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ISSN 0250 3638

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by

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PREPRINT SERIES IN MATHEMATICS
No.63/1986

BUCURESTI

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November 1986

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STABLE RANK FOR CERTAIN TYPE I C^*-ALGEBRAS

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Victor NISTOR

In [11] M.A.Riefell introduced the notion of topological stable rank of a C^*-algebra A as the least integer n such that any n-tuple $(x_1,\ldots$ $\ldots,x_n)\in A^n$ can be approximated arbitrarily close by an n-tuple of elements of A wich generate A as a left ideal (if no such integer exists we take the topological stable rank of A to be ∞). One of the reasons to study the topological stable rank is that it can be used to obtain cancellation theorems for projective modules as done in [12, 14, 15]. As shown in [3] the topological stable rank and the Bass stable rank coincide for C^*-algebras. We shall denote their common value for a C^*-algebra A by sr(A) (the stable rank of A).

It is known [1] that for a separable type I C^*-algebra A there exists a composition series with continuous trace subquotients. We shall find the value of the stable rank of A for a separable C^*-algebra with a finite such composition series with locally trivial quotients (theorem 7). This result generalyses results from [11, 14]. We also improve a theorem of [11] concerning the value of sr(A) in terms of sr(I) and of sr(A/I) for I a certain continuous trace ideal and show that $sr(A\otimes B)\leq sr(A)+sr(B)$ for certain separable C^* algebras of type I.

I want to express my gratitude to my adviser, professor Dan Voiculescu for his constant support.

The following facts can be found in [1]. Let I be a C^*-algebra. We shall denote by $\hat{I}$ the spectrum of I and by m(I) the linear span of the set of those $x\in I_+$ such that the function $\pi\to tr\pi(x)$ is continuous on $\hat{I}$ (3.1.5, 4.5.2). One says that I is of continuous trace if m(I) (wich is an ideal) is dense in I(4.5.3). In this case $\hat{I}$ is separated and I is isomorphic to a C^*-algebra corresponding to a continuous field $A=((I_t)_{t\in\hat{I}},\Gamma)$ of elementary C^*-algebras on I. Moreover a satisfies Fell's condition (10.5.4, 10.5.7, 10.5.8).

Let M(I) be the algbra of multipliers of I ([9]). If $b\in M(I)$ then it can be identified with a certain function $t\to b(t)\in M(I_t)$ on $\hat{I}$.

We recall that for a topological space T the covering dimension, dim (T) is the least integer n, such that each open cover of T has an open rafinement such that each point is contained in at most n+1 sets. If no such integer exists $\dim(T)=\infty$, If T is a compact metric space then all definitions of dimension are equivalent (see [6]).

We shall suppose that T, the spectrum of I, has finite covering dimension.

Let $t\to a(t)\in(I_t)_+$, $t\to b(t)\in M(I_t)_+$ be two positive elements of I and of M(I), respectively. We shall suppose that b(t) is not of finite rank for any $t\in T$.

We shall denote by χ_A the characteristic function of the set A.

<u>LEMMA 1.</u> Under the above hypothesis there exists a function $t\to v(t)\in I_t$ defined on T, for T compact, wich gives an element of I satisfying:

(1.1) $\chi_{[1/2,\infty)}(a(t))\leq v^*(t)v(t)\leq\chi_{(0,\infty)}(a(t))=s(a(t))$

(1.2) $v(t)v(t)^*\leq\chi_{(0,\infty)}(b(t))=s(b(t))$

for any $t\in T$.

<u>Proof.</u> The assumptions and lemma 10.7.11 of [1] give for I and $T=\hat{I}$:

(i) A finite open cover $(T_1,\dots,T_n)$ of T, with T_j closed.

(ii) For any $j\in\{1,\dots,n\}$ a continuous field $((\mathcal{H}_j(t))_{t\in T_j},\Gamma_j)$ of Hilbert spaces and isomorphisms h_j from A/T_j onto $A(\mathcal{H}_j)$ - the CCR-C^*-algebra induced by $\mathcal{H}_j$ ([1], 10.7.2)

(iii) For any $i,j\in\{1,\dots,n\}$ an isomorphism $g_{ij}(t):\mathcal{H}_j(t)\to\mathcal{H}_i(t)$ for $t\in T_{ij}=T_i\cap T_j$ wich induces $h_i^{-1}h_j$ from $A(\mathcal{H}_j/T_{ij})$ onto $A(\mathcal{H}_i/T_{ij})$.

(iv) For any $j\in\{1,\dots,n\}$ two numbers $0<a_j<b_j<1/2$ such that $(a_j,b_j)\cap\sigma(a(t))=\emptyset$ on T_j.

Denote by $c_j=(a_j+b_j)/2$ and by $p_j(t)=h_j(\chi_{[c_j,\infty)}(a(t)))$ wich belongs to $A(\mathcal{H}_j)$ due to (IV).

We shall solve the following technical problem:

Problem (P). To construct for any $j\in\{1,\dots,n\}$ a continuous function $t\to u_j(t)\in K(\mathcal{H}_j(t))$ wich gives a partial isometry in $A(\mathcal{H}_j)$ with the proprieties:

(a) $u_j^*(t)u_j(t)=p_j(t)$ on T_j

(b) $u_i(t)^*g_{ij}(t)u_j(t)=0$ on T_{ij} for $i\neq j$

(c) $u_j(t)u_j^*(t)\leq s(b(t))$ on T_j.

Let us observe that if we can solve problem (P) then we can solve the corresponding problem with p_j replaced in (a) by q_j such that $0\leq q_j\leq p_j$, for $u_j'=u_jq_j$ will satisfy (a), (b), (c) in this new form. We may suppose then that p_j defines a trivial vector bundle of rank r_j on T_j. Then problem (P) is equivalent to:

Problem (P_1). To construct continuous sections $\xi_i^j\in\Gamma_j$ for $j\in\{1,\dots,n\}$, $i\in\{1,\dots,r_j\}$ such that

(a') $(\xi_i^j(t),\ g_{jk}(t)\xi_e^k(t))=\delta_{ie}\delta_{jk}$ on T_{jk}

(b') $\xi_i^j(t)\in\overline{b(t)\mathcal{H}_j(t)}$ on T_j.

We shall solve now problem (P_1).

Let us suppose that we have defined the sections ξ_i^k for $k<j$ and $1\leq i\leq r_k$ and that we have extended the sections $\zeta_i^k=g_{jk}\xi_i^k$, $k<m$, $1\leq i\leq r_k$ from T_{jk} to all of T_j such that $(\xi_i^k(t),\xi_e^{m'}(t))=\delta_{km'}\delta_{ie}$ for $t\in T$. Let $p(t)$ be the orthogonal projection onto the linear span of the vectors $\xi_i^k(t)$ for $k<m$, $1\leq i\leq r_k$. Then $(1-p(t))b(t)\mathcal{H}_j(t)$ defines a continuous field of Hilbert spaces on T_j of infinite dimension in each point. The proof of 10.8.7 of [1] shows, using Michael's theorem [4], that we can extend $\xi_1^m,\dots,\xi_{r_m}^m$ to T_j, or, if $m=j$, that we can find sections $\xi_1^j,\dots,\xi_{r_j}^j$ with the desired properties. Problem (P_1) is thus solved.

To obtain the function v we shall choose a partition of unity $(\varphi_j)_{j=1,n}$ subordinated to the cover $(T_1,\dots,T_n)$. Then $\varphi_j^{1/2}h_j^{-1}(u_j)$ are well defined elements of I and their sum v satisfies our requirements.

Let $0\to I\to A\to B\to 0$ be a short exact sequence of C^*-algebras, I as above.

To any point $t\in T=\hat{I}$ corresponds an ideal $\mathcal{J}_t\subset B$ in the following way: the representation t has a unique (up to equivalence) extension to a representation of A on $\mathcal{H}_t$ (the Hilbert space of t). The Kernel of the induced map $B\to\mathcal{L}(\mathcal{H}_t)/\mathcal{K}(\mathcal{H}_t)$ will be denoted by $\mathcal{J}_t$ (remember that $t(I)\subset\mathcal{K}(\mathcal{H}_t)$ because any C^*-algebra of continuous trace is a CCR-C^*-algebra). See also [10], definition 1.7.

LEMMA 2. Let I a closed two - sided ideal of A with continuous trace. We shall suppose that $\dim(\mathcal{H}_t)=\infty$ for any $t\in T=\hat{I}$. Then $sr(A)\leq\max\{sr(A/I),2\}$.

<u>Proof</u>. Let us suppose that $s=\max\{sr(A/I),2\}<\infty$, otherwise the lemma is obvious. Also we may suppose that A has unit.

Let $x_1,\dots,x_s\in A$, π the quotient map $A\to A/I$, $\varepsilon>0$. We may suppose that, after a small perturbation, $\pi(x_1),\dots,\pi(x_s)$ generate A/I as a left ideal We want to show that there exist $x_1',\dots,x_s'$ wich generate A as a left ideal and such that $\|x_j-x_j'\|<\varepsilon$ for any $j\in\{1,\dots,s\}$. This will show that $sr(A)\leq s$.

Let $x=\begin{bmatrix} x_1 \\ \vdots \\ x_s \end{bmatrix} \in M_{s,1}(A)$, $y=x^*x=x_1^*x_1+\dots+x_s^*x_s$.

By the assumption there exists $\eta>0$ such that $\pi(y)\geq\eta$. Let $f:[0,\infty)\to[0,1]$ be a continuous function $f(t)=1$ for $t\in[0,\eta/2]$, supp $f\subset[0,\eta]$, $z=f(y)\in I$. The set of points $t\in T$ on wich $\|z(t)\|<\eta/4$ has a compact complement K_1 in T. Let K be a compact neighborhood of K_1, φ a continuous function with values in $[0,1]$, $\varphi=1$ on K_1, $\varphi=0$ off K.

Let $\delta,\mu>0$ to be specified later and let $g:[0,\infty)\to[0,1]$ be a continuous function vanishing off $[0,\delta]$ such that $g(0)=1$. We want to apply lemma 1 for $M_s(I)|_K$, $a=z|_K$, $b=g(xx^*)|_K$ to obtain a v such that

(2.1) $\quad \chi_{[1/2,1]}(z(t))\leq v^*(t)v(t)$

and if h is a continuous function on $[0,\infty)$ with values in $[0,1]$ such that $[0,\delta]\subset h^{-1}(\{1\})$, $[2\delta,\infty)\subset h^{-1}(\{0\})$ then

(2.2) $\quad h(xx^*)(t)v(t)=v(t)$

(we have denoted by $z(t)$ $(h(xx^*)(t))$ the image of $z(h(xx^*))$ in $I_t(M(I_t)))$. All we have to check is that $b(t)$ is nowhere of finite rank. Let us suppose that $b(t)$ is of finite rank for some $t\in K$. Let B denote A/I, T the operator $X(t)$ and $[\mathcal{K}]$ the orthogonal projection onto the closure of the space $\mathcal{K}$. If $b(t)$ is of finite rank then $b(t)\geq 1-[\mathrm{Ran}\ T]$ and ker T is finite dimensional from the assumption that $\pi(y)\geq\eta>0$. This means that T is invertible in the Calkin algebra. Since we have an injection $B/J_t\to\mathcal{L}(\mathcal{H}_t)/\mathcal{K}(\mathcal{H}_t)$ by the very definition of J_t, we obtain that the image of $x(t)$ in $M_{s,1}(B/J_t)$ is invertible. Since $s\geq2$ this means that $M_s(B/J_t)$ contain two isometries with orthogonal ranges. $\mathcal{H}_t$ is infinite dimensional and B has unit, hence $J_t\neq B$. Proposition 6.5 of [11] shows that $sr(M_s(B/J_t))=\infty$ and hence ([11] theorems 6.1 and 4.3) $sr(B)=\infty$, contradicting our assumption.

Denote by $u=\varphi v\in I$, $x'=x+\mu u=\begin{bmatrix} x'_1 \\ \vdots \\ x'_s \end{bmatrix}$.

$$(2.3)\quad (x'^*x')(t)=(x^*x)(t)+\gamma(u^*x+x^*u+\gamma u^*u)$$
$$(\eta/2-z(t))-2\gamma(\|x\|+1)\geq\eta/4-2(\|x\|+1)\gamma.$$

For $t\in K_1$ $\varphi(t)=1$ and hence, by functional calculus

$$(2.4)\quad (x'^*x')(t)=(x^*x)(t)+\gamma^2u^*u+\gamma(u^*x+x^*u)\geq$$
$$\geq (x^*x)(t)+\gamma^2\chi_{[0,\eta/2]}(x^*x)-2\gamma\|v^*x\| \geq$$
$$\geq\gamma^2-2\gamma\|v^*x\| \qquad \text{for } \gamma^2\leq\eta/2$$

By (2.2) we have

$$(2.5)\quad \|v^*x\|=\|v^*h(xx^*)x\|\leq\|h(xx^*)x\|\leq 2\delta$$

Let us choose γ and δ such that $0<\gamma<\varepsilon$, $2\gamma(\|x\|+1)<\eta/8$, $4\delta<\gamma$ and such that $\|x'-x\|<2\gamma$ implies that $\pi(x_1'),\dots,\pi(x_s')$ still generate A as a left ideal. Then (2.3), (2.4) and (2.5) show that there exists $\lambda>0$ such that $(x'^*x')(t)\geq\lambda$ for $t\in T$. Let φ be a pure state, π_φ the GNS representation associated with φ. If $\pi_\varphi\in\hat{I}$ then $\varphi(x'^*x')\geq\lambda>0$, if $\pi_\varphi\in(A/I)\hat{}$ then $\varphi(x'^*x')=$ $=\varphi'(\pi(x'^*x'))>0$ since $x_1',\dots,x_s'$ generate A/I as a left ideal (φ' is the induced state on A/I). We may conclude then that there exists $\lambda'>0$ such that $x_1'^*x_1'+\dots$ $\dots+x_s'^*x_s'\geq\lambda'$ and hence that $x_1',\dots,x_s'$ generate A as a left ideal.

The following lemma is an unpublished result of G.Nagy.

LEMMA 3. Let $0\to I\to A\to B\to 0$ be an exact sequence of C^*-algebras, such that sr(I)=sr(B)=1. Then sr(A)=1 if and only if the index morphism $\delta: K_1(B)\to K_0(I)$ is zero.

<u>Proof</u>. Suppose first that sr(A)=1. Choose u a unitary in $M_n(\tilde{B})$ and $v\in M_n(\tilde{A})$ a lifting of u. Choose $w\in M_n(\tilde{A})$ an invertible element close enough to v such that $\pi(w)$ represents the same class as u does in $K_1(B)$. Obviously $\delta([\pi(w)])=0$.

Conversely, we know that sr(A)=1 if and only if $sr(A\otimes\mathcal{K})=1$ ([11], theorem 3.6). Let $u\in\widetilde{\mathbb{K}\otimes A}$, $\varepsilon>0$. There exists an invertible element $v\in\widetilde{\mathbb{K}\otimes B}$ such that $\|\pi(u)-v\|<\varepsilon$. Since $\delta([v])=0$ there exists an invertible element $w\in\widetilde{\mathbb{K}\otimes A}$ such that $\pi(w)=v$. Let $w_o\in\widetilde{\mathbb{K}\otimes A}$ be such that $\pi(w_o)=v=\pi(w)$ and $\|u-w_o\|<\varepsilon$, then $w^{-1}w_o\in 1+\mathbb{K}\otimes I$. Choose an invertible element $x\in 1+\mathbb{K}\otimes I$ such that $\|x-w^{-1}w_o\|<\|w\|^{-1}(\varepsilon-\|u-w_o\|)$ then wx is invertible and

$$\|wx-u\|\leq\|wx-w_o\|+\|w_o-u\|<\|w\|\,\|w\|^{-1}(\varepsilon-\|w_o-u\|)+\|w_o-u\|=\varepsilon$$

We shall study next the opposite case, namely for I a twosided ideal of continuous trace such that the associated field of elementary C^*-algebras $J=((I_t)_{t\in T},\Gamma)$ $(T=\hat{I}$ - the spectrum of I, $I_t=I/\mathrm{Ker}\ t)$, be locally trivial with I_t a finite dimensional simple C^*-algebra. Let $T_n\subset T$ be the set of those $t\in T$ such that $I_t=M_n(C)$. By the assumption of locally triviality each T_n is open. Since $T=\bigcup_{n=1} T_n$ T_n is also closed. Let T_n correspond to the ideal $I_n\subset I$, $T_n=\hat{I}_n$, then I is the c_o-direct sum of the C^*-algebras I_n.

We notice that for a separable C^*-algebra I of continuous trace the spectrum $T=\hat{I}$ (wich is a locally compact Hausdorff space [1]) is a separable σ - compact metric space.

We shall use the following technical result due to A.J.-L.Shen ([14], proposition 3.15):

LEMMA 4. Let $\{J_\lambda\}_{\lambda\in\Lambda}$ be a net of closed ideal (ordered by inclusion) of a unital C^*-algebra A with J = the closure of the union of J_λ's. If K_λ are closed ideals of A such that $J_\lambda . K_\lambda=0$ for all $\lambda\in\Lambda$ then $sr(A)=$ $=\max\{sr(A/J),\ sr(A/K_\lambda)\,|\,\lambda\in\Lambda\}$.

LEMMA 5. a) Let A be a C^*-algebra, $I\subset A$ a closed two-sided ideal as above then

(5.1) $\quad sr(A)=\max\{sr(I),\ sr(A/I)\}$

b) If I is separable then

(5.2) $sr(I)=\sup_{n\geq 1}\{\{(\dim(T_n)-1)/2n\}'+1\}$

(Here $\{x\}'$ denotes the least integer m, $m\geq x$).

Proof. a) Let $\Lambda=\{U\subset T \mid U$ open and relatively compact in $T\}$, J_U the ideal of A corresponding to U, K_U the ideal of A corresponding to $\hat{A}\setminus U$.

We want to show that A/K_U identifies naturaly with a quotient of I. This will follow if we show that $K_U + I = A$ or equivalently that $\hat{K}_U \cup \hat{I}=(\hat{A}\setminus \bar{U})\cup T=\hat{A}$.

We have to prove that $\bar{U}\subset T$.

In the following exact sequence

$$0\to (I+K_U)/K_U \to A/K_U \to A/(I+K_U)\to 0$$

A/K_U has the spectrum U and $(I+K_U)/K_U$ has the spectrum $\bar{U}\cap T$. Using the compactness of $\bar{U}\cap T$ and the local triviality of J we obtain that $(I+K_U)/K_U$ has a unit. This shows that $\bar{U}\cap T$ is closed in $\bar{U}$ and hence closed. Since $U\subset \bar{U}\cap T$ it follows that $\bar{U}\subset \overline{\bar{U}\cap T}=\bar{U}\cap T$ and hence $\bar{U}\subset T$. We obtained isomorphisms $A/K_U\cong (I+K_U)/K_U \cong I/I\cap K_U$. Theorem 4.3 of [11] shows that $sr(A/K_U)\leq sr(I)$.

We shall use lemma 4: $I=\bigcup_{\lambda\in\Lambda} J_\lambda$ and hence $sr(A)=\max\{sr(A/I), sr(A/K_U) \mid U\in\Lambda\}\leq \max\{sr(A/I), sr(I)\}$.

b) Suppose first that $I=I_n$ and T_n is compact. Cover T_n by a finite number of open sets $V_1,\ldots,V_m$ such that $T|\bar{V}_k$ is trivial for any $k\in\{1,\ldots,m\}$. If J_k is the ideal of I corresponding to $T\setminus V_k$ the last statement is equivalent to the fact that $I/J_k=M_n(C(V_k))$. By [8], corollary 2.7

$$sr(I)=sr(I/J_1\cap\ldots\cap J_m)=\max\{sr(I/J_k) \mid 1\leq k\leq m\}$$

By [11] Theorem 6.1 $sr(I/J_k)=\{(\dim(V_k)-1)/2n\}'+1$. By the sum theorem [6] $\dim(T)=\max\{\dim(\bar{V}_k)) \mid 1\leq k\leq m\}$ and hence $sr(I)=\{(\dim(T)-1)/2n\}'+1$.

The general statement can be obtained as follows: $sr(I)=\max\{sr(I_n) \mid n\in N\}$ ([11], Theorem 5.2). All we have to prove is that $sr(I_n)=\{(\dim(T_n)-1)/2n\}'+1$. We shall use Lemma 4 in the following setting: let $L_1,L_2,\ldots,L_m,\ldots$ be compact subsets of T_n such that $T_n=\bigcup_{m=1} L_m$, $L_m \subset \mathring{L}_{m+1}$, $\Lambda = N$ J_m the ideal corresponding to L_m in I_n, K_m the ideal corresponding to $(T_n \cup \{\infty\}) \setminus L_m$ in I_n ($\tilde{I}_n$ denotes the algebra $I_n + C1$) $\overline{\bigcup_{m\geq 1} J_m} = I_n$ and $\tilde{I}_n / I_n \cong C$ hence

$$sr(I_n)=sr(\tilde{I}_n)=\max_{m\geq 1}\{ sr(\tilde{I}_n/K_U)\}$$

$$=\sup_{m\geq 1}\{\{(\dim(L_m)-1)/2n\}'+1\}=\{(\dim(T_n)-1)/2n\}'+1$$

since $\dim(T_n)=\sup_{m\geq 1}\dim(L_m)$ by the sum theorem [6].

DEFINITION 6. Let A be a separable C^*-algebra with a composition series $\{o\}= I_o \subset I_1 \subset \ldots \subset I_{n+1}=A$ such that each of the subquotients I_{k+1}/I_k for $0\leq k\leq n$ is of continuous trace and it satisfies either:

1° I_{k+1}/I_k has only finite dimensional irreducible representations and the corresponding field of elementary C^*-algebras is locally trivial; or

2° I_{k+1}/I_k has only infinite dimensional irreducible representations and the spectrum $(I_{k+1}/I_k)^\wedge$ has finite dimension. Then we say that A satisfies condition $\mathcal{A}$.

THEOREM 7. a) Let I be a separable C^*-algebra of continuous trace such that the corresponding field of elementary C^*-aleqebras is locally trivial, $I=c_o\text{-}\bigoplus_{k\in N\cup\{\infty\}} I_k$ with I_k homogeneous of degree k. Then

$$sr(I)=\sup\{s_\infty\}\cup\{\{(\dim(I_k)-1)/2k\}'+1 \mid k\in N\}$$

Here $s_\infty=1$ if $\dim(\hat{I}_\infty)\leq 1$ and $s_\infty=2$ else.

b) Let A satisfy condition $\mathcal{A}$.

If $sr(I_{k+1}/I_k)=1$ for $0\leq k\leq n$ and at least one of the index homomorphisms $\delta: K_1(I_{k+1}/I_k) \to K_o(I_k)$ for $1\leq k\leq n$ is not zero then $sr(A)=2$, else

$$sr(A)=\max_{0\leq k\leq n}\{ sr(I_{k+1}/I_k)\}$$

Proof. a) follows from lemma 5 b) (for I_∞ we use an identical device and [11] theorem 3.6).

b) follows by induction on n using lemma 2, lemma 3 and lemma 5.

THEOREM 8. Let A and B satisfy condition $\mathcal{A}$ then

$$sr(A \otimes B) \leq sr(A) + sr(B)$$

Proof. Let $\{0\} = I_o \subset I_1 \ldots I_{n+1} = A$ and $0 = J_o \subset J_1 \ldots J_{m+1} = B$ be composition series as in definition 4 then $A \otimes B$ has a composition series with quotients isomorphic to $(I_{k+1}/I_k) \otimes (J_{e+1}/J_e)$.

If $sr(A \otimes B) \in \{1,2\}$ then (5.2) is obvious. Let $sr(A \otimes B) \geq 3$ then $sr(A \otimes B) = sr((I_{k+1}/I_k) \otimes (J_{e+1}/J_e))$ for some k and e. It is obvious that I_{k+1}/I_k and J_{e+1}/J_e satisfy 1^o of definition 4. Let $I_{k+1}/I_k = c_o$-direct sum of the ideals L_j, $j \in N$, $J_{e+1}/J_e = c_o$-direct sum of the ideals K_i, $i \in N$ with L_j and K_j homogeneous of degree j. Since

$$sr(L_j \otimes K_i) = \{(\dim(\hat{L}_i \times \hat{K}_i) - 1)/2ij\}' + 1 \leq$$
$$\leq (\{(\dim(\hat{L}_j) - 1)/2j\}' + 1) + (\{(\dim(\hat{K}_i) - 1)/2i\}' + 1) =$$
$$= sr(L_j) + sr(K_i)$$

we obtain

$$sr(A \otimes B) = sr((I_{k+1}/I_k) \otimes (J_{e+1}/J_e)) = \sup_{i,j} sr(K_i \otimes L_i) =$$
$$= \sup_{i,j}(sr(K_i) + sr(L_j)) = \sup_i sr(K_i) + \sup_j sr(L_j) \leq$$
$$\leq sr(I_{k+1}/I_k) + sr(J_{e+1}/J_e) \leq sr(A) + sr(B).$$

REMARK. Theorem 8 answers question 7.3 of [11] in a particular case.

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