.

Proof. By reasons concerning the primitive spectrum of L , it is enough to show that L is not $*$ -isomorphic to $C(S^2) \otimes A$. To get a contradiction assume that $K_0(L)$ is isomorphic to $K_0(C(S^2) \otimes A)$ as in Theorem 4.6. Since any homeomorphism $\phi: S^2 \rightarrow S^2$ has the degree ± 1 , it follows, with the notation of Theorem 4.6, that $J = K^0(\phi): K^0(S^2) \rightarrow K^0(S^2)$ is given by $J(s+tx) = s \pm tx$. We shall consider only the case $J(s+tx) = s - tx$. The case $J = \text{id}$ is simpler. By Theorem 4.6 and Lemma 4.2 we must have a commutative diagram of the form (we have deleted the spaces $K^0(S^2)^2$):



where $\gamma_1 = (c+dx)J^{(2)}$, $\delta_1 = (e+fx)J^{(2)}$, $\gamma_2 = (c'+d'x)J^{(2)}$,
 $\delta_2 = (e'+f'x)J^{(2)}$ and $J^{(2)} = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$.

The following computations use the identities $ab=ba=-b$ and $J^{(2)}(g+hx)J^{(2)}=g-hx$, $g, h \in M_2(\mathbb{Z})$. The commutativity of the above diagram implies:

$$ec = a_1 \dots a_k \dots a_n \quad (4)$$

$$fc - ed = b(a_2 a_3 \dots a_n + a_1 a_3 \dots a_n + \dots + a_1 a_2 \dots a_{n-1}) \quad (5)$$

$$c'e = a_{k+1} \dots a_n \dots a_m \quad (6)$$

$$d'e - c'f = 0 \quad (7)$$

$$e'c'ec = a_1 \dots a_k \dots a_n \dots a_m \dots a_r \quad (8)$$

From (4), (5), (6) and (7) we get:

$$d'a_1 \dots a_n - a_{k+1} \dots a_n \dots a_m d = c'(p_2 p_3 \dots p_n + \dots + p_1 p_2 \dots p_{n-1}) (-1)^{n-1} b$$

so that we infer that $\begin{bmatrix} p_n & 0 \\ 0 & p_n \end{bmatrix}$ divides $c'b$ in $M_2(\mathbb{Z})$. It

follows that p_n divides $\det(c')$. We obtain from (4) that p_n^2 divides $\det(ec)$ hence p_n^3 divides $\det(e'c'ec)$ which contradicts (8) since $\det(a) = -3$.

In contrast with the above Proposition we have the following:

5.2. PROPOSITION. Let X be a connected finite CW-complex of dimension ≤ 3 and let A be a UHF-algebra, $A = \varinjlim A_i$ where each A_i is a finite discrete factor. Then $L = \varinjlim (C(X) \otimes A_i, \phi_i)$ is $*$ -isomorphic by a $C(X)$ -linear isomorphism to $C(X) \otimes A$, for any choice of $\phi_i \in \text{Hom}_{C(X)}(C(X) \otimes A_i, C(X) \otimes A_{i+1})$.

Proof. By hypothesis we have $\text{Vect}(X) \cong \mathbb{N} \times H^2(X, \mathbb{Z}) = \{s + \eta x : s \in \mathbb{N}, \eta \in H^2(X, \mathbb{Z}), x^2 = 0\}$. Hence

$$\begin{aligned} \text{Hom}_{C(X)}(C(X) \otimes A_i, C(X) \otimes A_{i+1}) / \sim &\cong \{E \in \text{Vect}(X) : E \otimes [n_i] = \\ &= [n_{i+1}]\} \cong \{s_i + \eta x : \eta \in H^2(X, \mathbb{Z}), n_i \eta = 0, x^2 = 0\} \end{aligned}$$

where $A_i = M_{n_i}$ and $s_i = n_{i+1}/n_i$ (see Corollary 2.2). Consider an arbitrary inductive limit $L' = \varinjlim (C(X) \otimes A_i, \phi'_i)$ of the same type as L . We shall apply Theorem 4.3 to show that $L \cong L'$ as $C(X)$ -modules. To prove that $(D(L), [1_L]) \cong (D(L'), [1_{L'}])$ as pointed $\text{Vect}(X)$ -modules we shall use Lemma 4.2 i.e. we shall construct a commutative diagram of the type:

$$\begin{array}{ccccccc} D_1 & \xrightarrow{s_1 + \eta_1 x} & \dots & \xrightarrow{s_m + \eta_m x} & D_{m+1} & \xrightarrow{\dots} & \xrightarrow{s_r + \eta_r x} D_{r+1} \xrightarrow{\dots} \\ & \searrow \gamma_1 & & \nearrow \delta_1 & & \searrow \gamma_2 & \nearrow \delta_2 \\ D'_1 & \xrightarrow{s'_1 + \eta'_1 x} & \dots & \xrightarrow{s'_k + \eta'_k x} & D'_{k+1} & \xrightarrow{\dots} & \xrightarrow{s'_q + \eta'_q x} D'_{q+1} \xrightarrow{\dots} \end{array}$$

where $D_i = D'_i = (N \times H^2(X, Z), n_i)$, $(\phi_i)_* = s_i + \eta_i x$, $(\phi'_i)_* = s_i + \eta'_i x$,
 $\gamma_1 = s_1 \dots s_k + \xi_1 x$, $\delta_1 = s_{k+1} \dots s_m + \xi_2 x$, $\gamma_2 = s_{m+1} \dots s_q + \xi_3 x$, etc.

Let $T_i := \{\eta \in H^2(X, Z) : n_i \eta = 0\}$. The torsion part of $H^2(X, Z)$ is finite. Hence the sequence $T_1 \subset T_2 \subset \dots$ stops. Since $\eta_i \in T_i$ we may assume that $\eta_i \in T_1$, $i \geq 1$. After dropping finitely many terms in the sequence s_1, s_2, s_3, \dots we may also assume that any class $\hat{s}_i \in \mathbb{Z}/n_1 \mathbb{Z}$ occurs infinitely many times. With these assumptions, the sequence $(\xi_i)_{i=1}^\infty$, $\xi_1 = 0$, is constructed inductively, using the following remark: given $u < v$ and $\xi \in T_1$ there are $w > v$ and $\xi' \in T_1$ such that if $\gamma = s_u \dots s_v + \xi x$ and $\delta = s_{v+1} \dots s_w + \xi' x$, the following diagram:

$$\begin{array}{ccccc} D_u & \xrightarrow{s_u + \eta_u x} & \dots & \xrightarrow{s_w + \eta_w x} & D_{w+1} \\ & \searrow \gamma & & \nearrow \delta & \\ & & D_{v+1} & & \end{array}$$

commutes, i.e.:

$$\prod_{i=u}^w (s_i + \eta_i x) = (s_{v+1} \dots s_w + \xi' x) (s_u \dots s_v + \xi x).$$

To prove this we choose w large enough such that

$$(s_u \dots s_v)^\wedge \text{ divides } (s_{v+1} \dots s_w)^\wedge \text{ in } \mathbb{Z}/n_1 \mathbb{Z}.$$

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