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1. INTRODUCTION

The aim of this note is to point out that some features which recently appear in the rather extensive literature concerning completion problems can be obtained using the Schur analysis of positive or contractive block-matrices.

The Schur analysis has its origin in [31], where the coefficients of contractive analytic functions in the unit disk were computed from the so-called "Schur sequences" via a Schur algorithm. A Schur analysis for Hankel block-matrices was done in the famous paper [1]. The general case of indexing all contractive intertwining dilations was started in [11], [12], and completed in [3] (see also [14]), using choice sequences (a generalization of Schur sequences) as free parameters. Schur analysis for positive Toeplitz block-matrices was done in [13]; the case of arbitrary positive block-matrices [16] asked for generalized choice sequences as free parameters. Schur-type algorithms are presented in all these generalizations.

The structure of contractive block-matrices is intimately connected with Schur analysis. This is illustrated by the intensive use of the structure of two-by-two matrix contractions ([6], [18], [27]) in various topics of dilation theory. A first attempt to generalize this for arbitrary matrix contractions was done in [17]. The Schur analysis of positive block-matrices [16] or the iterative use of the structure of the elementary rotation associated to a two-by-two matrix contraction [2] provide a complete description of the structure of an arbitrary matrix contraction; this description, together with the Schur analysis of positive matrices will be our main tools.

The completion problems we are talking about are the following:

(A) Consider the operators $(K_{ij})_{1\leq i,j\leq n}$ with $|i-j|\leq m$, where $0\leq m\leq n-1$. Analyse the positive band extensions of (K_{ij}). (A completion $T=(T_{ij})_{1\leq i,j\leq n}$ of (K_{ij}) is a band exension if T is invertible and $(T^{-1})_{ij}=0$ for $|i-j|>m$.)

(B) Consider the operators $(K_{ij})_{1\leq j\leq i\leq n}$. Give necessary and sufficient conditions such that there exists an $n\times n$ contractive block-matrix T such that $T_{ij}=K_{ij}$ for $1\leq j\leq i\leq n$. Describe all solutions T and analyse isometric, coisometric or unitary ones.

(C) Describe the structure of upper triangular contractions $T = (T_{ij})$ (i.e. $T_{ij} = 0$ for $i > j$) and their realizability as transfer operators for time-variant linear systems.

We do not intend to make a complete description of the history of these problems. Here are some bibliographical remarks concerning them.

For Problem (A) see [20], [19]; the maximum determinant problem for it is discussed in [19] and [26]. The permanence principle for Problem (A) is given in [21], while the maximum distance problem is studied in [10] and [22].

For Problem (B) see [19], [7], [8]; the parametrization of solutions was done in [8] (via a linear fractional map, as a consequence of a more general lifting theorem) or in [17]. The unitary case was considered in [9].

For transfer operators of time-variant linear system see for example [24].

The structure of the present paper is the following. In Section 2 we completely describe the Schur analysis for contractive block-matrices (finite or infinite – Theorems 2.4 and 2.7); this analysis comprises a Schur-type algorithm for computing the entries of the matrix from the parameters and the description of the defect spaces and of the elementary rotation in terms of the parameters. A certain connection between positiveness and contractiveness is pointed out in Remark 2.5(3) and Theorem 2.6. Some Szegö-type limit theorems for this contractive case are also presented (Theorem 2.8).

The next three sections deal with Problems (A), (B) and (C), respectively.

In Section 3, we analyse positive completions using the parametrization of positive matrices by choice triangles. It is shown that all cases of positive completions consist in the Schur analysis of given data and the continuations of the parameters already determined (Corollary 3.2 and Remark 3.4(2)). The trivial continuation – with zero entries – of the determined parameters corresponds to the solution of maximum determinant (or maximum entropies) – see Corollary 3.5 – ; the known structure of the inverse of the whole matrix for this case is also reobtained (in general dimensional case) – see Theorem 3.6 – . By contrast, it is shown that the maximum distance problem is equivalent to all these only in the one dimensional case (Corollary 3.9).

Section 4 presents a similar study for contractive completions using the parametrization given in Theorem 2.4; the results are summarized in Corollary 4.1. The existence (and the description of the solutions) of isometric completions is carefully analysed in Corollary 4.2, showing the importance of the last given diagonal.

Section 5 gives a variant of Theorem 2.4 for upper triangular contractions (Theorem 5.2). The realizability of such operators as transfer operators for unitary time-variant systems is deduced (Theorem 5.4).

2. SCHUR ANALYSIS OF CONTRACTIVE BLOCK-MATRICES

For two (complex) Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, the set of all (linear, bounded) operators from $\mathcal{H}$ into $\mathcal{H}'$ is denoted by $L(\mathcal{H},\mathcal{H}')$; $L(\mathcal{H})$ stands for $L(\mathcal{H},\mathcal{H})$. For a contraction $T \in L(\mathcal{H},\mathcal{H}')$, denote as usual [32] by D_T and $\mathcal{D}_T$ the defect operator, resp. the defect space of T, i.e.:

$$D_T = (I - T^*T)^{\frac{1}{2}} \tag{2.1}$$

$$\mathcal{D}_T = \overline{D_T(\mathcal{H})}. \tag{2.2}$$

The unitary operator:

$$J(T) : \mathcal{H} \oplus \mathcal{D}_{T^*} \longrightarrow \mathcal{H}' \oplus \mathcal{D}_T$$

$$J(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix} \tag{2.3}$$

is called the elementary rotation of T. For an arbitrary operator $S \in L(\mathcal{H},\mathcal{H}')$, $\mathcal{R}(S)$ stands for the closed range of S.

We first recall the Schur analysis for positive block-matrices ([16]); this will be explicitely used in Section 3, and also provides a way for obtaining the structure of contractive block-matrices. Consider a string $\{\mathcal{H}_i\}_{i=1}^n$ of Hilbert spaces, $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$, and fix a positive operator $S = (S_{ij})_{1\le i,j\le n}$ in $L(\mathcal{H})$, where $S_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$.

Clearly, S_{ii}, $1 \le i \le n$, are arbitrary positive operators in $L(\mathcal{H}_i)$. A $\{(S_{ii})_{i=1}^n\}$--choice triangle (called "generalized choice sequence" in [16]) is a set of contractions $\mathcal{G} = (G_{ij})_{1\le i\le j\le n}$ such that $G_{ii} = 0 \in L(\mathcal{R}(S_{ii}))$, $1 \le i \le n$ and otherwise G_{ij} acts between $\mathcal{D}_{G_{i+1,j}}$ and $\mathcal{D}_{G^*_{i,j-1}}$. We need the following operators associated with a choice triangle:

a) the *row contractions* $R_{ij}(\mathcal{G}) = R_{ij}$, $1 \le i < j \le n$, where

$$\begin{cases} R_{ij} : \bigoplus_{k=i+1}^{j} \mathcal{D}_{G_{i+1,k}} \longrightarrow \mathcal{R}(S_{ii}) \\ R_{ij} = (G_{i,i+1}, D_{G^*_{i,i+1}} G_{i,i+2}, \ldots, D_{G^*_{i,i+1}} \cdots D_{G^*_{i,j-1}} G_{ij}); \end{cases} \tag{2.4$_{ij}$}$$

b) the *column contractions* $C_{ij}(\mathcal{G}) = C_{ij}$, $1 \le i < j \le n$, where

$$\begin{cases} C_{ij} : \mathcal{R}(S_{jj}) \longrightarrow \bigoplus_{k=-(j-1)}^{-i} \mathcal{D}_{G^*_{-k,j-1}} \\ C_{ij} = (G_{j-1,j}, G_{j-2,j} D_{G_{j-1,j}}, \ldots, G_{ij} D_{G_{i+1,j}} \cdots D_{G_{j-1,j}})^t \end{cases} \tag{2.5$_{i,j}$}$$

("t" stands for matrix transpose);

c) the *generalized rotations* $U_{ij}(G) = U_{ij}$, $1 \le i \le j \le n$, where

$$U_{ii} = I_{\mathcal{R}(S_{ii})} \tag{2.6$_{ii}$}$$

and for $j > i$

$$U_{ij}: \bigoplus_{k=-j}^{-i} \mathcal{D}_{G^*_{-k,j}} \to \bigoplus_{k=i}^{j} \mathcal{D}_{G_{i,k}}$$

$$U_{ij} = J_e(G_{i,i+1})J_e(G_{i,i+2}) \dots J_e(G_{ij})(U_{i+1,j} \oplus I_{\mathcal{D}_{G^*_{ij}}}), \tag{2.6$_{ij}$}$$

where the subscript "e" at $J(G_{i,i+k})$ means that $J_e(G_{i,i+k})$ is $J(G_{i,i+k})$ on $\mathcal{D}_{G_{i+1,i+k}} \oplus \mathcal{D}_{G^*_{i,i+k}}$, and the identity elsewhere;

d) the *triangular operators* $F_{ij}(G) = F_{ij}$, $1 \le i \le j \le n$, where

$$F_{ii} = S_{ii}^{\frac{1}{2}} \tag{2.7$_{ii}$}$$

and for $j > i$

$$F_{ij}: \bigoplus_{k=i}^{j} \mathcal{R}(S_{kk}) \to \bigoplus_{k=i}^{j} \mathcal{D}_{G_{i,k}}$$

$$F_{ij} = \begin{bmatrix} F_{i,j-1} & U_{i,j-1}C_{ij}S_{jj}^{\frac{1}{2}} \\ 0 & D_{G_{ij}} \cdots D_{G_{j-1,j}} S_{jj}^{\frac{1}{2}} \end{bmatrix}. \tag{2.7$_{ij}$}$$

Note that the operators F_{ij} also verify the relations:

$$F_{i,j} = \begin{bmatrix} S_{ii}^{\frac{1}{2}} & R_{ij}F_{i+1,j} \\ 0 & \tilde{D}(R_{ij})F_{i+1,j} \end{bmatrix}, \tag{2.7$'_{i,j}$}$$

where

$$\tilde{D}(R_{ij}) = \begin{bmatrix} D_{G_{i,i+1}} & -G^*_{i,i+1}G_{i,i+2} & -G^*_{i,i+1}D_{G^*_{i,i+2}}G_{i,i+3} & \cdots & -G^*_{i,i+1}D_{G^*_{i,i+2}} \cdots D_{G^*_{i,j-1}}G_{ij} \\ 0 & D_{G_{i,i+2}} & -G^*_{i,i+2}G_{i,i+3} & \cdots & -G^*_{i,i+2}D_{G^*_{i,i+3}} \cdots D_{G^*_{i,j-1}}G_{ij} \\ 0 & 0 & D_{i,i+3} & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & D_{G_{i,j}} \end{bmatrix} \tag{2.7$''_{i,j}$}$$

is the operator which appears in the identification of $D_{R_{ij}}$ with $\bigoplus_{k=i+1}^{j} D_{G_{i,k}}$ (see (2.15)).

With these notations, the Schur analysis for positive block-matrices ([16]) is the following:

2.1. THEOREM. *There exists a one-to-one correspondence between the positive operators* $S = (S_{ij})_{1\le i,j\le n}$ *and the pairs* $\{(S_{ii})_{1\le i\le n}, G\}$, *where* $S_{ii} \in L(H_i)$ $(1 \le i \le n)$ *are positive operators and* $G = (G_{ij})_{1\le i\le j\le n}$ *is an* $\{(S_{ii})_{i=1}^{n}\}$-*choice triangle. Between corresponding elements the following formulas hold:*

$$(2.8)_{i,i+1} \qquad S_{i,i+1} = S_{ii}^{\frac{1}{2}} G_{i,i+1} S_{i+1,i+1}^{\frac{1}{2}},$$

$(1 \le i \le n-1)$ *and for* $i + 1 < j \le n$

$$(2.8)_{ij} \qquad S_{ij} = S_{ii}^{\frac{1}{2}}(R_{i,j-1}U_{i+1,j-1}C_{i+1,j} + D_{G^*_{i,i+1}} \cdots D_{G^*_{i,j-1}} G_{ij} D_{G_{i+1,j}} \cdots D_{G_{j-1,j}}) S_{jj}^{\frac{1}{2}}.$$

Moreover, for $1 \le i \le j \le n$ *the following factorizations hold:*

$$(2.9)_{i,j} \qquad (S_{kl})_{i\le k,l\le j} = F^*_{ij} F_{ij}.$$

2.2. REMARKS. 1) There are now no difficulties in describing positive block-kernels on **N** or **Z**, using infinite (uni- or bilateral) choice triangle; the Toeplitz case corresponds to choice strings or sequences.

2) The reason why U_{ij}, $1 \le i \le j \le n$, from (2.6) are called generalized rotations will be clear from Remark 2.5.

3) With respect to the decomposition of H as $\bigoplus_{i=1}^{n} H_{n-i+1}$, the matrix of S is $(\hat{S}_{ij})_{1\le i,j\le n}$ where $\hat{S}_{ij} = S_{n-i+1,n-j+1} = S^*_{n-j+1,n-i+1}$. If $(\hat{G}_{ij})_{1\le i\le j\le n}$ is the $\{(\hat{S}_{ii})_{i=1}^{n}\}$--choice triangle of S with respect to this decomposition, then it is plain that $\hat{G}_{ij} = G^*_{n-j+1,n-i+1}$, for $1 \le i \le j \le n$.

4) If the spaces H_i, $1 \le i \le n$, are finite dimensional, then the formulas (2.9) and (2.7) imply that

$$(2.10) \qquad \det S = \Big(\prod_{i=1}^{n} \det S_{ii}\Big)\Big(\prod_{1\le i<j\le n} \det D^2_{G_{ij}}\Big).$$

5) The formula (2.10) illustrates very clear the classical inequality of Hadamard, i.e.

$$(2.11) \qquad \det S \le \prod_{i=1}^{n} \det S_{ii},$$

and the fact that the equality in (2.11) occurs if and only if S is diagonal (for this last assertion see the algorithm from (2.8)). Moreover, the generalized inequality of Hadamard also follows. Indeed, let $1 \leq p < n$; then from (2.10) it results that

$$(2.12)_p \qquad \det S \leq \left(\det (S_{ij})_{1\leq i,j\leq p}\right)\left(\det (S_{ij})_{p+1\leq i,j\leq n}\right).$$

The equality in $(2.12)_p$ occurs if and only if $G_{ij} = 0$ for $1 \leq i \leq p$ and $p+1 \leq j \leq n$. An inspection of the algorithm (2.8) shows that this is the case if and only if $S_{ij} = 0$ for $1 \leq i \leq p$ and $p+1 \leq j \leq n$. (For Hadamard inequalities see for example [23]).

We will give now the structure of a contractive block-matrix and of its elementary rotation.

For this, let $H = \bigoplus_{i=1}^{m} H_i$, $H' = \bigoplus_{j=1}^{n} H'_j$ and $T = (T_{ij})_{1\leq i\leq n,\, 1\leq j\leq m}$ a contraction in $L(H, H')$, where $m,n \in \mathbf{N}$ (the finite case). The structure of T can be described using the analysis of positive block-matrices via the fact that $\|T\| \leq 1$ if and only if the block-matrix

$$(2.13) \qquad S = \begin{bmatrix} I & & & & T_{n1} & & \cdots & T_{nm} \\ & \ddots & 0 & & \vdots & & & \vdots \\ & 0 & \ddots & & T_{21} & & & \vdots \\ & & & I & T_{11} & T_{12} & \cdots & T_{1m} \\ T^*_{n1} & \cdots & & T^*_{11} & I & & & \\ \vdots & & & \vdots & & \ddots & 0 & \\ \vdots & & & \vdots & & 0 & \ddots & \\ T^*_{nm} & \cdots & & T^*_{1m} & & & & I \end{bmatrix}$$

(acting in $H'_n \oplus \ldots \oplus H'_1 \oplus H_1 \oplus \ldots \oplus H_m$) is positive. The previous analysis suggests the following:

2.3. DEFINITION. A $(\bigoplus_{i=1}^{m} H_i, \bigoplus_{j=1}^{n} H'_j)$-*choice matrix* is a set of contractions $G = (G_{i,j})_{0\leq i\leq n,\, 0\leq j\leq m}$, where $G_{0,j} = 0 \in L(H_j, H_{j-1})$, for $1 \leq j \leq m$, $(H_o = \{0\})$, $G_{i0} = 0 \in L(H'_{i-1}, H'_i)$ for $1 \leq i \leq n$ $(H'_o = \{0\})$ and for $ij > 0$, G_{ij} is in $L(D_{G_{i-1,j}}, D_{G^*_{i,j-1}})$.

To any choice matrix we attach the following objects (similar to (2.4), (2.5) and (2.6)):

a) the *row contractions* $R_{ij}(G) = R_{ij}$, $1 \leq i \leq n$, $0 \leq j \leq m$, $i+j > 1$, where

$$(2.14)_{ij}\quad \begin{cases} R_{ij}: H'_{i-1}\oplus\cdots\oplus H'_1 \oplus D_{G_{i-1,1}} \oplus\cdots\oplus D_{G_{i-1,j}} \to H'_i \\ R_{ij} = (\overbrace{0,\ldots,0}^{i-1 \text{ times}}, G_{i1}, D_{G^*_{i1}} G_{i2}, \ldots, D_{G^*_{i1}} \cdots D_{G^*_{i,j-1}} G_{ij}); \end{cases}$$

b) the *column contractions* $C_{ij}(G) = C_{ij}$, $0 \leq i \leq n$, $1 \leq j \leq m$, $i+j>1$, where

$$(2.15)_{ij}\quad \begin{cases} C_{ij}: H_j \to H_{j-1}\oplus\cdots\oplus H_1 \oplus D_{G^*_{1,j-1}} \oplus\cdots\oplus D_{G^*_{i,j-1}} \\ C_{ij}: (\overbrace{0,\ldots,0}^{j-1 \text{ times}}, G_{1j}, G_{2j} D_{G_{1j}}, \ldots, G_{ij} D_{G_{i-1,j}} \cdots D_{G_{1j}})^t; \end{cases}$$

c) the *generalized rotations* $U_{ij}(G) = U_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$, $i+j>2$, where

$$\begin{aligned} &(2.16)_{1j} && U_{1j}: H_{j-1}\oplus\cdots\oplus H_1 \to H_1 \oplus\cdots\oplus H_{j-1},\ 2 \leq j \leq m \\ &(2.16)_{i1} && U_{i1}: H'_1\oplus\cdots\oplus H'_{i-1} \to H'_{i-1} \oplus\cdots\oplus H'_1,\ 2 \leq i \leq n, \end{aligned}$$

are the matrices with I on the cross diagonal and zero elsewhere, and for $2 \leq i \leq n$ and $2 \leq j \leq m$,

$$(2.16)_{i,j}\quad \begin{cases} U_{ij}: H_{j-1}\oplus\cdots\oplus H_1 \oplus D_{G^*_{1,j-1}} \oplus\cdots\oplus D_{G^*_{i-1,j-1}} \to \\ \qquad \to H'_{i-1}\oplus\cdots\oplus H'_1 \oplus D_{G_{i-1,1}} \oplus\cdots\oplus D_{G_{i-1,j-1}} \\ U_{ij} = J_e(0_{H'_{i-2}\oplus\cdots\oplus H'_1, H'_{i-1}}) J_e(G_{i-1,1}) \cdots J_e(G_{i-1,j-1})(U_{i-1,j} \oplus I_{D_{G^*_{i-1,j-1}}}), \end{cases}$$

where the subscript "e" has the same meaning as in (2.6), and $0_{H_1,H_2}$ is the zero operator between H_1 and H_2. With these notations, we have the following:

2.4. THEOREM. a) *There exists a one-to-one correspondence between the contractions* $T \in L(\bigoplus_{i=1}^{m} H_i, \bigoplus_{j=1}^{n} H'_j)$ *and the set of* $(\bigoplus_{i=1}^{m} H_i, \bigoplus_{j=1}^{n} H'_i)$*-choice matrices* $G = (G_{ij})_{0 \leq i \leq m,\ 0 \leq j \leq n,\ i+j>0}$. *Between corresponding elements, the following formulas hold:*

$$(2.17)_{11}\qquad T_{11} = G_{11}$$

and for $i+j>2$

$$(2.17)_{ij}\qquad T_{ij} = R_{i,j-1} U_{ij} C_{i-1,j} + D_{G^*_{i,0}} \cdots D_{G^*_{i,j-1}} G_{ij} D_{G_{i-1,j}} \cdots D_{G_{0,j}}$$

(Schur algorithm for block-contractions).

b) *Moreover, if* T *and* G *correspond to each other, then the following unitary identifications hold:*

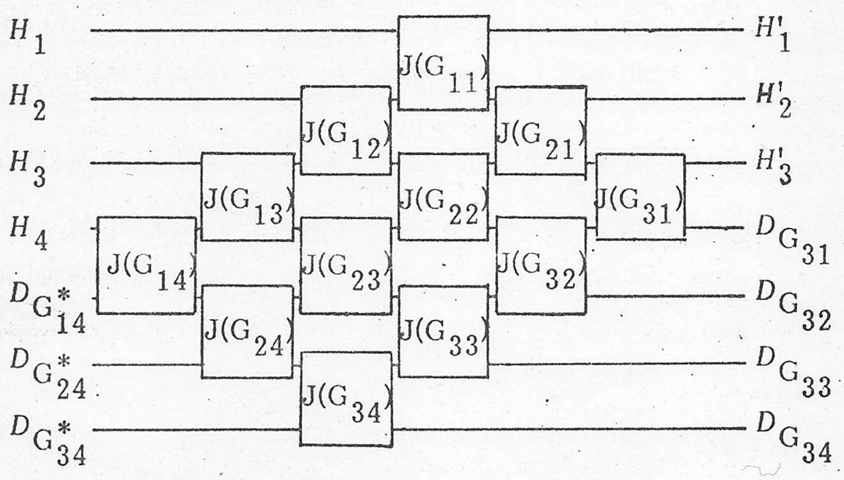
(2.18) $$D_T \overset{\alpha(T)}{\simeq} D_{G_{n1}} \oplus \ldots \oplus D_{G_{nm}}$$

(2.19) $$D_{T^*} \overset{\alpha^*(T)}{\simeq} D_{G^*_{1m}} \oplus \ldots \oplus D_{G^*_{nm}} ,$$

where $\alpha(T)D_T$ $(\alpha_*(T)D_{T^*})$ *is an* $m \times m$ *(resp.* $n \times n$*) upper triangular matrix having on the diagonal the products* $D_{G_{nj}} D_{G_{n-1,j}} \cdots D_{G_{1j}}$, $1 \leq j \leq m$ *(resp.* $D_{G^*_{im}} D_{G^*_{i,m-1}} \cdots D_{G^*_{i1}}$, $i \leq i \leq n$*).*

c) *With these identifications of the defect spaces of* T, *its elementary rotation* J(T) *has the following cell-structure* (for clarity, the picture is drawn for m = 4 and n = 3):

(2.20)



Part (a) follows directly from Theorem 2.1. The defect identifications in Part (b) can be deduced using (2.9) applied to S from (2.13). However, (b) and (c) can be obtained using the structure of defect spaces of row and column contractions ([3], and [14], Lemmas 1.3 and 1.4) and of two-by-two matrix contractions ([6], Theorem 1.3), as well as the form of their elementary rotation ([4], Lemma 2.2 and [2], Proposition 1.1). A careful inspection of these results show that the general case of (2.18), (2.19) and (2.20) can be reduced to the basic ingredients mentioned above. Let us also note that the factorizations in [17] can be simply read out from (2.20).

2.5. REMARKS. 1) A useful property (already alluded in the previous reduction of the general case) of the description of matrix contractions by choice matrices is that one can easily handle the operations of "glueing" or "broking" some of the spaces $\{H_i\}_{i=1}^m$ and $\{H'_j\}_{j=1}^n$; these operations require exactly the *same action* on the associated choice matrix. This feature is important in iterative analysis, as those performed in Section 4 on completion to isometries.

2) Let us mention also another feature of the choice matrix G associated to a contraction T. The strings $\{G_{i1}, G_{i2}, \dots, G_{im}\}_{i=1}^{n}$ are fitted to produce the row contractions

$$(2.21)_{ij} \qquad \begin{cases} R_i = R((G_{ij}); 1 \leq j \leq m) : \bigoplus_{j=1}^{m} \mathcal{D}_{G_{i-1,j}} \to H'_j \\ R_i = (G_{i1}, D_{G^*_{i1}} G_{i2}, \dots, D_{G^*_{i1}} \cdots D_{G^*_{i,m-1}} G_{im}) \end{cases}$$

$1 \leq i \leq n$, (see (2.14)). Because D_{R_i} (via α_i) can be identified with $\mathcal{D}_{G_{i1}} \oplus \dots \oplus \mathcal{D}_{G_{im}}$, the string $\{\overline{R}_1 = R_1, \overline{R}_2 = R_2\alpha_1, \overline{R}_3 = R_3\alpha_2\alpha_1 | D_{\overline{R}_2}, \dots, \overline{R}_n = R_n\alpha_{n-1} \cdots \alpha_1 | \mathcal{D}_{\overline{R}_{n-1}}\}$ is fitted to produce the column contraction $C(\overline{R}_i; 1 \leq i \leq n) = (\overline{R}_1, \overline{R}_2 D_{\overline{R}_1}, \dots, \overline{R}_n D_{\overline{R}_{n-1}} \cdots D_{\overline{R}_1})^t$, which is exactly T. So, with a slight abuse of notation, we can write

$$(2.22) \qquad T = C((R(G_{ij}); 1 \leq j \leq m); 1 \leq i \leq n) = R((C(G_{ij}); 1 \leq i \leq n); 1 \leq j \leq m).$$

This is another justification of the fact that choice matrices behave nicely at "broking" or "glueing" components in the direct sum decompositions of H and H'.

3) We are now able to explain the term "generalized rotation" which appears in (2.6) and (2.16). First, it is easy to verify that the operator U_{ij} from (2.16) is (modulo some obvious space-manipulations) the elementary rotation of the matrix $(T_{k\ell})_{1 \leq k \leq i-1,\, 1 \leq \ell \leq j-1}$.

The positive case deserves more attention. Consider again the notation from the beginning of this section: $S \in L(H)$ is a positive operator with the matrix $(S_{ij})_{1 \leq i,j \leq n}$ with respect to the decomposition $H = \bigoplus_{i=1}^{n} H_i$ and the $\{(S_{ii})_{i=1}^{n}\}$-choice triangle $G = (G_{ij})_{1 \leq i \leq j \leq n}$. Construct now the $\{\bigoplus_{i=2}^{n} R(S_{ii}), \bigoplus_{i=1}^{n-1} R(S_{ii})\}$-choice matrix $\hat{G} = (\hat{G}_{ij})_{0 \leq i \leq n-1,\, 0 \leq j \leq n-1,\, i+j>0}$, where

$$\hat{G}_{ij} = G_{n-i,\, 2n-2i-j+1},$$

for $i+j \geq n+1$, and zero on the other positions; this means that to the choice triangle

$$\begin{array}{cccc} 0 & G_{12} & G_{13} & G_{14} \\ & 0 & G_{23} & G_{24} \\ & & 0 & G_{34} \\ & & & 0 \end{array}$$

we associate the choice matrix

$$
\begin{array}{c|ccc}
0 & 0 & 0 & G_{34} \\
0 & 0 & G_{23} & G_{24} \\
0 & G_{12} & G_{13} & G_{14} \\
\hline
 & 0 & 0 & 0
\end{array} .
$$

Denote by $T = (T_{ij})_{1\le i,j\le n-1}$ the contraction associated to $\hat{G}$ by Theorem 2.4. Then the operator U_{ij} $(j>i)$ which appear in (2.6) is (neglecting the identity on some spaces) the elementary rotation of the matrix:

$$(T_{k\ell})_{n-j+1\le k\le n-1,\ i\le \ell\le j-1} .$$

We think that is worth to isolate a part of the preceding remark as the following (see also Theorem 3.2 of [5]):

2.6. THEOREM. *There exists a one-to-one correspondence between the positive matrices* $S = (S_{ij})_{1\le i,j\le n} \in L(\bigoplus_{i=1}^{n} H_i)$ *and the pairs* $\{(S_{ii})_{i=1}^{n}, T\}$, *where* $S_{ii} \in L(H_i)$ *are arbitrary positive operators and* $T \in L(\bigoplus_{i=2}^{n} R(S_{ii}), \bigoplus_{i=1}^{n-1} R(S_{ii}))$ *is a contraction with* $T_{k\ell} = 0$ *for* $k + \ell \le n$. *This correspondence is given by* $S \leftrightarrow G \leftrightarrow \hat{G} \leftrightarrow T$ *from Remark* 2.5(3).

We shortly discuss now the infinite case (i.e., where m and/or n are infinite; the case of matrix contractions indexed by $\mathbb{Z}$ can be also inferred from this). First, let us note that the algorithm described in Part (a) of Theorem 2.4 (and the representation 2.22) works in the infinite case, with corresponding infinite choice matrices.

For the identification of defect spaces in the infinite case, we need the analysis for infinite row contractions as done in [15], Propositions 1.4 and 1.6. For recalling this, suppose $n = 1$ and $m = \infty$ in the previous considerations. Then $T = (G_{11}, D_{G_{11}^*} G_{12}, D_{G_{11}^*} D_{G_{12}^*} G_{13}, \dots)$. For each $1 \le k < \infty$, denote by $T_k = T \,|\, \bigoplus_{j=1}^{k} H_j$, and by P_k the orthogonal projection of H onto $\bigoplus_{j=1}^{k} H_j$. Then the operator

$$
(2.23) \qquad \begin{cases} \alpha(T) : D_T \to \bigoplus_{j=1}^{\infty} D_{G_{1j}} \\ \alpha(T) D_T = D_\infty(T) , \end{cases}
$$

where

$$
(2.24) \qquad D_\infty(T) = s - \lim_{k\to\infty} \alpha(T_k) D_{T_k} P_k ,
$$

(see (2.18) for $\alpha(T_k)$ and (2.7)'' for $\alpha(T_k)D_{T_k}$) is a unitary operator (Proposition 1.4 in [15]).

For computing $\mathcal{D}_{T^*}$, define

$$(2.25)\qquad \begin{cases} \alpha_*(T) : \mathcal{D}_{T^*} \to \mathcal{F}(T) \\ \alpha_*(T)D_{T^*} = D_{\infty,*}(T), \end{cases}$$

where

$$(2.26)\qquad D^2_{\infty,*}(T) = \text{s-}\lim_{k\to\infty} |\alpha_*(T_k)D_{T_k^*}|^2,$$

($\alpha_*(T_k)D_{T_k^*} : H' \to \mathcal{D}_{G^*_{1k}}$ is $\alpha_*(T_k)D_{T_k^*} = D_{G^*_{1k}} \dots D_{G^*_{11}}$), and

$$(2.27)\qquad \mathcal{F}(T) = \mathcal{R}(D^2_{\infty,*}(T));$$

then $\alpha_*(T)$ is a unitary operator (Proposition 1.6 in [15]).

Of course, similar facts hold for column contractions.

In the case $n < \infty$, the space $\mathcal{D}_T$ can be easily identified (using (2.22) and (2.23)) with $\bigoplus_{j=1}^{\infty} \mathcal{D}_{G_{nj}}$, the unitary operator which realizes this identification being an infinite upper triangular matrix having on the diagonal the products $D_{G_{nj}} \cdots D_{G_{1j}}$ $(1 \leq j < \infty)$. Something similar is true for $\mathcal{D}_{T^*}$ when $m < \infty$ and $n = \infty$. In general, using (2.22), (2.25) and (2.24), we have:

2.7. THEOREM. *Let* $T = (T_{ij})_{1\leq i,j<\infty}$ *be a contraction and let* $G = (G_{i,j})_{0\leq i,j<\infty,\, i+j>0}$ *be its choice matrix. Then the following are unitary operators:*

$$(2.28)\qquad \begin{aligned} \alpha(T) &: \mathcal{D}_T \to \bigoplus_{j=1}^{\infty} \mathcal{F}(C((G_{ij});\ 1 \leq i < \infty)) \\ \alpha_*(T) &: \mathcal{D}_{T^*} \to \bigoplus_{i=1}^{\infty} \mathcal{F}(R((G_{ij});\ 1 \leq j < \infty)). \end{aligned}$$

Simple identifications show that Theorem 2.7 contains the cases where m (or n) is finite. On the other hand, the facts from Remark 2.5 and Theorem 2.6 have obvious "infinite" variants.

We end this section by pointing out some Szegö-type limit phenomena (see [25]) for a contraction $T = (T_{ij})_{1\leq i,j<\infty}$. (Similar facts for the positive case were obtained in [16] and [5] as nonstationary variants for the classical Szegö limit theorems; see [13], [15] and [4] for the Toeplitz case.)

These phenomena are based on the fact that if $T = (T_{ij})_{1\leq i\leq n,\, 1\leq j\leq m}$, and all Hilbert spaces involved are of finite dimension, then Theorem 2.4 (b) implies that

(2.29) $$\det(I - T^*T) = \prod_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \det D^2_{G_{ij}}.$$

Suppose now that $T = (T_{ij})_{1\leq i,j<\infty}$, that all T_{ij} act on finite dimensional Hilbert spaces, and that the choice matrix is "nondegenerate", i.e. $\det D_{G_{ij}} \neq 0$ for all $1\leq i,j<\infty$. For simplifying the writing, let us consider the contractions

$(2.30)^{\gamma\delta}_{\alpha\beta}$ $$M^{\gamma\delta}_{\alpha\beta} = (T_{ij})_{\alpha\leq i\leq\beta,\ \gamma\leq j\leq\delta},$$

where $\alpha,\beta,\gamma,\delta \in \mathbf{N}$; if $\alpha>\beta$ or $\gamma>\delta$, take $M^{\gamma\delta}_{\alpha\beta} = 0$. Also, for a contraction W on finite dimensional spaces denote by

(2.31) $$d(W) = \det D^2_W.$$

There are quite a few convergence phenomena resulting from nonstationary Szegö-type limit theorems in [16], [5], and from the specific form of (2.13). The importance of them comes, e.g., from their geometric significance as convergence of angles in nonstationary processes and from connections with maximal entropy spectral analysis. Due to row-column symmetry of T, we give here only "row" formulas.

2.8. THEOREM. *For* $T = (T_{ij})_{1\leq i,j<\infty}$ *(with all Hilbert spaces of finite dimensions), we have:*

a) For every $q \in \mathbf{N}$,

$(2.32)_q$ $$\lim_{n\to\infty} (d(M^{q,n}_{1,1})/d(M^{q-1,n}_{1,1})) = \lim_{n\to\infty} (d(M^{n,n}_{q,1})/d(M^{n,n}_{q+1,1})) =$$
$$= \prod_{n=1}^{\infty} (d(M^{q,n}_{1,n})/d(M^{q-1,n}_{1,n})) = \prod_{n=1}^{\infty} d(G_{qn}).$$

Denote this limit by $g^{(r)}_q(T)$ *(the symmetric notation* $g^{(c)}_q(T)$ *for columns is clear).*

b) *For a fixed* $q \in \mathbf{N}$ *and* $p \geq q$,

$(2.33)_{q,p}$ $$\lim_{n\to\infty} (d(M^{p,n}_{1,1})/d(M^{q-1,n}_{1,1})) \Big(= \lim_{n\to\infty} \prod_{i=q}^{p} (d(M^{i,n}_{1,1})/d(M^{i-1,n}_{1,1}))\Big) =$$
$$= \lim_{n\to\infty} (d(M^{n,n}_{q,1})/d(M^{n,n}_{p+1,1})) \Big(= \lim_{n\to\infty} \prod_{i=q}^{p} (d(M^{n,n}_{i,1})/d(M^{n,n}_{i+1,1}))\Big) =$$
$$= \prod_{n=1}^{\infty} (d(M^{p,n}_{1,n})/d(M^{q-1,n}_{1,n})) = \prod_{i=q}^{p}\prod_{n=1}^{\infty} d(G_{in}) = \prod_{i=q}^{p} g^{(r)}_i(T).$$

c) *For a fixed* $q \in \mathbf{N}$, *the limit for* $p \to \infty$ *in* $(2.33)_{q,p}$ *is*

$$(2.34)_q \qquad \prod_{i=q}^{\infty}\prod_{n=1}^{\infty} d(G_{in}) = \prod_{i=q}^{\infty} g_i^{(r)}(T) = 1/\lim_{p\to\infty} (d(M_{1,1}^{q,p})/g_1^{(c)}\cdots g_p^{(c)}) .$$

Part (a) is the analog of the first Szegö limit theorem; it is natural to call $g_q^{(r)}(T)$ the r-*geometrical mean of order* q for T. Part (c) is the analog of the second Szegö limit theorem, and Part (b) is a "scale" of Szegö limit theorems which connects the first one (for $p = q$) with the second one (for $p \to \infty$). See [16], Corollaries 5.6 and 5.7. Note also the expression of the r-*entropy of order* q for T ([16], Relation 5.14))

$$(2.35)_q \qquad h_q^{(r)}(T) = -\tfrac{1}{2}\ln g_q^{(r)}(T) = -\sum_{n=1}^{\infty} \ln\det D_{G_{qn}} .$$

The limits in Theorem 2.8 can be multiplied starting by deleting the first s ($\in \mathbb{N}$) columns of T; in this way one obtains $g_{q,s}^{(r)}(T)$, and so on.

3. POSITIVE COMPLETIONS

This section is devoted to the study of positive block-matrices using their structure given in Theorem 2.1. The problems to be considered come from Problem A, and include topics as positive completions of band (or more general) matrices, permanence principle, extremal (band) completions, maximum determinant problem, and maximum distance problems. We reobtain - in an unified way - some results from [19], [20], [21], [26], [10], [22] (some in a more general setting), and we offer explanations of these results using the choice triangle associated to a positive matrix. We consider only the "finite" case which contains all the necessary ingredients; see also Remark 2.2 (1) for the "infinite" case.

Let $H = \bigoplus_{i=1}^{n} H_i$ be a Hilbert space. For a positive operator $S = (S_{ij})_{1\le i,j\le n} \in L(H)$, we consider its associated $\{(S_{ii})\}_{1\le i\le n}$-choice triangle $G(S) = G = (G_{ij})_{1\le i\le j\le n}$ (see Theorem 2.1) and the row contractions, column contractions, generalized rotations and triangular operators associated to G by (2.4), (2.5), (2.6), and (2.7) respectively.

To formulate a more general completion problem than Problem (A) we need the following:

3.1. DEFINITION. A set E $\{(i,j);\ i\le j,\ 1\le i\le n,\ 1\le j\le n\}$ is called a *quasi-triangle* if $j_i = \max\{j \mid i\le j\le n,\ (i,j)\in E\}\ge i$ for each $1\le i\le n$, and for every (k,ℓ) with $i\le k\le \ell\le j_i$, $(k,\ell)\in E$.

Note that the sequence $\{j_i\}_{i=1}^{n}$ appearing in the previous definition is non-decreasing.

Now we can consider the problem:

$(A)_E$ $\left\{\begin{array}{l}\text{Consider E a quasi-triangle and for each } (i, j) \in E \text{ choose an } S_{ij} \in L(H_j, H_i)\text{. Give} \\ \text{necessary and sufficient conditions such that there exists a positive } S \in L(H) \\ \text{with the chosen entries for } (i, j) \in E\text{, and describe all such S.}\end{array}\right.$

A necessary condition for the existence of a positive completion for $(S_{ij})_{(i,j)\in E}$ – where E is a quasi-triangle – is the following:

$(*)$ $\left\{\begin{array}{l}\text{For each } 1 \le i \le n\text{, the block-matrix } (S_{k\ell})_{i\le k,\ell\le j_i} \text{ is positive, where for } i<j\text{,} \\ S_{ij} = S_{ji}^*.\end{array}\right.$

If Condition $(*)$ is verified, then one can apply Theorem 2.1 for analysing the structure of positive matrices which appear in it. The "overlapings" which appear in Condition $(*)$ and the one-to-one feature of Theorem 2.1 imply that $(S_{ij})_{(i,j)\in E}$ verifying $(*)$ is characterized by a set of contractions $(G_{ij})_{(i,j)\in E}$ such that $G_{ii} = 0 \in L(R(S_{ii}))$, $1 \le i \le n$ and otherwise G_{ij} acts between $D_{G_{i+1,j}}$ and $D_{G^*_{i,j-1}}$. (If $(i, j) \in E$, $i<j$, then easily follows that $(i + 1, j)$ and $(i, j - 1)$ are in E).

3.2. COROLLARY. *Let* E *be a quasi-triangle and* $(S_{ij})_{(i,j)\in E}$ *a set of operators. Then* $(S_{ij})_{(i,j)\in E}$ *admits a positive completion if and only if it verifies* $(*)$. *In this case, the positive completions of* $(S_{ij})_{(i,j)\in E}$ *are in one-to-one correspondence with the completions of the set of contractions* $(G_{ij})_{(i,j)\in E}$ *to a* $\{(S_{ii})\}_{i=1}^n$*-choice triangle.*

This follows from the fact that the quasi-triangle of contractions $(G_{ij})_{(i,j)\in E}$ can be always completed to a $\{(S_{ii})\}_{i=1}^n$ - choice triangle (taking, at least, zero if $(i, j) \notin E$, $i<j$), and for Theorem 2.1. The algorithm connecting corresponding elements (under the correspondence of the corollary) is that given in (2.8).

3.3. REMARK. Let $1 \le m_1 \le m_2 \le n$. It is clear that if E is quasi-triangle, then $E(m_1, m_2) = \{(i, j) \in E;\ m_1 \le i \le j \le m_2\}$ is also a quasi-triangle. So, if $(S_{ij})_{(i,j)\in E}$ has a positive completion, the same is true for any $(S_{ij})_{(i,j)\in E(m_1,m_2)}$ with $1 \le m_1 \le m_2 \le n$, the Condition $(*)$ being also hereditary. This is the so-called *permanence principle* [21].

3.4. REMARKS. 1) When there exists an $m \in \mathbf{N}$ such that in the quasi-triangle E, $j_i = \min\{i + m, n\}$ for any $1 \le i \le n$, we reobtain the band situation considered (in the finite dimensional case) in [19], [20], [21]. This situation includes the Toeplitz case.

2) It is clear that the existence of positive completions does not depend upon the order of the subspaces $\{H_i\}_{i=1}^n$. The characterization of the configurations obtained from quasi-triangles by reordering was given in [26]; these are so-called "chordal graphs". In [26], it was shown that the hypothesis that the given entries form a chordal graph is necessary even in the scalar case considered there. So, Corollary 3.2 covers the

most general situation: in the arbitrary dimensions case, if the given entries form a chordal graph, reorder the spaces to obtain a quasi-triangle and then apply the analysis from the quoted corollary.

The interpretation of the maximum determinant problem ([19], [26]) is the following:

3.5. COROLLARY. *Consider again the situation of Corollary 3.2, and that the Hilbert spaces involved are finite-dimensional. Then there exists one and only one positive completion having a maximum determinant, namely that obtained from the completion with zeros of the quasi-triangle associated to the given data. This is also the unique solution which maximizes all the entropies of order* i, $1 \leq i \leq n$ *(see Relations* (5.14) *in* [5] *for the definition of the entropies).*

The result follows from the formula (2.10).

In the quoted papers, the solution to the maximum determinant problem is obtained as the unique positive completion whose inverse has zeros on the places where the entries of the initial matrix are not prescribed.

In our setting this phenomenon follows from the study of "nonstationary" orthogonal polynomials associated to a positive matrix. We present here only a "shortcut" for obtaining this characterization of the solution of the maximum determinant problem from our Corollary 3.5.

3.6. THEOREM. a) *Let* $S = (S_{ij})_{1\leq i,j\leq n} \in L(\bigoplus_{i-1}^{n} H_i)$ *be a positive operator with the choice triangle* $G = (G_{ij})_{1\leq i\leq j\leq n}$. *If the operators* S_{ii}, $1 \leq i \leq n$, *and* $D_{G_{ij}}$, $1 \leq i < j \leq n$ *are invertible, then* S *is invertible.*

b) *In this case, if* E *is a quasi-triangle and* $G_{ij} = 0$ *for* $(i,j) \notin E$, *then* $(S^{-1})_{ij} = 0$ *for* $(i,j) \notin E$.

PROOF. Part (a) follows by induction over n from (2.9) and (2.7).

(b) Let E be a quasi-triangle and suppose $G_{ij} = 0$ for $(i,j) \notin E$. From $(2.9)_{1n}$ it follows that the conclusion would follows from the fact that $(F_{1n}^{-1})_{ij} = 0$ for $(i,j) \notin E$ (remember that F_{1n} is an upper triangular matrix). The relation $(2.7)_{1n}$ implies that (we consider the nontrivial situation $n \geq 2$)

$$
(3.2) \qquad F_{1n}^{-1} = \begin{bmatrix} F_{1,n-1}^{-1} & -F_{1,n-1}^{-1}U_{1,n-1}C_{1n}D_{G_{n-1,n}}^{-1} \cdots D_{G_{1n}}^{-1} \\ 0 & S_{nn}^{-\frac{1}{2}}D_{G_{n-1,n}}^{-1} \cdots D_{G_{1n}}^{-1} \end{bmatrix},
$$

with respect to the decompositions $(\bigoplus_{i=1}^{n-1} \mathcal{D}_{G_{1,i}})\oplus\mathcal{D}_{G_{1n}}$ and $(\bigoplus_{i=1}^{n-1} \mathcal{R}(S_{ii}))\oplus\mathcal{R}(S_{nn})$. From (3.2) it follows that we can reach our conclusion by induction over n, provided we handle the operator

$$(3.3)\qquad \tilde{C}_{1n} = F^{-1}_{1,n-1}U_{1,n-1}C_{1n} \quad .$$

So, it remains to prove that the first n' components of the column $\tilde{C}_{1n}$ are zero, where $n' = \max\{i;\ (i,n) \notin E\}$. For $n = 2$, $\tilde{C}_{12} = C_{12} = G_{12}$, and the result follows immediately. Suppose now that $n \geq 3$ and that $n' \geq 1$. We will show that the first component of $\tilde{C}_{1n}$ is zero, and that the proof can go on by induction. From $(2.7)'_{1,n-1}$, we obtain:

$$(3.4)\qquad F^{-1}_{1,n-1} = \begin{bmatrix} S_{11}^{-\frac{1}{2}} & -S_{11}^{-\frac{1}{2}}R_{1,n-1}\tilde{D}^{-1}(R_{1,n-1}) \\ 0 & F^{-1}_{2,n-1}\tilde{D}^{-1}(R_{1,n-1}) \end{bmatrix} ,$$

with respect to the decompositions $\mathcal{D}_{G_{11}}\oplus(\bigoplus_{i=2}^{n-1} \mathcal{D}_{G_{1i}})$ and $\mathcal{R}(S_{11})\oplus(\bigoplus_{i=2}^{n-1} \mathcal{R}(S_{ii}))$. From $(2.6)_{1,n-1}$ we have

$$(3.5)\qquad U_{1,n-1} = V_{1,n-1}\begin{bmatrix} U_{2,n-1} & 0 \\ 0 & I \end{bmatrix},$$

where

$$(3.6)\qquad V_{1,n-1} = J_e(G_{12})J_e(G_{13})\ldots J_e(G_{1,n-1}) ,$$

and the matrix is written with respect to the decompositions $(\bigoplus_{i=1-n}^{-2} \mathcal{D}_{G^*_{-i,n-1}})\oplus\mathcal{D}_{G^*_{1,n-1}}$ and $(\bigoplus_{i=2}^{n-1} \mathcal{D}_{G_{2,i}})\oplus\mathcal{D}_{G^*_{1,n-1}}$. But $V_{1,n-1}$ is, modulo the canonical identifications of defect spaces (see Theorem 2.4), the elementary rotation of $R_{1,n-1}$, so

$$(3.7)\qquad V_{1,n-1} = \begin{bmatrix} R_{1,n-1} & * \\ \tilde{D}(R_{1,n-1}) & * \end{bmatrix}.$$

with respect to the decompositions $(\bigoplus_{i=2}^{n-1} \mathcal{D}_{G_{2,i}})\oplus\mathcal{D}_{G^*_{1,n-1}}$ and $\mathcal{D}_{G_{11}}\oplus(\bigoplus_{i=2}^{n-1} \mathcal{D}_{G_{1i}})$; the entries noted by $*$ will not be used explicitely. Finally

(3.8) $$C_{1n} = (C_{2n}, 0)^t,$$

because of $(2.5)_{1n}$ and of the fact that $G_{1n} = 0$.

Now, using (3.4), (3.5), (3.7) and (3.8), the relation (3.3) implies that:

(3.9) $$\tilde{C}_{1n} = (0, F_{2,n-1}^{-1} U_{2,n-1} C_{2n}) = (0, \tilde{C}_{2n}).$$

The relation (3.9) shows indeed that the first component of $\tilde{C}_{1n}$ is zero, and that the proof of the fact that the first n' components are zero can be done inductively.

The proof of theorem is now completed.

3.7. REMARK. The implication in Theorem 3.6 (b) is in fact an equivalence (due to the one-to-one correspondence of Corollary 3.2), and does not depend on the finite dimensionality of the spaces $\{H_i\}_{i=1}^n$. In the later case, it gives the characterization of the solution to the maximum determinant problem obtained in [19] and [26].

We end this section with an analysis of the maximum distance problem from [10], [22]. A generalization of this problem as considered in [22] is the following. Consider a family $\{H_n\}_{n=1}^{\infty}$ of Hilbert spaces and for each $n \in \mathbf{N}$ let $S_{nn} \in L(H_n)$ be a fixed positive operator. Fix also a positive block-matrix $S = (S_{ij})_{1 \leq i,j \leq m}$. A pair $V = \{(V(n))_{n \in \mathbf{N}}; K\}$, where K is a Hilbert space and $V(n) \in L(H_n, K)$, is called an S-*admisible pair* if

(3.10) $$S_{ij} = V(i)^* V(j), \qquad 1 \leq i, j \leq m.$$

For an S-admisible pair V define $K_n(V) = K_n = \bigvee_{k=1}^{n} V(k) H_k \subset K$ and $P_n(V) = P_n = P_{K_n}^K$. The pair V is called *extremal* if for every $n \in \mathbf{N}$ the quantity

$(3.11)_n$ $$\|(I - P_n) V(n+1)\|$$

is equal with the supremum, taken over all admissible pairs, of similar quantities. The general maximum distance problem (MDP) means to describe all extremal S-admissible pairs.

The *Toeplitz case* (TMDP) asks – when $H_n = H_1$ and $S_{nn} = S_{11}$ for every n, and S is Toeplitz – the same extremal problem taken over all admissible pairs which verifies instead of (3.10) the conditions:

$(3.10)_T$ $$S_{i-j+1,1} = V(i)^* V(j), \qquad \text{for } |i - j| < m.$$

The *scalar case* (both for (MDP) and for (TMDP)) means that $\dim H_n = 1$ for every $n \in \mathbf{N}$. (In [22] is considered the scalar case of (TMDP).)

Our approach to (MDP) is the following. First, we show that all S-admissible pairs can be (esentially) described using all completions of the $\{(S_{ii})_{i=1}^{m}\}$-choice triangle (associated to S) to a $\{(S_{ii})_{n=1}^{\infty}\}$-choice triangle.

To do this, we have to recall from [16] the description of so-called *Kolmogorov decomposition* for a positive block-matrix $\hat{S} = (S_{ij})_{1\le i,j<\infty}$ in terms of its associated $\{S_{nn}\}_{n=1}^{\infty}$-choice triangle $G = \{(G_{ij})_{1\le i\le j<\infty}\}$. For this, consider the infinite row--contractions

$$(3.12)_i \qquad R_i = R_{i\infty}, \quad 1 \le i < \infty,$$

defined by (2.4), and the identifications of defect spaces of R_i as done in (2.23)-(2.27) (following [15]); recall that $F(R_i)$ (defined in (2.27) and (2.23)) is an identification of the space $D_{R_i^*}$, and denote

$$(3.13)_i \qquad D(R_i) = \alpha(R_i)D_{R_i} = \bigoplus_{j=i+1}^{\infty} D_{G_{ij}}, \quad 1 \le i < \infty.$$

Define the spaces:

$$(3.14)_1 \qquad K^{(1)} = \ldots \oplus F(R_1) \oplus F(R_1) \oplus R(S_{11}) \oplus D(R_1),$$

and

$$(3.14)_i \qquad K^{(i)} = \ldots \oplus F(R_1) \oplus F(R_1) \oplus F(R_2) \oplus \ldots \oplus F(R_{i-1}) \oplus R(S_{ii}) \oplus D(R_i), \quad 2 \le i < \infty,$$

and for $1 \le i < \infty$ the unitary operators:

$$(3.15)_i \qquad \begin{cases} W_i : K^{(i+1)} \longrightarrow K^{(i)} \\ W_i = I \oplus W_i^o, \end{cases}$$

where

$$(3.16)_i \qquad \begin{gathered} W_i^o : F(R_i) \oplus (R(S_{i+1,i+1}) \oplus D(R_{i+1})) \longrightarrow R(S_{ii}) \oplus D(R_i) \\ W_i^o = \begin{bmatrix} I & 0 \\ 0 & \alpha(R_i) \end{bmatrix} J(R_i) \begin{bmatrix} 0 & I \\ \alpha_*^*(R_i) & 0 \end{bmatrix}. \end{gathered}$$

Consider now the pair $V_G = \{(V_G(i))_{i\varepsilon N}, K^{(1)}\}$, where

$$(3.17)_i \qquad \begin{cases} V_G(i) : H_i \to K^{(1)} \\ V_G(i) = \begin{cases} (P^{K_1}_{R(S_{11})})^* S_{11}^{\frac{1}{2}} & i = 1 \\ [W_1 W_2 \ldots W_{i-1} | R(S_{ii})] S_{ii}^{\frac{1}{2}} & i > 1. \end{cases} \end{cases}$$

The pair V_G is then the minimal Kolmogorov decomposition of $\hat{S}$, i.e. $S_{ij} = V_G(i)^* V_G(j)$ for all $1 \leq i, j \leq \infty$ and $K_1 = \bigvee_{i \in \mathbf{N}} V(i)H_i$; V_G is clear an S-admissible pair for $S = (S_{ij})_{1 \leq i,j \leq m}$.

The next lemma is the basis of our analysis of (MDP):

3.8. LEMMA. *With previous notation*

$$(3.18)_i \qquad \| (I - P_n(G))V_G(n+1) \| = \| D_{G_{1,n+1}} D_{G_{2,n+1}} \cdots D_{G_{n,n+1}} S^{\frac{1}{2}}_{n+1,n+1} \|$$

for each $n \geq 1$.

PROOF. For n = 1, we have

$$\| (I - P_1(G))V_G(2) \| = \| (I - P^{K^{(1)}}_{R(S_{11})})(W_1 \mid R(S_{22}))S^{\frac{1}{2}}_{22} \| = \| D_{G_{12}} S^{\frac{1}{2}}_{22} \| ,$$

simply looking at the matrix of W_1^o (see $(3.16)_1$ and $(2.7)''_{1,\infty}$). The proof can be completed by induction, applying the induction hypothesis to the matrix obtained from $\hat{S}$ deleting the first row and the first column, and finally making the last product with W_1.

Let us remark that the previous lemma also follows from the formula of the product $W_1 W_2 \dots W_n$ obtained via the analysis in Theorem 3.2 from [5].

Our discussion of (MPD) is now transparent. Any S-admissible pair V for the fixed positive block-matrix $S = (S_{ij})_{1 \leq i,j \leq m}$ produces a positive extension $\hat{S}$ of S (note that the main diagonal is fixed). This extension $\hat{S}$ has a choice triangle G and a unique Kolmogorov decomposition V_G. It is clear that $\mathcal{V}_G$ is, essentially, the minimal part of V. Thus it follows that the supremum in $(3.11)_n$ can be taken only over all V_G, obtained from various positive extensions of $\hat{S}$ (or, which is equivalent, from all completions of the choice triangle of S to an infinite $\{(S_{nn})_{n=1}^{\infty}\}$-choice triangle). So, we obtain:

3.9. COROLLARY. a) *For* (MDP) *the supremum in* $(3.11)_n$, $n \geq m$ *is equal to* $\| S^{\frac{1}{2}}_{n+1,n+1} \|$, *which is attained, for example, by the completion of the choice triangle of* S *with zero entries. However, in the nonscalar case,* (3.18) *shows that this solution is far from being unique.*

b) *For* (TMDP) *the supremum in* $(3.11)_n$, $n \geq m$ *is equal to* $\| D_{G_{n-m+1,n+1}} \cdots D_{G_{n,n+1}} S^{\frac{1}{2}}_{n+1,n+1} \| = \| D_{G_{m-1}} \cdots D_{G_1} S^{\frac{1}{2}}_{11} \|$ *(where* $G_j = G_{i,i+j}$ *for every* $i, j \geq 1$), *which is attained, for example, by the completion of the choice sequence of* S *with zero entries. However, in the nonscalar case,* (3.18) *shows that this solution is far from being unique.*

c) *In the scalar case both* (MDP) *and* (TMDP) *have unique solutions, namely those indicated in* a) *resp.* b).

3.10. REMARK. 1) Corollary 3.9 shows that in the scalar case (MDP) or (TMDP) is equivalent with the maximum determinant problem (see [22]). It also shows that, in general, this is no longer true: the unique solution to the maximum determinant problem is a solution to (MDP) or (TMDP), which have many others.

2) The formula (3.2) from [22], which expresses the distance from (3.11) – for the scalar case – in terms of a ratio of Gramians follows easily from Lemma 3.8 and from our formula (3.1).

4. CONTRACTIVE COMPLETIONS

In this section we apply the Schur analysis of contractive block-matrices (given in Section 2) to Problem B. Following a notation which is usual in this context (see e.g., [8], [9]) we define for a fixed $n \in \mathbb{N}$, for two families of Hilbert spaces $\{H_i\}_{1\leq i\leq n}$ and $\{H'_i\}_{1\leq i\leq n}$, and for $-n \leq m \leq n$, the sets

$$(4.1)_m \qquad \Omega_\ell(m) = \{K \in L(H, H');\ K = (K_{ij}) \text{ with } K_{ij} = 0 \text{ for } i < j - m\}$$

and

$$(4.2)_m \qquad \Omega_u(m) = \{K \in L(H, H');\ K = (K_{ij}) \text{ with } K_{ij} = 0 \text{ for } i > j - m\},$$

where $H = \bigoplus_{i=1}^{n} H_i$, $H' = \bigoplus_{i=1}^{n} H'_i$. Clearly $\Omega_\ell(-n) = \{0\}$, $\Omega_\ell(n) = \Omega_\ell(n-1) = L(H, H')$, and $\Omega_\ell(m) = \Omega_u(-m)^*$, for $-n \leq m \leq n$. Due to the last fact, we present only the case of Ω_ℓ, the other one being completely similar. With these notations, a generalization of Problem (B) reads as follows:

$(B')_m$ Consider $K \in \Omega_\ell(m)$, $-n \leq m \leq n-1$. Give necessary and sufficient conditions such that there exist contractions in the coset $K + \Omega_u(m+1)$ and describe all solutions. Same problem for contractions replaced by isometric, coisometric, or unitary operators.

More general completion problems can be handled, provided the given data can be "moved" in the upper left corner (a fact also remarked in [17]); moreover the basic facts appear in the study of Problem (B') and the interested reader can adapt the analysis done for positive completions in Corollary 3.2. The aspect that only "square" matrices are considered here is also inesential.

We apply now Theorem 2.4 for obtaining some of the results on Problem (B') from [19], [7], [8, Theorem 2] and [9]. For this, let $K = (K_{ij}) \in \Omega_\ell(m)$, where $-n+1 \leq m \leq n-2$ (to avoid trivial cases). The following condition is clearly necessary for the existence of a contraction in $K + \Omega_u(m+1)$:

(*) $\left\{\begin{array}{l}\text{For each } p \text{ with } \max\{1, 1-m\} \leq p \leq n, \text{ the block-matrix } (K_{ij})_{p\leq i\leq n,\ 1\leq j\leq \min\{p+m,n\}} \\ \text{is a contraction.}\end{array}\right.$

Suppose K verifies (*); then one can apply Theorem 2.4 for analysing the structure of the contractions which appear in Condition (*). For this it is necessary to "move" the low-left corner into the upper-left one; we do this by writing $H' = \bigoplus_{i=1}^{n} H'_{n-i}$, so the q-row becomes the n - q one ($1 \leq q \leq n$). We keep, however, the initial notation of the matrix entries. Taking into consideration the "overlapings" which appear in Condition (*), and the one-to-one feature of Theorem 2.4, it follows that K verifying (*) is characterized by a set of contractions

(4.3) $$G(K) = \{G_{ij}\}_{\max\{1,1-m\}\leq i\leq n+1,\ 0\leq j\leq \min\{i+m,n\},\ (n+1-i)+j>0}$$

where

$(4.4)_j$ $$G_{n+1,j} = 0 : H_j \to H_{j-1}, \quad \text{for } 1 \leq j \leq \min\{n+m+1, n\}$$

(with $H_o = \{0\}$)

$(4.5)_i$ $$G_{i,0} = 0 : H'_{i+1} \to H'_i, \quad \text{for } \max\{1, 1-m\} \leq i \leq n$$

(with $H'_{n+1} = \{0\}$), and

$(4.6)_{ij}$ $$G_{ij} : D_{G_{i+1,j}} \to D_{G^*_{i,j-1}}, \quad \text{for } (n+1-i)j > 0;$$

the connection between K and $G(K)$ is given by the algorithm (2.17).

It is clear that $G(K)$ can be completed to a $(\bigoplus_{i=1}^{n} H_i, \bigoplus_{j=1}^{n} H'_{n-j})$-choice matrix, and that the "freedom" in starting this completion depends on the defect spaces of the elements of the "last diagonal" in $G(K)$. We have then:

4.1. COROLLARY. *Let* $K \in \Omega_\ell(m)$ *where* $-n+1 \leq m \leq n-2$.

a) *There exist contractions in* $K + \Omega_u(m+1)$ *if and only if* K *verifies* (*). *In this case, there exists a one-to-one correspondence between the contractions in* $T \in K + \Omega_u(m+1)$ *and the completions of* $G(K)$ *to* $(\bigoplus_{i=1}^{n} H_i, \bigoplus_{j=1}^{n} H'_{n-j})$*-choice matrices* $G(T)$*; the defect spaces* D_T *and* D_{T^*} *can be (unitary) identified with* $\bigoplus_{k=1}^{n} D_{G_{1k}}$*, resp.* $\bigoplus_{k=1}^{n} D_{G^*_{kn}}$.

Moreover, if K *verifies* (∗):

b) *There exists only one contraction in* $K + \Omega_u(m + 1)$ *if and only if from each pair* $\{\mathcal{D}_{G_{i+1,\, i+m+1}}, \mathcal{D}_{G^*_{i,\, i+m}}\}$, *for* $\max\{1, -m\} \leq i \leq \min\{n - m - 1, n\}$, *at least one space is zero.*

c) *If the Hilbert spaces involved are of finite dimensions, then there exists one and only one contraction in* $K + \Omega_u(m + 1)$ *which has maximum entropy for each row (or column); that is the completion of* $G(K)$ *with zero elements.*

Part (a) follows form Theorem 2.4; the algorithm connecting corresponding elements is that given in (2.14). Part (b) results from the way $G(K)$ is completed to a choice matrix. Part (c) follows from (2.35).

We call the completion of $G(K)$ with zero elements – even in the infinite dimensional case – the *maximum entropy completion.*

We analyse now the existence of isometries in $K + \Omega_u(m + 1)$ reobtainig some results in [9], Theorem 3.1 (without the finite dimensional assumption). The cases of coisometric or unitary completions can be inferred from this.

For $K \in \Omega_\ell(m)$, $-n + 1 \leq m \leq n - 2$, consider $G(K)$ the set of contractions associated to K by (4.3)-(4.6). Denote:

$$(4.7)_{i,j} \qquad d_{ij} = \dim \mathcal{D}_{G_{i,j}}, \quad d^*_{ij} = \dim \mathcal{D}_{G^*_{i,j}},$$

for $\max\{1, 1 - m\} \leq i \leq n + 1$, $0 \leq j \leq \min\{i + m, n\}$. Then we have:

4.2. COROLLARY. *With previous notation, there exist isometries in* $K + \Omega_u(m + 1)$ *if and only if the following conditions are fulfilled:*

$$(4.8)_- \qquad \begin{cases} \sum_{j=1}^{i} d_{j-m,j} \leq \sum_{k=1}^{-m} d^*_{k,0} + \sum_{j=1}^{i-1} d^*_{j-m,j}, & \text{for all } 1 \leq i \leq n + m, \\ \sum_{j=1}^{n+m} d_{j-m,j} + \sum_{j=n+m+1}^{n} d_{n+1,j} \leq \sum_{k=1}^{-m} d^*_{k,0} + \sum_{j=1}^{n+m} d^*_{j-m,j}, & \end{cases}$$

if $m < 0$, *or*

$$(4.8)_+ \qquad \begin{cases} d_{1,1} = d_{1,2} = \ldots = d_{1,m+1} = 0 \\ \sum_{j=2}^{i} d_{j,j+m} \leq \sum_{j=1}^{i-1} d^*_{j,j+m}, & \text{for all } 2 \leq i \leq n - m, \end{cases}$$

if $m \geq 0$.

In this case, there exists a one-to-one correspondence between the isometries in $K + \Omega_u(m+1)$ *and the completions of* $G(K)$ *to* $(\bigoplus_{i=1}^{n} H_i, \bigoplus_{j=1}^{n} H'_{n-j})$*-choice matrices with* G_{1j} $(1 \le j \le n)$ *being isometries.*

PROOF. Note first the following fact concerning two-by-two matrix contractions. Suppose that the (1,1), (1,2) and (2,1) entries are given, i.e.

$$\begin{bmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & * \end{bmatrix}, \tag{4.9}$$

with A, Γ_1, and Γ_2 contractions. Then the (2,2) entry can be chosen such that the whole matrix to be an isometry if and only if

$$\Gamma_2 \text{ is an isometry} \quad \text{and} \quad \dim \mathcal{D}_{\Gamma_1} \le \dim \mathcal{D}_{\Gamma_2^*}. \tag{4.10}$$

This follows from the structure of the (2,2) entry as being $-\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*}\Gamma D_{\Gamma_1}$, where $\Gamma \in L(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$ is a contraction, and from the fact that the defect of the matrix is unitary equivalent with $\mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_{\Gamma}$ (see Theorem 2.4).

Coming back to the situation of the corollary, we will need the notation from (2.30); we intertwine the indices indicating the rows in order to emphasize the way we are looking at the matrices in this section.

Let K be given as in the statement, and let $G(K)$ its associated set of contractions (by Corollary 4.1). For clarity, we analyse separately the case $m < 0$ and the case $m \ge 0$.

Case I : m < 0. We try to construct an isometry in $K + \Omega_u(m+1)$ by completing inductively the contractions $M_{n,1}^{i-m,i}$ to H' - valued isometries, $1 \le i \le n+m$. After this, the last step will be to complete $M_{n,1}^{1,n+m}$ to an isometry defined on H. The first step is to complete the column $M_{n,1}^{1-m,1}$ to an H'-valued isometry. From $(4.8)_-$ for i = 1 we have

$$d_{1-m,1} \le \sum_{k=1}^{-m} d^*_{k,0} \quad \left(= \sum_{k=1}^{-m} \dim H'_k\right), \tag{4.11$_1$}$$

which is exactly the necessary and sufficient condition for the existence of an isometric completion of $M_n^{1-m,1}$ to a matrix $M_{n,1}^{1,1}$, via an isometry $W_1 : \mathcal{D}_{G_{1-m,1}} \to \sum_{k=1}^{-m} H'_k$. We have, in general, that (see (2.3))

$$(4.12)_1 \qquad d_{1-m,1} + \dim D_{W_1^*} = \sum_{k=1}^{-m} d^*_{k,0};$$

in case $d_{1-m,1}$ is not finite, we take W_1 such that

$$(4.13)_1 \qquad \dim D_{W_1^*} = \sum_{k=1}^{-m} d^*_{k,0}.$$

The next step consists in completing $M_{n,1}^{2-m,2}$ to an H'-valued isometry, respecting the already chosen $M_{n,1}^{1,1}$. After obvious identifications, this reduces to a problem similar to that presented in (4.9) and (4.10): in our case, one can take $A = K_{2-m,1}$, $\Gamma_1 = G_{2-m,2}$, $\Gamma_2 = (G_{1-m,1}, D_{G^*_{1-m,1}} W_1)^t$. The operator Γ_2 is in this case an isometry because W_1 is so; thus (4.10) implies that the necessary and sufficient condition for the solution of this step is that

$$(4.11)_2 \qquad d_{2-m,2} \leq d^*_{1-m,1} + \dim D_{W_1^*}.$$

From $(4.12)_1$ and $(4.13)_1$ it follows that $(4.11)_2$ is equivalent with the relation $(4.8)_-$ for $i = 2$. Denote by W_2 the isometry from $D_{G_{2-m,2}}$ into $D_{W_1^*}$ which appear in the structure of the upper-right corner of the completion of $M_{n,1}^{2-m,2}$. Then we have

$$(4.12)_2 \qquad d_{2-m,2} + \dim D_{W_2^*} = \dim D_{W_1^*};$$

in case $d_{2-m,2}$ is not finite, we take W_2 such that

$$(4.13)_2 \qquad \dim D_{W_2^*} = \dim D_{W_1^*}.$$

The proof of Case I can now be completed by induction.

Case II : $m \geq 0$. From $(4.8)_+$ it follows that $M_{n,1}^{1,m+1}$ is already an isometry. The proof can be now done in same manner as in Case I, completing inductively the matrices $M_{n,1}^{i,m+i}$ $(2 \leq i \leq n-m)$ to H'-valued isometries, respecting the existing $M_{n,1}^{1,m+1}$-part.

The last part of the corollary is immediate from Corollary 4.1.

4.3. REMARKS. (i) It is easy to show that in the finite dimensional case, the conditions (4.8) are equivalent with those appearing in Theorem 3.1 (ii) from [9]. For example, if $m < 0$ the condition in $(4.8)_-$ for $i = 1$ is equivalent with $\dim D_{M_{n,1}^{1-m,1}} \leq \sum_{k=1}^{-m} \dim H'_k$, the condition for $i = 2$ is equivalent with $\dim D_{M_{n,1}^{2-m,2}} \leq \sum_{k=1}^{-m+1} \dim H'_k$, and so on. Some other quantities appearing in [9] can be computed from the choice matrix. For example, in [9] there were introduced the partial rank indices r_{pq} $(1 \leq p,q \leq n)$ by the formula:

$$r_{pq} = \phi_{pq} - \phi_{p,q-1} - \phi_{p+1,q} + \phi_{p+1,q-1},$$

where

$$\phi_{pq} = \dim \ker D_{M_{n,1}^{p,q}}$$

for $1 \leq p,q \leq n$, and $\phi_{pq} = 0$ for $p = n+1$ or $q = 0$. From Theorem 2.4 it follows easily that

$$r_{pq} = \dim \ker D_{G_{pq}} .$$

(ii) It is plain that when $\dim H_i^* = \dim H_i' = \kappa < \infty$ for any $1 \leq i \leq n$, the conditions in $(4.8)_-$ and the second line in $(4.8)_+$ are automatically fulfilled.

(iii) The form of the conditions (4.7) strenghtens the importance of the "last diagonal" in this completion problem.

(iv) The analysis of completions can be very well understood on the diagram of the elementary rotation of the whole matrix (see (2.17)).

5. UPPER TRIANGULAR CONTRACTIONS

This section is devoted to the presentation of a variant of Theorem 2.4 for lower triangular (square) matrix contractions; this structure theorem will be used for the description of the realizability of such contractions as transfer operators for linear, unitary, time-variant systems. In fact, the realizability theorem will provide another form of the algorithm in (2.17).

The structure theorem given here will take into account the simplifications introduced by the zero elements of the upper triangular matrices. Its origins may be found in a similar analysis for lower triangular Toeplitz contractions which appeared in the study of contractive intertwining dilations (see [3] and [14]).

Let $H = \bigoplus_{i=1}^{\infty} H_i$, $H' = \bigoplus_{j=1}^{\infty} H_j'$, and let $T = (T_{ij})_{1 \leq i,j < \infty}$ be a contraction in $L(H, H')$ which is upper triangular, i.e., $T_{ij} = 0$ for $i > j$ (in other words, $T \in \Omega_u(0)$ in the terminology of Section 4). For such contractions it is useful to give (besides the direct application of Theorem 2.4) a structure theorem, which, in the finite case, is obtained by moving the zero entries in the left-upper corner. This means that for each finite section $(T_{ij})_{1 \leq i,j \leq n}$ of T we change the order in the codomain to be $H_n' \oplus \ldots \oplus H_1'$, and apply Theorem 2.4 for the resulting matrix contraction; then we put together the obtained information for giving the structure of T.

This procedure asks for the following considerations.

5.1. DEFINITION. A $(\bigoplus_{i=1}^{\infty} H_i, \bigoplus_{j=1}^{\infty} H'_j)$-*choice cotriangle* is a set of contractions $G = (G_{ij})_{1\leq i\leq j+1<\infty}$, where $G_{i,i-1} = 0 \in L(H_{i-1}, H'_i)$ for $i = 1,2,\dots$, $(H_0 = \{0\})$, and for $i \leq j$ G_{ij} is in $L(D_{G_{i+1,j}}, D_{G^*_{i,j-1}})$.

To each choice cotriangle we associate the following objects:

a) the *row contractions* $R_{ij}(G) = R_{ij}$, $i \geq 1$, $j \geq i$, defined by

$$(5.1)_{ij}\quad \begin{cases} R_{ij} : H'_{i+1}\oplus\dots\oplus H'_j\oplus H_1\oplus\dots\oplus H_{i-1}\oplus D_{G_{i+1,i}}\oplus\dots\oplus D_{G_{i+1,j}} \to H'_i \\ R_{ij} = (\overbrace{0,\dots,0}^{j-1 \text{ times}}, G_{ii}, D_{G^*_{ii}}G_{i,i+1},\dots, D_{G^*_{ii}}\cdots D_{G^*_{i,j-1}}G_{ij}); \end{cases}$$

b) the *column contractions* $C_{ij}(G) = C_{ij}$, $i \geq 1$, $j \geq i$, where

$$(5.2)_{ij}\quad \begin{cases} C_{ij} : H_j \to H_{j-1}\oplus\dots\oplus H_1\oplus D_{G^*_{j,j-1}}\oplus\dots\oplus D_{G^*_{i,j-1}} \\ C_{ij} = (\overbrace{0,\dots,0}^{j-1 \text{ times}}, G_{jj}, G_{j-1,j}D_{G_{jj}},\dots, G_{ij}D_{G_{i+1,j}}\cdots D_{G_{jj}})^t; \end{cases}$$

c) the *generalized rotations* $U_{ij}(G) = U_{ij}$, $i \geq 1$, $j \geq i$, $i+j > 2$, where

$$(5.3)_{jj}\qquad U_{jj} : H_{j-1}\oplus\dots\oplus H_1 \to H_1\oplus\dots\oplus H_{j-1}$$

is, for each $j \geq 2$, the operator of reversing the order, and for $j > i \geq 1$,

$$(5.3)_{ij}\quad \begin{cases} U_{ij} : H_{j-1}\oplus\dots\oplus H_1\oplus D_{G^*_{j,j-1}}\oplus\dots\oplus D_{G^*_{i+1,j-1}} \to \\ \quad \to H'_{i+1}\oplus\dots\oplus H'_j\oplus H_1\oplus\dots\oplus H_{i-1}\oplus D_{G_{i+1,i}}\oplus\dots\oplus D_{G_{i+1,j-1}} \\ U_{ij} = J_e(0_{H'_{i+2}\oplus\dots\oplus H'_j\oplus H_1\oplus\dots\oplus H_{i-1}, H'_{i+1}})\cdot \\ \quad \cdot J_e(G_{i+1,i})\cdots J_e(G_{i+1,j-1})(U_{i+1,j}\oplus I_{D_{G^*_{i+1,j-1}}}), \end{cases}$$

where the conventions are those from (2.16).

These are, of course, the forms of (2.14), (2.15), and (2.16) which are necessary for the present situation. Thus, from Theorem 2.4 we obtain:

5.2. THEOREM. a) *There exists a one-to-one correspondence between the upper triangular contractions* $T \in L(\bigoplus_{i=1}^{\infty} H_i, \bigoplus_{j=1}^{\infty} H'_j)$ *and the set of* $(\bigoplus_{i=1}^{\infty} H_i, \bigoplus_{j=1}^{\infty} H'_j)$*-choice cotriangle* $G = (G_{ij})_{1\le i\le j+1<\infty}$. *Between corresponding elements, the following formulas hold:*

$$(5.4)_{ii} \qquad T_{ii} = G_{ii},$$

for $i = 1, 2, \ldots,$ *and for* $1 \le i < j < \infty$,

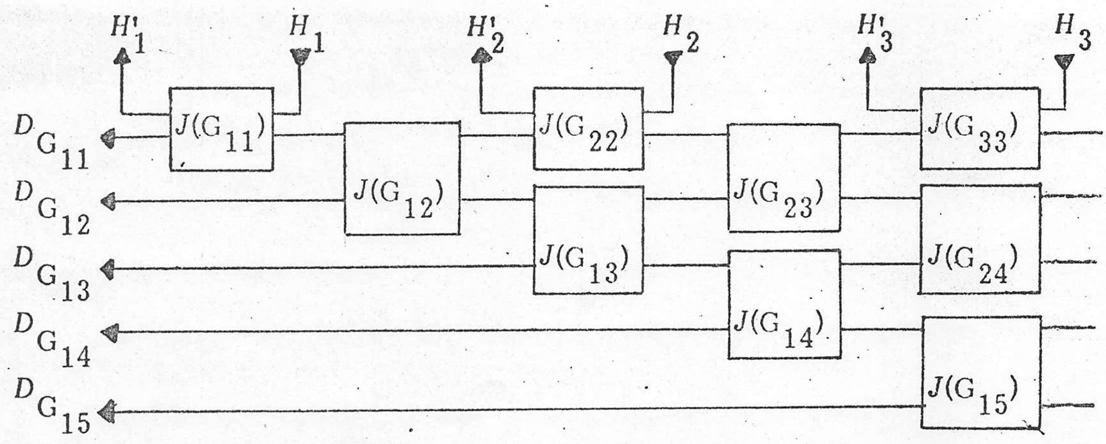
$$(5.4)_{ij} \qquad T_{ij} = \overline{R}_{i,j-1} U_{ij} C_{i+1,j} + D_{G^*_{ii}} \cdots D_{G^*_{i,j-1}} G_{ij} D_{G_{i+1,j}} \cdots D_{G_{jj}},$$

where $\overline{R}_{i,j-1} = R_{ij} | H'_{i+1} \oplus \ldots \oplus H'_j \oplus H_1 \oplus \ldots \oplus H_{i-1} \oplus D_{G_{i+1,i}} \oplus \ldots \oplus D_{G_{i+1,j-1}}$.

b) *Moreover, if* T *and* G *correspond to each other then there exists a unitary operator* $\alpha(T)$ *acting between* D_T *and* $\bigoplus_{n=1}^{\infty} D_{G_{1n}}$. *The identification of* D_{T^*} *can be done using the method from* [14, Theorem 4.3] *where the Toeplitz case was considered.*

c) *The algorithm* (5.4) *– and the cell-structure of any finite part of the elementary rotation of* T *– can be read out on the infinite scheme:*

(5.5)



5.3. REMARK. In the Toeplitz case, a slight modification of the diagram (5.5) produces a transmission-line structure (see [30]) which is the "flow graph" or "block-diagram" of the classical Schur algorithm (see [28]).

Let us give now the realizability procedure mentioned at the beginning of the section. Consider the linear, time-variant system

(5.6)
$$\begin{cases} x_n = A_n x_{n+1} + B_n u_n \\ y_n = C_n x_{n+1} + D_n u_n , \qquad n \geq 1 , \end{cases}$$

where $u_n \varepsilon U_n$ (the input spaces), $y_n \varepsilon Y_n$ (the output spaces), and $x_n \varepsilon X_n$ (the state spaces), and for which the operators

(5.7)$_n$
$$\begin{cases} \Lambda_n : X_{n+1} \oplus U_n \to X_n \oplus Y_n \\ \Lambda_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} , \end{cases}$$

$n \geq 1$, are unitary operators. It is easy to show that the transfer operator of the system (5.6) is an upper triangular *contraction* T, where

(5.8)
$$\begin{cases} T : \bigoplus_{i=1}^{\infty} U_i \to \bigoplus_{i=1}^{\infty} Y_i \\ T = (T_{ij}) \end{cases}$$

is given by $T_{ij} = 0$ for $i < j$, and

(5.9)$_{ii}$ $\qquad T_{ii} = D_i , \qquad (1 \leq i < \infty),$

(5.9)$_{i,i+1}$ $\qquad T_{i,i+1} = C_i B_{i+1} \qquad (1 \leq i < \infty),$

and for $1 \leq i < j - 1$,

(5.9)$_{i,j}$ $\qquad T_{ij} = C_i A_{i+1} \cdots A_{j-1} B_j$

(T is a contraction because the part of T acting between $\bigoplus_{i=1}^{n} U_i$ and $\bigoplus_{j=1}^{n} Y_j$ is a part of a unitary matrix between $X_{n+1} \oplus (\bigoplus_{i=1}^{n} U_i)$ and $X_n \oplus (\bigoplus_{j=1}^{n} Y_j)$, for each $n \in \mathbf{N}$).

Conversely, consider T an upper triangular contraction acting between $H = \bigoplus_{i=1}^{\infty} H_i$ and $H' = \bigoplus_{j=1}^{\infty} H'_j$, and let $G = (G_{ij})_{1 \leq i \leq j+1 < \infty}$ be its associated choice cotriangle. For the infinite row contractions

(5.10)$_i$
$$\begin{cases} R_i : H_i \oplus \bigoplus_{j=i+1}^{\infty} D_{G_{i+1,j}} \to H'_i \\ R_i = (G_{ii} , D_{G_{ii}^*} G_{i,i+1} , \dots) \end{cases}$$

we repeat the analysis of defect spaces and of the elementary rotation done in (2.20)-(2.24) and (3.13)-(3.16). We are led to consider the spaces

$$(5.11)_i \qquad \begin{cases} K_i = \ldots \oplus F(R_1) \oplus F(R_1) \oplus F(R_2) \oplus \ldots \oplus F(R_i) \oplus H_i \oplus \bigoplus_{j=i+1}^{\infty} D_{G_{i+1,j}} \\ K_i' = \ldots \oplus F(R_1) \oplus F(R_1) \oplus F(R_2) \oplus \ldots \oplus F(R_{i-1}) \oplus H_i' \oplus \bigoplus_{j=1}^{\infty} D_{G_{i,j}}, \end{cases}$$

and the unitary operators

$$(5.12)_i \qquad W_i : K_i \to K_i',$$

which act as $J(R_i)$ – via the identifications of defect spaces of R_i – between $F(R_i) \oplus (H_i \oplus \bigoplus_{j=i+1}^{\infty} D_{G_{i+1,j}})$ and $H_i' \oplus \bigoplus_{j=1}^{\infty} D_{G_{i,j}}$ and as the identity on the rest of components (see (3.15) - (3.16)).

The system attached to T is defined as follows: take

$$(5.13)_n \qquad \begin{cases} U_n = H_n, \\ Y_n = H_n' \\ X_n = K_n' \ominus H_n', \end{cases}$$

for each $n \geq 1$. Note that for $n > 1$, we have

$$(5.14)_n \qquad X_n = K_n' \ominus H_n' = K_{n-1} \ominus H_{n-1}.$$

Finally, define for each $n \geq 1$

$$(5.15)_n \qquad \begin{cases} A_n = P_{X_n}^{K_n'} W_n | X_{n+1} \\ B_n = P_{X_n}^{K_n'} W_n | U_n \\ C_n = P_{Y_n}^{K_n'} W_n | X_{n+1} \\ D_n = G_{nn}. \end{cases}$$

Then it is clear that

$$(5.16)_n \qquad W_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} : X_{n+1} \oplus U_n (= K_n) \to X_n \oplus Y_n (= K_n'),$$

so the system defined in (5.15) is a unitary one.

5.4. THEOREM. *Consider an upper triangular contraction* T *and the spaces and operators defined from* T *by* (5.13) *and* (5.15). *Then the system* (5.6) *(with the spaces* (5.13) *and operators* (5.15)) *is unitary and has* T *as its transfer operator.*

PROOF. Remark that $\{W_i\}_{i=1}^{\infty}$ defined in (5.12) are not exactly the unitary operators which appear in the Kolmogorov decomposition of the positive operator associated to T (see Theorem 5.2 and Section 2); however, they differ only by some rows with the only nonzero entry equal to the identity. Moreover, the way of obtaining the transfer operator of a system (5.6) requires exactly the same computations on W_n as those appearing in Kolmogorov decomposition of T (see also (3.17)), and the theorem follows.

5.5. REMARK. (a) The formulas (5.9) are – via Theorem 5.4 – another way of looking at the Schur algorithm for T contained in Theorem 5.2.

(b) For time-invariant systems (i.e., when T is a Toeplitz operator) such kind of state-representation of T was obtained in [29] using the Naimark dilation of T.

(c) Several other notions and facts from system theory have interpretations in the setting of Theorem 5.4.

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