INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

THE TOPOLOGY OF SOME AFFINE HYPERSURFACES

by

András NÉMETHI and Alexandru ZAHARIA

PREPRINT SERIES IN
MATHEMATICS
No.65/1986

BUCURESTI

THE TOPOLOGY OF SOME AFFINE HYPERSURFACES

by

András NÉMETHI*) and Alexandru ZAHARIA*)

November 1986

*) The National Institute for Scientific and Technical Creation,

Department of Mathematics, Bd. Păcii nr.220,

79622 Bucharest, ROMANIA

THE TOPOLOGY OF SOME AFFINE HYPERSURFACES

by

András NÉMETHI and Alexandru ZAHARIA

§1. Introduction. Statement of results.

Let $f: C^n \longrightarrow C$ be a polynomial map. Let us define the bifurcation points B_f of f as those points of C around which f is not a C^∞ locally trivial fibration. It is well known that the set of bifurcation points of f is a finite subset of C ([1],[3],[8],[13]) consisting of the set of all critical values of f(denoted by Σ_f) and perhaps some other points. A fiber $f^{-1}(c)$ is generic if $c \notin B_f$ and special if $c \in B_f$. Let $\phi: S_R - f^{-1}(0) \longrightarrow S^1$, $\phi(z) = f(z)/|f(z)|$ be the Milnor map at infinity, where $S_R = \{z : \|z\| = R\}$ with R sufficiently large.

We consider the following questions:

(Q1) How can be determined a.) the bifurcation set B_f ?

b.) the topology of the generic fiber?

c.) the topology of the special fibers?

(Q2) When is the Milnor map at infinity the projection map of a smooth fiber bundle? (in analogy to the local Milnor fibration [7])

(Q3) What is the relation between

a.) the generic fiber of f and the fiber of this fiber bundle.

b.) the monodromy of f around the bifurcation points and the monodromy of this fiber bundle.

The purpose of this paper is to answer these questions in the case of some special classes of polynomials.

We recall some definitions and notations:

Let $f = \sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be a polynomial (where $\nu = (\nu_1, \dots, \nu_n)$ and $z^\nu = z_1^{\nu_1} \dots z_n^{\nu_n}$ as usual) and we take the polyhedron $\tilde{\Gamma}_-(f) = \{$the convex closure of $\{0\} \cup \operatorname{supp} f\} \subset R^n_+$ where $\operatorname{supp} f = \{\nu \in N^n : a_\nu \neq 0\}$. The Newton boundary $\tilde{\Gamma}(f)$ of the polynomial f at infinity is by definition the union of the closed faces of the polyhedron $\tilde{\Gamma}_-(f)$ which do not contain the origin. We say that f is nondegenerate on the face Δ if the equations $\frac{\partial f_\Delta}{\partial z_1} = \dots = \frac{\partial f_\Delta}{\partial z_n} = 0$ have no solution in $(C^*)^n$ (where $f_\Delta(z) = \sum_{\nu \in \Delta} a_\nu z^\nu$). The polynomial f is called convenient if the intersection of suppf with each coordinate axe is non-empty. We consider the following classes of polynomials:

$\mathcal{N}\mathcal{W} = \{ f \in C[z] \mid f(0) = 0$ and f has a nondegenerate Newton principal part at infinity, that is to say f is nondegenerate on every face Δ of $\tilde{\Gamma}(f)\}$.

$\mathcal{N}\mathcal{B} = \{ f \in C[z] \mid f \in \mathcal{N}\mathcal{W}$ and f convenient$\}$.

These classes of polynomials were introduced by A. G. Kouchnirenko in [4] and he also answer to (Q.1.b) in the case of $\mathcal{N}\mathcal{B}$ [4, Théorème V.].

On the other hand, S. A. Broughton in [1] has answered to (Q.1.) in the case of tame polynomials:

$\mathcal{T} = \{ f \in C[z] \mid$ there exist no sequence $\{z_k\}_k \in C^n$ with $\lim_{k\to\infty} \|z_k\| = \infty$ and $\lim_{k\to\infty} \partial f(z_k) = 0 \}$ (where $\partial f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$).

In [8] A. Némethi has extended the class of tame polynomials to the class of quasi-tame polynomials and answer to (Q1):

$Q\mathcal{T} = \{ f \in C[z] \mid$ there exist no (f, C^n)- sequence $\}$

where for any subset A of C^n and $f \in C[z]$ an (f,A) - sequence is a sequence $\{z_k\}_k \subset A$ such that $\lim_{k\to\infty} \|z_k\| = \infty$, $\lim_{k\to\infty} \partial f(z_k) = 0$ and $\{c_k\}_k = \{ f(z_k) - \langle z_k, \overline{\partial f(z_k)} \rangle \}_k$ is convergent.

Therefore, there are two possible approaches in the study of polynomial maps: the first is to use the Newton boundary, the second to consider behaviour of $\partial f(z)$ at infinity. In this paper we unify these two approaches and we extend the results about the local Milnor fibration of hypersurface singularities.

We introduce the following notations and classes of polynomials.

Let $f \in C[z]$ and denote by $\overline{\mathrm{supp} f}$ the convex closure of suppf. A face Δ of $\overline{\mathrm{supp} f}$ is called <u>bad</u> if there exists a hyperplane H with equation $a_1x_1 + \ldots + a_nx_n = 0$ ($x_1, \ldots, x_n$ are the coordinates in R^n_+) with:

a.) there exist i and j such that $a_i < 0$ and $a_j > 0$

b.) $H \cap \overline{\mathrm{supp} f} = \Delta$

c.) the affine subvariety of dimension = $\dim\Delta$ spaned by Δ contains the origin.

$$\mathcal{NN}_0 = \{ f \in C[z] \mid f \in \mathcal{NN},\ f \text{ has only isolated singularities on } C^n - f^{-1}(0) \text{ and } \overline{\mathrm{supp} f} \text{ without bad faces} \}$$

$$\mathcal{MT} = \{ f \in C[z] \mid \text{there exist no } (f, M(f)) - \text{sequence} \}$$

$$\mathcal{ST} = \{ f \in C[z] \mid \text{if } \{z_k\}_k \text{ is a } (f, M(f)) - \text{sequence, then } \lim_{k \to \infty} c_k = 0 \}$$

Here $M(f) = \{ z \in C^n \mid \text{there exist } \lambda \in C \text{ such that } \partial f(z) = \lambda \cdot \bar{z} \}$

If $f \in \mathcal{MT}$ (respectively $f \in \mathcal{ST}$) we say that f is M - tame, (semi-tame).

In trying to keep track of the different classes of polynomials it may be helpful to look at the following diagram of

<u>Inclusions:</u>

$$\begin{array}{ccccccc} \mathcal{NC} & \subset & \mathcal{T} & \subset & \mathcal{QT} & \subset & \mathcal{MT} \\ \cap & & & & & & \cap \\ \mathcal{NN}_0 & & & \hookrightarrow & & & \mathcal{ST} \\ \cap & & & & & & \\ \mathcal{NN} & & & & & & \end{array}$$

(the proof of the non-obvious ones will be given in the sequel)

By the following examples we prove that these classes are distinct (possibly excepting $\mathcal{QT} \subset \mathcal{MT}$).

<u>Examples</u>

1.) If $f = xy + x^2y^2 + x^2$, then $f \in \mathcal{NN}$, but $f \notin \mathcal{ST}$ and $f \notin \mathcal{NN}_0$.

2.) If $f = xy$, then $f \in \mathcal{T}$, $f \in \mathcal{NN}_0$, but f isn't convenient.

3.) If $f = (x + y^2)^2 + y^2$, then $f \in \mathcal{T}$, f is convenient, but $f \notin \mathcal{NN}$.

4.) If $f = x^2y + xy^2 + x^5y^3 + x^3y^5$ then $f \in \mathcal{QT}$ but $f \notin \mathcal{T}$

5.) If $f = xy + xy^4$, then $f \in \mathcal{NN}_0$ (hence $f \in \mathcal{ST}$) but $f \notin \mathcal{MT}$

We prove the following theorems:

Theorem 1.

(M) if $f \in \mathcal{MT}$, then

a.) $B_f = \Sigma_f = \{c_1, \ldots, c_r\}$ and $f^{-1}(c_i)$ has only isolated singularities (say at the points z_{ij} for $j = 1, \ldots, k_i$) for all $c_i \in \Sigma_f$.

b.) For any $c_i \in \Sigma_f$ there exists a closed disc D_{c_i} centered at c_i and a deformation retract $r : f^{-1}(D_{c_i}) \longrightarrow f^{-1}(c_i)$. In addition there exist closed balls B_{ij} such that $f : f^{-1}(D_{c_i}) - \bigcup_j B_{ij} \longrightarrow D_{c_i}$ is a C^∞ trivial fibration and $f : (f^{-1}(D_{c_i} - \{c_i\}), f^{-1}(D_{c_i} - \{c_i\}) \cap \bigcup_j B_{ij}) \longrightarrow D_{c_i} - \{c_i\}$ is a C^∞ locally trivial fibration of pairs of spaces. The fibers in the balls are exactly the local Milnor fibers of the isolated singularities $(f^{-1}(c_i), z_{ij})$ (with Milnor number $\mu(z_{ij})$).

c.) The generic fiber $f^{-1}(c)$ has the homotopy type of a bouquet of μ spheres of dimension (n-1). The special (singular) fiber $f^{-1}(c_i)$ ($c_i \in \Sigma_f$) has the homotopy type of a bouquet of $\mu - \sum_j \mu(z_{ij})$ spheres of dimension (n-1). The number μ can be calculated by

$$\mu = \sum_{i,j} \mu(z_{ij}) = \dim_C \mathbb{C}[z] / \langle \partial f(z) \rangle$$

(S). If $f \in \mathcal{ST}$, then

a.) $B_f \subset \Sigma_f \cup \{0\}$

b.) If $c \in B_f - \{0\}$, then $f^{-1}(c)$ has only isolated singularities with proprieties as in (M.b).

Theorem 2.

(S). If $f \in \mathcal{ST}$, and R is sufficiently large, then the Milnor map at infinity ϕ is the projection map of a smooth fiber bundle.

(M). If $f \in \mathcal{MT}$, then in addition we have the following results:

a.) S_R has a natural spinnable structure (open book decomposition). In particular the closure of each fiber $\phi^{-1}(\theta)$ in S_R is a smooth $(2n-2)$dimensional manifold with boundary with interior $\phi^{-1}(\theta)$ and boundary $K = f^{-1}(0) \cap S_R$.

b.) The space K is $(n-3)$-connected

c.) Each fiber has the homotopy type of a bouquet of $\tilde{\mu}$ spheres, where $\tilde{\mu}$ = degree of the mapping $z \longmapsto \frac{\partial f(z)}{\|\partial f(z)\|}$ from S_R to the unit sphere of C^n.

Theorem 3.

(M) If $f \in \mathcal{MT}$ and R is sufficiently large, then

a.) The generic fiber $f^{-1}(c)$ is diffeomorphic with the fiber $\phi^{-1}(\theta)$ of the fiber bundle. In particular $\mu = \tilde{\mu}$.

b.) The fiber bundles $\phi : S_R - f^{-1}(0) \longrightarrow S^1$ and $f : f^{-1}(S^1_r) \longrightarrow S^1_r$ $(S^1_r = \{c \in C \mid |c| = r\})$ with r sufficiently large are equivalent bundles.

(S). Let $f \in \mathcal{ST}$ and D be a disc centered at $0 \in C$. Then for R sufficiently large $\phi : S_R - f^{-1}(D) \longrightarrow S^1$ is a fiberbundle equivalent with the fiber bundle $f : f^{-1}(S^1_r) \longrightarrow S^1_r$ (with r sufficiently large). In particular $\phi^{-1}(\theta) - f^{-1}(D)$ is diffeomorphic with the generic fiber $f^{-1}(c)$.

Remark.

In general the generic fiber is not diffeomorphic with $\phi^{-1}(\theta)$ in the case of the semi-tame polynomials. For example, if we take $f = x^2y + x$, then $f \in \mathcal{ST}$, $B_f = \{0\}$, the generic fiber is diffeomorphic with C^* and $\phi^{-1}(\theta)$ is diffeomorphic with $C - \{\text{two points}\}$.

What can we say about the general case or about the polynomials with nondegenerate Newton principal part at infinity? The answer to our questions is far from being simple, but we can answer partially to (Q.1.a).

Let $S_f = \{c \in C \mid$ there exist a sequence $\{z_k\}_k \in M(f)$ such that $\lim\limits_{k\to\infty} \partial f(z_k) = 0$ and $\lim\limits_{k\to\infty} (f(z_k) - \langle z_k, \overline{\partial f(z_k)} \rangle) = c \}$

Theorem 4. Let $f \in C[z]$. Then $B_f \subset S_f$.

Remark.

In [8] is proved that $B_f \subset \Lambda_f$ where $\Lambda_f = \{c \in C \mid$ there exist a sequence $\{z_k\}_k \in C^n$ such that $\lim_{k\to\infty} \partial f(z_k) = 0$ and $\lim(f(z_k) - \langle z_k, \overline{\partial f(z_k)}\rangle) = c\}$

It is clear that $S_f \subset \Lambda_f$ and in general $S_f \neq \Lambda_f$. If we take $f = x^5z^2 + x^5y^3 + x^{11}y^3z^2 + x$, than $-54\sqrt{\frac{2}{15}} \in \Lambda_f - S_f$.

If $f \in \mathcal{NW}$, then S_f can be specified in the following way:

Let $B = \{$the set of bad faces of $\overline{\text{supp} f}\}$. If $\Delta \in B$, then we define the set:

$$\Sigma_\Delta = \{f_\Delta(z^0) \mid z_i^0 \neq 0 \text{ for all } i = 1,\dots,n \text{ and } \partial f_\Delta(z^0) = 0\}$$

It is clear that $\Sigma_\Delta \subset \Sigma_{f_\Delta}$, hence Σ_Δ is a finite set.

Theorem 5. Suppose that $f \in \mathcal{NW}$. Then

$$B_f \subset \Sigma_f \cup \{0\} \cup \bigcup_{\Delta \in B} \Sigma_\Delta$$

§2. Proofs.

An important tool in the proofs is the Curve Selection Lemma :

Let $f_1,\dots,f_k$, $g_1,\dots,g_l$, $h_1,\dots,h_p \in R[x_1,\dots,x_n]$ be polynomial maps with real coefficients. Let $U = \{x \in R^n : f_i(x) = 0, i = 1,\dots,k\}$ and $V = \{x \in R^n : g_i(x) > 0, i = 1,\dots,l\}$. Suppose that there exists a sequence $\{x_k\}_k \subset U \cap V$ such that $\lim_{k\to\infty} \|x_k\| = \infty$ and $\lim_{k\to\infty} h_j(x_k) = 0$ $(j=1,\dots,p)$. Then there exists a real analytic curve $p : (0,\varepsilon) \longrightarrow U \cap V$ with $\lim_{t\to 0} \|p(t)\| = \infty$ and $\lim_{t\to 0} h_j(p(t)) = 0$ $(j=1,\dots,p)$ and of the form $p(t) = at^\alpha + a_1 t^{\alpha+1} + \dots$ $(\alpha < 0)$.

Proof. We consider R^n in natural way in the projective space $P R^n$, and we use the Lemma 3.1 of [7] in an affine neighbourhood of an accumulation point x of the sequence $\{x_k\}_k$.

Lemma 1. Let $f \in \mathcal{ST}$ (respectively $f \in \mathcal{MT}$ or $f \in C[z]$) and we take $c \in C - \{0\}$. (respectively $c \in C$ or $c \in C - S_f$). Then for a small closed disc D_c the set $f^{-1}(D_c) \cap M(f)$ is bounded.

Proof. Let $f \in \mathcal{ST}$, $c \in C - \{0\}$ and we choose D_c such that $D_c \cap \{0\} = \emptyset$.

We assume that the assertion is not true. By Curve Selection Lemma we find $p(t) \in f^{-1}(D_c) \cap M(f)$ such that $f(p(t)) = b + b_1 t^l + \dots$ with $b \neq 0$. The contradiction is obtained exactly as in proof of Lemma2, the case $\beta = 0$.

If $f \in \mathcal{MT}$ or $f \in C[z]$ we act similarly.

Proof of Theorem 4.

Let $c \notin S_f$. We take a small closed disc D_c centered at c and R sufficiently large such that $D_c \cap S_f = \emptyset$, $f^{-1}(c')$ meets transversally S_R for all $c' \in D_c$ and $f^{-1}(D_c) \cap M(f) \cap \{z : \|z\| \geq R\} = \emptyset$. By Lemma 1 the vectors $\bar{z}$ and $\partial f(z)$ are C - linearly independent vectors for all $z \in f^{-1}(D_c) \cap \{z : \|z\| \geq R\}$ therefore there exists a smooth vector field v_1 such that $\langle v_1, z \rangle = 0$ and $\langle v_1, \overline{\partial f(z)} \rangle = 1$. This vector field can be extended by Ehresmann's Fibration Theorem [2] on a smooth vector field v on $f^{-1}(D_c)$ such that $\langle v, \overline{\partial f(z)} \rangle = 1$ for all $\{z : \|z\| < R\}$. If w is a vector field on D_c, then $\tilde{v}(z) = w(f(z)) \cdot v(z)$ is tangent to $S_{\|z\|}$ if $\|z\| \geq R$ and $df_z \tilde{v}(z) = w(f(z))$. The integral curves of the vector field $\tilde{v}$ may be used to construct a trivialization of f over D_c.

Proof of Theorem 1.

(M.a) and (S.a) is an immediate consequence of Theorem4.

The proof of (M.b,c) and (S.b) is based on Lemma 1 and is almost similar with the proofs of [1] or [5 , § 5]. The difference consist in the construction of the vector field in a neighbourhood of the infinity (and here we use Lemma 1).

Let $f \in \mathcal{MT}$, $c_i \in \Sigma_f$. We take sufficiently small closed balls B_{ij} centered at z_{ij}, a small disc D_{c_i} centered at c_i and sufficiently large R such that in B_{ij} we can apply the Milnor theory of isolated singularities, $f(c')$ meets transversally S_R and ∂B_{ij} $(j=1,\dots k_i)$ for all $c' \in D_{c_i}$ and $f^{-1}(D_{c_i}) \cap M(f) \cap \{z : \|z\| \geq R\} = \emptyset$. We construct the smooth vector fields on $f^{-1}(D_{c_i}) \cap \{z : \|z\| \geq R\}$ as in the proof of Theorem 4, on

$f^{-1}(D_{c_i}) \cap \{z : \|z\| \leq R$ and $z \notin B_{ij}\}$ by Ehresmann's Fibration Theorem, and on $f^{-1}(D_{c_i} - \{c_i\}) \cap B_{ij}$ by the local proprieties of isolated singularities. For the other details see [1, Theorem1, Theorem2].

The relation $\sum_{i,j} \mu(z_{ij}) = \dim_C \mathbb{C}[z]/\langle \partial f(z) \rangle$ is proved in , [4, Proposition 1.14].

Proof of Theorem 5.

Let $f \in \mathcal{W}$. Let $p(t) \in M(f)$ be an analytic curve with $\lim_{t\to 0} \partial f(p(t))=0$ and $c(t) = f(p(t)) - \langle p(t) , \overline{\partial f}(p(t))\rangle$ convergent. If $\lim_{t\to 0} p(t) = z^0 \in C^n$, then $\partial f(z^0) = 0$, hence $c(0) \in \sum_f$. Suppose that $\lim_{t\to 0} \|p(t)\| = \infty$. We may assume that $p(t) = (z_1^0 t^{a_1}+\ldots,\ldots,z_k^0 t^{a_k}+\ldots,0,\ldots,0)(z_1^0 \neq 0,\ldots,z_k^0 \neq 0)$ and $a_1 < 0$. Since $p(t) \in M(f)$ we have $\frac{\partial f}{\partial z_i}(p(t)) = \lambda(t) \cdot \overline{p_i}(t)$ and $\lambda(t) \equiv 0$ iff $f(p(t)) = c(t) = \text{constant} \in \sum_f$. If $\lambda(t) \not\equiv 0$ then $\lambda(t) = \lambda_0 t^{o(\lambda)} +$ + higher ($\lambda_0 \neq 0$) and $\operatorname{suppf} \cap R^k \neq \emptyset$ ($R^k_k = \{x \in R^n \mid x_{k+1}=\ldots=x_n=0\}$). Consider the continuous function $l_a(x) = \sum_{j=1}^{k} a_j x_j$ on $\overline{\operatorname{suppf}} \cap R^K$ and let Δ be the unique face of $\overline{\operatorname{suppf}} \cap R^K$ where $l_a(x)$ takes the minimal value, say d. Then $f(p(t)) = f_\Delta(z^0)t^d$ + higher and $\frac{\partial f}{\partial z_i}(p(t)) = \frac{\partial f_\Delta}{\partial z_i}(z^0)t^{d-a_i}$+higher

If $d > 0$, then $c(t) = c_0 t^d+\ldots$, hence $\lim_{t\to 0} c(t) = 0$.

If $d = 0$ and $a_i \leq 0$ for each $i =1,\ldots,k$, then $f(z_1,\ldots,z_k,0,\ldots 0)$ not depend on the z_1, hence $\frac{\partial f}{\partial z_1}(p(t)) \equiv 0$. Since $p_1(t) \not\equiv 0$ we obtain $\lambda(t) \equiv 0$, hence $c(t) \in \sum_f$.

Suppose that $d =0$, there exist $j \in \{1,\ldots,k\}$ with $a_j > 0$ and Δ is not a bad face, or $d < 0$. Then Δ is a face of $\tilde{\Gamma}(f)$. Since f is non-degenerate on the face Δ , there exists $l \in \{1,\ldots,k\}$ such that $\frac{\partial f_\Delta}{\partial z_l}(z^0) \neq 0$ (evidently $d-a_l > 0$). Let $I = \{i \mid a_i = a_l\}$. If $i \in I$, then $d - a_i = o(\lambda)+a_i$, otherwise $d - a_i < o(\lambda) + a_i$, hence for $i \notin I$: $\frac{\partial f_\Delta}{\partial z_i}(z^0) = 0$. By homogeneity $a_l \cdot \sum_{i\in I} z_i^0 \frac{\partial f_\Delta}{\partial z_i}(z^0) = d \cdot f_\Delta(z^0)$. If $d < 0$, then by the convergence of

$c(t): f_{\Delta}(z^0) = \sum_{i\in I} z_i^0 \cdot \frac{\partial f_\Delta}{\partial z_i}(z^0)$. Hence in both cases we obtain the absurd equality

$$0 = \sum_{i\in I} z_i^0 \cdot \frac{\partial f_\Delta}{\partial z_i}(z^0) = \lambda_0 \cdot \sum_{i\in I} |z_i^0|^2.$$

If $d = 0$, and Δ is a bad face, then $\frac{\partial f_\Delta}{\partial z_i}(z^0) = 0$ for all $i = 1,\dots,k$ (otherwise we obtain a contradiction as above). Hence $c(t) = f_{\Delta}(z^0) + c_1 t + \dots$ Thus $\lim_{t\to 0} c(t) = \{ f_{\Delta}(z^0) \mid z_i^0 \neq 0$ and $\partial f_{\Delta}(z^0) = 0 \}$.

Proof of Inclusions

The proof of inclusion $\mathcal{N}\mathcal{N}_0 \subset \mathcal{S}\mathcal{T}$ is contained in the proof of Theorem 5. (If $\lambda(t) \equiv 0$, since $\dim \operatorname{Sing} f^{-1}(c(t)) \geqslant 1$: $c(t) = 0$).

The proof of inclusion $\mathcal{N}\mathcal{b} \subset \mathcal{T}$:

Let $p(t) \in C^n$ an analytic curve with $\lim_{t\to 0} \partial f(p(t)) = 0$ and $\lim_{t\to 0} \|p(t)\| = \infty$ (we preserve the notations of the above proof). By convenience condition there exist $m_i \in N^*$ such that $(0,\dots,m_i,\dots,0) \in \operatorname{supp} f$, hence $d \leqslant m_i a_i$ for all $i = 1,\dots,n$. Therefore $d < 0$. If $a_i \geqslant 0$, then $d - a_i < 0$, if $a_i < 0$, then $d \leqslant m_i a_i \leqslant a_i$. Since $\lim_{t\to 0} \frac{\partial f_\Delta}{\partial z_i}(z^0) t^{d-a_i} + \dots) = 0$ we obtain that $\frac{\partial f_\Delta}{\partial z_i}(z^0) = 0$ for all $i = 1,\dots,n$. in contradiction with the nondegenerate condition on Δ.

The other incusions are trivial.

Proof of Theorem 2

(S). The model of the proof is [7, § 4]. We begin with some lemmas:

Lemma 2. Let $f \in \mathcal{S}\mathcal{T}$ (respectively $f \in \mathcal{M}\mathcal{T}$) and $p: (0,\varepsilon) \longrightarrow C^n$ an analytic curve (as in Curve Selection Lemma) such that $\lim_{t\to 0} \|p(t)\| = \infty$, the number $f(p(t))$ is non-zero and the vector grad $\log f(p(t))$ is a complex multiple $\lambda(t) \cdot p(t)$. Then the argument of the complex number $\lambda(t)$ tends to zero or π (respectively to zero) as $t \to 0$.

Proof: Let $f \in \mathcal{S}\mathcal{T}$. Consider $p(t) = at^{\alpha} + a_1 t^{\alpha+1} + \ldots \qquad (\alpha < 0,\ a \neq 0)$

$$f(p(t)) = bt^{\beta} + b_1 t^{\beta+1} + \ldots \qquad (b \neq 0)$$

$$\operatorname{grad} f(p(t)) = ct^{\gamma} + c_1 t^{\gamma+1} + \ldots \qquad (c \neq 0)$$

Since graf $f(p(t)) = \lambda(t) \cdot p(t)\, \overline{f(p(t))}$, we have $p(t) \in M(f)$ and $ct^{\gamma} + c_1 t^{\gamma+1} + \ldots = \lambda(t) \cdot \bar{b} \cdot a\, t^{\alpha+\beta} + \ldots$

Hence $\lambda(t) = \lambda_0 t^{\gamma-\alpha-\beta} + \ldots$ and $c = \lambda_0 \bar{b} a$. From the identity $\frac{df}{dt} = \langle \frac{dp}{dt}, \operatorname{grad} f \rangle$ we obtain $\beta b t^{\beta-1} + \ldots = \alpha \|a\|^2 \bar{\lambda}_0 b t^{\alpha-1+\gamma} + \ldots$ If $\beta = 0$, then $\alpha + \gamma \geqslant 1$, hence $\lim\limits_{t \to 0} (f(p(t)) - \langle p(t), \overline{\partial f(p(t))} \rangle = b$ in contradiction with the definition of $f \in \mathcal{S}\mathcal{T}$. Therefore $\beta \neq 0$, hence $\beta = \alpha \|a\|^2 \bar{\lambda}_0$, which proves that λ_0 is a real number. In the case of $f \in \mathcal{M}\mathcal{T}$: $\beta < 0$, hence λ_0 is a positive real number.

An immediate consequence is the

Lemma 3. Let $f \in \mathcal{S}\mathcal{T}$ (respectively $f \in \mathcal{M}\mathcal{T}$). Then there exists a sufficiently large $R_0 \in (0, \infty)$ so that for all $z \in C^n - f^{-1}(0)$ with $\|z\| \geqslant R_0$ the two vectors z and grad log $f(z)$ are either C - linearly independent or else grad log $f(z) = \lambda \cdot z$ where $\lambda \neq 0$ and $|\operatorname{Im} \lambda| < |\operatorname{Re} \lambda|$ (respectively $\arg \lambda \in (-\frac{\pi}{4}, \frac{\pi}{4})$)

Proof. We use the Curve Selection Lemma and Lemma 2.

(See also the proof of Lemma 4.3 from Lemma 4.4. [7])

Lemma 4. Let $f \in \mathcal{S}\mathcal{T}$ and R_0 as in Lemma 3. Then for each $R \geqslant R_0$ there exists a smooth tangential vector field $v(z)$ on $S_R - f^{-1}(0)$ so that for each $z \in S_R - f^{-1}(0)$ the complex inner product $\langle v(z), i \operatorname{grad} \log f(z) \rangle$ is non-zero and has argument less then $\pi/4$ in absolute value.

Proof. The proof is similar as the proof of Lemma 4.6 [7], with a minor modification.

We construct the vector field locally. If the vectors z and grad log $f(z)$ are C - linearly independent, then we find $v(z)$ such that

$\langle v(z), z\rangle = 0$ and $\langle v(z), i \operatorname{grad} \log f(z)\rangle = 1$. If $\operatorname{grad} \log f(z) = \lambda z$, we take $v = \sigma i z$ with $\sigma \in \{+1,-1\}$ such that $\operatorname{Re} \langle \sigma i z, i \operatorname{grad} \log f(z)\rangle > 0$. Using a partition of unity we obtain the desired vector field.

If we replace Lemma 4.4, Lemma 4.3 and Lemma 4.6 [7] by Lemma 2, Lemma 3 and Lemma 4, we obtain the proof of Theorem 2 (S), as in [7, § 4].

(M). Let $f \in \mathcal{MT}$. Replaceing S_ε in [7] by S_R all the arguments of §5, §6, §7 [7] remain valid. (See the proof of Theorems 5.1, 5.2, 6.5, 7.2, Appendix B).

Proof of Theorem 3.

(M) The proof is similar with Theorem 5.11. [7].

(S) We use a variant of Lemmas 2 and 3

Lemma 5. Let $f \in \mathcal{ST}$ and D a closed disc centered at 0. If R is sufficiently large, then there exist a closed disc D' centered at 0, $D' \supsetneqq D$ such that if $\operatorname{grad} \log f(z) = \lambda z$, then either $f(z) \in D$ or $f(z) \notin D'$ and in the second case $\arg \lambda \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

Proof. Let $p(t) \to \infty$ such that $\operatorname{grad} \log f(p(t)) = \lambda(t)\, p(t)$. If we preserve the notations of Lemma 2, then $\beta \neq 0$. If $\beta > 0$, then $f(p(t)) \longrightarrow 0$, otherwise $f(p(t)) \to \infty$ and $\lambda_0 = \frac{\alpha \|a\|^2}{\beta} > 0$.

Thus, if $f \in \mathcal{ST}$, the functions $a(z) = a_\theta(z) = \log |f(z)|$ used by Milnor in §5 has two types of critical points. If we consider the restriction of a on $S_R - f^{-1}(D)$ respectivelly a_θ on $F_\theta - f^{-1}(D)$, then the arguments of §5 remain valid.

§3. Stability of Milnor fiberings at infinity

The following theorem is the global analogous of the local case proved by Oka [9], [10], [11].

Theorem 6. Let $f \in \mathcal{N}\ell$. Then the Milnor fibration at infinity is determined by the Newton boundary $\tilde{\Gamma}(f)$. In particular is independent of a particular choice of the coefficients of $f(z)$.

In fact, we prove the following result:

Let $f \in \mathcal{NB}$ and a closed disc, D, centered at the origin such that $\Sigma_f \subset \operatorname{int} D$. Then the diffeomorphic type of the generic fiber of f and the monodromy action of the fiber boundle $f: f^{-1}(\partial D) \longrightarrow \partial D$ depend only on the Newton boundary of f.

Proof of Theorem 6.

Suppose that we have two polynomials $f, g \in \mathcal{NB}$ such that $\tilde{\Gamma}(f) = \tilde{\Gamma}(g) = \Gamma$. Because the non-degenerate condition of the Newton principal part is an open condition ([4, Theoreme II, iii]), we can take a piecewise analytic family F(z,t) such that F(z,0) = f(z), F(z,1) = g(z) and $F(z,t) = F_t(z)$ as a function of z is a convenient polynomial with non-degenerate Newton principal part at infinity and $F_t(0) = 0$ for each t.

We denote by $\Sigma_t = \{$the set of critical values of $F_t\}$. Then F_t is a C^∞ locally trivial fibration over the complement of Σ_t.

Lemma 6. There exists a compact disc D centered at the origin such that $\Sigma_t \subset \operatorname{int} D$ for each t.

Proof. Assume the contrary. Then using the Curve Selection Lemma we can find a real analytic curve (p(s), t(s)) such that $\partial F_{t(s)}(p(s)) = 0$, and $\lim_{s\to 0} F_{t(s)}p(s) = \infty$. Hence $\lim_{s\to 0}\|p(s)\| = \infty$. We may assume, that $p(s) = (z_1^0 \cdot s^{a_1} + \dots, z_k^0 s^{a_k} + \dots, 0, 0, \dots 0)(a_1 < 0\ ; z_1^0 \neq 0, \dots, z_k^0 \neq 0)$ $t(s) = t^0$ + higher. Since f is convenient $\Gamma \cap R^k \neq \phi$ and we consider the continuous function $l_a(x)$ on $\Gamma \cap R^k$ defined by $l_a(x) = a_1x_1 + \dots + a_kx_k$ (where $R^k = \{x \in R^n : x_{k+1} = \dots = x_n = 0\}$). Let Δ be the unique face of $\Gamma \cap R^k$ where $l_a(x)$ takes the minimal value d. Then $\frac{\partial F}{\partial z_j}(p(s), t(s)) = \frac{\partial F_\Delta}{\partial z_j}(z^0, t^0) s^{d-a_j}$ + higher. Therefore $\frac{\partial F_\Delta}{\partial z_j}(z^0, t^0) = 0$ for all $j = 1, \dots, k$ which is in contradiction with the non-degeneracy assumption for $F_{t^0}(z)$. (Note that $F_\Delta(z,t)$ is a function of $z_1, \dots, z_k$ and t)

Lemma 7. Let D a compact disc as in Lemma 6. Then for R sufficiently large $F_t^{-1}(c)$ meets transversally the sphere S_R for each t and $c \in D$.

Proof. If the assertion is not true, then by Curve Selection Lemma there exists an analytic curve (p(s), t(s)) such that $\lim_{s\to 0}\|p(s)\| = \infty$,

$F_{t(s)}(p(s)) \in D$ for each s and

$\partial F_{t(s)}(p(s)) = \lambda(s).\,\bar{p}(s)$, where $\lambda(s) \in C$. Let $\lambda(s) = \lambda_0 s^{o(\lambda)} + \dots (\lambda_0 \neq 0)$

If we preserve the notations of lemma 6, then

$$\frac{\partial F}{\partial z_j}(p(s),\ t(s)) = \frac{\partial F_\Delta}{\partial z_j}(z^0,t^0)s^{d-a_j}+\dots = \lambda_0 \bar{z}_j^0\, s^{o(\lambda)+a_j}+\dots$$

Let $I = \{i \mid d-a_i = o(\lambda) + a_i\}$. Then $i \notin I$ iff $\frac{\partial F_\Delta}{\partial z_j}(z^0,t^0) = 0$.

By conveniente assumption $d < 0$, hence $\Delta \in \tilde{\Gamma}(f)$. Therefore by non-degeneracy assumption for $F_{t^0}(z)$ we get that $I \neq \emptyset$. On the other hand $F(p(s), p(s)) =$ $= F_\Delta(z^0,t^0)s^d$ + higher $\in D$, hence $F_\Delta(z^0,t^0) = 0$. By homogeneity

$$\frac{d-o(\lambda)}{2}\cdot \sum_{i \in I} z_i^0 \frac{\partial F_\Delta}{\partial z_i}(z^0,t^0) = d \circ F_\Delta(z^0,t^0) = 0.$$

But $d \neq o(\lambda)$ because $d - a_1 \leqslant o(\lambda) + a_1$ and $a_1 < 0$. Thus we obtain the absurd equality

$$0 = \sum_{i \in I} z_i^0 \frac{\partial F_\Delta}{\partial z_i}(z^0,t^0) = \lambda_0 \cdot \sum_{i \in I} |z_i^0|^2$$

Let $F_t(z)$ an analytic family, where $t \in J = [0,1]$. We define $E = \{(z,t) \in C^n \times J \mid F_t(z) \in \partial D,\ \|z\| \leqslant R\}$, $\varphi : E \longrightarrow \partial D$ by $\varphi(z,t) = F_t(z)$ and $\pi : E \longrightarrow J$ by $\pi(z,t) = t$. Then φ is a fiber boundle and π is non-degenerate on each fiber of φ. Thus using a fiber - preserving vector field for π we have the following commutative diagram

$$\begin{tikzcd}
F_0^{-1}(\partial D) \cap B_R \arrow[rr, "\Psi"] \arrow[dr, "F_0"'] & & F_1^{-1}(\partial D) \cap B_R \arrow[dl, "F_1"] \\
& \partial D &
\end{tikzcd}$$

where Ψ is a diffeomorphism and $B_R = \{z : \|z\| \leqslant R\}$. By Theorem 3 there is a fiber-preverving diffeomorphism Ψ_i $(i=0,1)$

$$\begin{tikzcd}
F_i^{-1}(\partial D) \cap B_R \arrow[r, "\Psi_i"] \arrow[d] & S_R - F_i^{-1}(0) \arrow[d, "\phi"] \\
\partial D \arrow[r, "\sim"'] & S^1
\end{tikzcd}$$

This completes the proof.

Acknowledgement. We wish to thank Professor A. Dimca for many interesting and stimulating discussions.

REFERENCES :

[1] S.A. Broughton : On the topology of polynomial hypersurfaces, Proceedings of Symposia in Pure Mathematics Volume 40 (1983), Part 1.

[2] C. Ehresmann : Sur l'espaches fibrés différentiables, C. R. Acad. Sci. Paris 224 (1947), 1611-1612.

[3] Hà, H. V. & Lê, D. T. : Sur la topologie des polynômes complexes, Acta Math. Vietnamica 9 (1984) 21-32.

[4] A. G. Kouchnirenko : Polyèdres de Newton et nombres de Milnor, Inventiones math. 32, 1-31 (1976).

[5] K. Lamotke : The topology of complex projective varieties after S. Lefschetz, Topology. Vol. 20 (1981), 15-51.

[6] Y. Matsumoto : On a class of deformations of holomorphic functions, J. Fac. Sci. Univ. Tokyo Sect IA., 20, no.2 (1973), 239-251.

[7] J. Milnor : Singular Points of Complex Hypersurfaces, Ann. of math. Studies 61, Princeton Univ. Press, 1968.

[8] A. Némethi : Théorie de Lefschetz pour les variétiés algebriques affines C. R. Acad. Sc. Paris, t. 303, Série I, n° 12, (1986)

[9] M. Oka : On the bifurcation of the multiplicity and topology of the Newton boundary, J. Math. Soc. Japan. Vol. 31, No. 3, (1979)

[10] M. Oka : On the topology of the Newton boundary II, J. Math. Soc. Japan 32 (1980), 65-92.

[11] M. Oka : On the topology of the Newton boundary III, J. Math. Soc. Japan Vol. 34, No. 3, (1982) 541-549.

[12] M. Oka : Deformation of Milnor fiberings, J. Fact Sci. Univ. Tokyo Sect.IA, 20, no. 3 (1973), 397-400.

[13] J. L. Verdier : Stratifications de Whitney et Théorème de Bertini-Sard, Invent. math. 36 (1976), 295-312.

András NÉMETHI

Alexandru ZAHARIA

The National Institute for
Scientific and Technical Creation
Department of Mathematics
Bd. Păcii 220, 79622 Bucharest, ROMANIA.