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by András NÉMETHI

Let $g(x)$ and $h(y)$ be holomorphic functions defined on neighbourhoods of the origins of $\mathbb{C}^m$ and $\mathbb{C}^n$ with $g(0)=h(0)=0$ and with isolated singularities at the origins. We define the holomorphic function f on a neighbourhood of the origin of $\mathbb{C}^m \times \mathbb{C}^n$ by $f(x,y)=g(x)+h(y)$. It is immediate, that f has isolated singularity at the origin with Milnor number $\mu(f)=\mu(g)\cdot\mu(h)$.

Denote by $F(g)$, $F(h)$ and $F(f)$ the corresponding Milnor fibers; m_g, m_h and m_f the characteristic maps of the Milnor fiberings; $\Gamma(g)$, $\Gamma(h)$ and $\Gamma(f)$ the Seifert matrices of Milnor fiberings; $K(g)$, $K(h)$ and $K(f)$ the links defined by g, h and f. By [5] and [6] there is a homotopy equivalence $a: F(g) * F(h) \longrightarrow F(f)$ such that the two maps $a\circ(m_g * m_h)$ and $m_f \circ a$ are homotopic to each other, and $\Gamma(f) \equiv (-1)^{mn}\, \Gamma(g) \otimes \Gamma(h)$. For other proprieties of the fiber $F(f)$ and characteristic map m_f see [4], [5], [6], [7].

The purpose of this paper is the study of the link $K(f)$. Our result is the following

Theorem :

1. The subspace $K(g,h)=\{(x,y)\in K(f) : g(x)=h(y)=0\}$ is homeomorphic to the join $K(g) * K(h)$.

2. The space $K(f)-K(g,h)$ is a smooth fiber bundle over S^1, with projection mapping $\theta(x,y)=g(x)/|g(x)|$. The fiber is diffeomorphic to $F(g)\times F(h)$ and the characteristic map (via this diffeomorphism) is $m_g \times m_h$.

Remarks :

i) The closure of each fiber F_α $(e^{i\alpha} \in S^1)$ is the union $F_\alpha \cup K(g,h)$. This space is not a manifold with boundary, hence our theorem does not determine a spinnable structure (open book decomposition) on $K(f)$. (For the definition and some proprieties of the spinnable structures see [2], [8].)

ii) In particular, when $f(x,y)=g(x)+y^k$, we obtain the special case studied by A.Durfee and L.Kauffman [1] and J. Stevens [8].

Corollary :

$K(g+y^k)-K(g)$ is a smooth fiber bundle over S^1 with fiber diffeomorphic to $F(g)$ and characteristic map m_g^k. In other words : $K(g+y^k)$ is the k-fold cyclic cover of S^{2m-1} branched over $K(g)$.

Proof of Theorem

By [5, § 3] there is a homeomorphism ψ from $S_\varepsilon^{2m-1} \times S_\varepsilon^{2n-1}$ onto $S_{2\varepsilon}^{2m+2n-1}$ with $(r\cdot f)\circ\psi = g*h$, where r is a positive real-valued continuous function on $S_{2\varepsilon}^{2m+2n-1}$.

We recall the construction of the map ψ :

There exists a continuous map $[0,1]\times(D_{2\varepsilon}^{2m}-\{0\}) \longrightarrow D_{2\varepsilon}^{2m}$, $(r,x) \longmapsto r\circ x$ such that

i) $1\circ x = x$, $0\circ x = 0$, $(r\cdot s)\circ x = r\circ(s\circ x)$

ii) $g(r\circ x) = r\cdot g(x)$

iii) $|r\circ x|$ is a strictly increasing function of r

iv) $$[0,1]\times S_\rho^{2m-1} \Big/ \{0\}\times S_\rho^{2m-1} \longrightarrow D_\rho^{2m} \quad , \quad [r,x] \longmapsto r\circ x$$

is homeomorphism $(0 < \rho \leq 2\varepsilon)$

Similar, we consider a continuous map $[0,1]\times(D_{2\varepsilon}^{2n}-\{0\}) \longrightarrow D_{2\varepsilon}^{2n}$ $(r,y) \longmapsto r\circ y$ with similar proprieties, in particular $h(r\circ y) = = r\cdot h(y)$.

Then $\varphi:(0,1]\times S_{2\varepsilon}^{2m+2n-1} \longrightarrow D_{2\varepsilon}^{2m+2n}-\{0\}$ defined by $\varphi(r,x,y)=(r\circ x, r\circ y)$ is homeomorphism.

Let $\sigma:(D_{\varepsilon}^{2m}-\{0\})*(D_{\varepsilon}^{2n}-\{0\}) \longrightarrow D_{2\varepsilon}^{2m+2n}-\{0\}$ defined by $\sigma([x,s,y])=(s\circ x,(1-s)\circ y)$. Then by definition $\psi=p_2\circ\varphi^{-1}\circ\sigma \mid S_{\varepsilon}^{2m-1}*S_{\varepsilon}^{2n-1}$, where p_2 is the second projection.

By an attentive analysis of the Sakamoto's proof we obtain that the map ψ is a diffeomorphism from $S=\{[x,s,y] : s\neq 0 , s\neq 1\}$ onto $\psi(S)$, where S is considered with its natural differentiable structure.

Let $K=(g*h)^{-1}(0)=\{[x,s,y]\in S_{\varepsilon}^{2m-1}*S_{\varepsilon}^{2n-1} : s\cdot g(x)+(1-s)\cdot g(h)=0\}$. It is easy to prove that $\{[x,s,y] : s\cdot g(x)=(1-s)\cdot h(y)=0\}=K(g)*K(h)$ and $K-K(g)*K(h)\subset S$. Then ψ induces a homeomorphism from K onto $K(f)$, a homeomorphism from $K(g)*K(h)$ onto $K(g,h)$ and a diffeomorphism from $K-K(g)*K(h)$ onto $K(f)-K(g,h)$.

We define the smooth maps $\theta': K-K(g)*K(h) \longrightarrow S^1$ by $\theta'([x,s,y])=g(x)/|g(x)|$, $\theta_g:S_{\varepsilon}^{2m-1}-K(g) \longrightarrow S^1$ by $\theta_g(x)=g(x)/|g(x)|$ and $\theta_h:S_{\varepsilon}^{2n-1}-K(h) \longrightarrow S^1$ by $\theta_h(y)=-h(y)/|h(y)|$.

Then the map $u_\alpha:\theta_g^{-1}(e^{i\alpha})\times\theta_h^{-1}(e^{i\alpha}) \longrightarrow (\theta')^{-1}(e^{i\alpha})$ defined by $u_\alpha(x,y)=[x,h(y)/(h(y)-g(x)),y]$ is a diffeomorphism (for all $\alpha\in[0,2\pi]$). Therefore θ is a smooth fiber bundle over S^1 isomorphic to the smooth fiber bundle $(\theta_g\times\theta_h)^{-1}(\Delta) \longrightarrow \Delta=S^1$, where Δ is the diagonal of $S^1\times S^1$. The verification of the relation $\theta\circ\psi=\theta'$ finishes the proof.

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