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On Perturbations of Reflexive Algebras

by

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We denote by $\mathcal{H}, \mathcal{L}(\mathcal{H})$, and $\mathcal{K}$ a complex Hilbert space, the algebra of bounded linear operators on $\mathcal{H}$, and the ideal of compact operators on $\mathcal{H}$; respectively. We recall that a subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is said to be reflexive if it contains every operator T such that $T\mathcal{M} \subset \mathcal{M}$ whenever $\mathcal{M}$ is closed invariant subspace for $\mathcal{A}$.

In this paper we answer in the negative the following two questions.

PROBLEM 1. Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a reflexive algebra. Is then $\mathcal{A} + \mathcal{K}$ norm-closed?

PROBLEM 2. Suppose that $\mathcal{A}_n$, $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ are similar reflexive algebras, $n \geq 0$, and $\lim_{n\to\infty} \mathrm{dist}(\mathcal{A}_n, \mathcal{A}) = 0$. Can we choose invertible operators X_n such that $X_n^{-1} \mathcal{A} X_n = \mathcal{A}_n$ and $\lim_{n\to\infty} \|X_n - I\| = 0$?

The distance mentioned in Problem 2 is, of course, the Pompeiu-Hausdorff distance between the unit balls of $\mathcal{A}_n$ and $\mathcal{A}$.

We note that Problem 1 has an affirmative answer if the invariant subspaces of $\mathcal{A}$ are totally ordered by inclusion (i.e., $\mathcal{A}$ is a nest algebra); see [5]. The answer to Problem 1 is not known for algebras with commutative invariant subspace lattice (CSL-algebras); see [1] and [9] for more details about such algebras.

The answer to Problem 2 is positive if $\mathcal{A}_n$ and $\mathcal{A}$ are nest algbras. Problem 2 is open if $\mathcal{A}$ is similar to a CSL-algebra, and it is also open for algebras acting on finite-dimensional spaces. See [2], [3], [4], [8] and [10] for more information about this problem.

We begin with our example concerning Problem 1; this example is related to that given in [4]. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_j : 0 \leq j < \infty\}$, and define operators $T, P_0, S \in \mathcal{L}(\mathcal{H})$ such that

$$P_0\, x = (x, e_0) e_0 \,, \quad x \in \mathcal{H} \,,$$
$$S\, e_j = e_{j+1} \,, \quad j \geq 0 \,,$$
$$T = S + P_0 \,.$$

Next, denote by $\mathcal{A}$ the weakly closed unital algebra generated by T.

PROPOSITION 3. The algebra $\mathcal{A}$ is reflexive and $\mathcal{A} + \mathcal{K}$ is not closed in the norm topology.

This result will be proved in several steps. Let us set $\Lambda = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{1\}$.

LEMMA 4. The function $f : \Lambda \longrightarrow \mathcal{H}$ defined by

$f(\lambda) = e_0 + \sum_{k=1}^{\infty} \lambda^{k-1}(\lambda - 1) e_k$ is analytic on $\operatorname{int}(\Lambda)$,

$\lim_{r \uparrow 1} f(r) = f(1)$, and $T^* f(\lambda) = \lambda f(\lambda)$, $\lambda \in \Lambda$.

Proof. The analyticity of f is immediate, and so is the relation $\|f(r) - f(1)\| = (1-r)(1-r^2)^{-1/2}$, $r \in (0,1)$. Since $T^* = S^* + P_0$, we have $T^* e_0 = e_0$ and $T^* e_j = e_{j-1}$, $j \geq 1$. Thus

$$\begin{aligned} T^* f(\lambda) &= e_0 + \sum_{k=1}^{\infty} \lambda^{k-1}(\lambda-1)e_{k-1} \\ &= e_0 + (\lambda-1)e_0 + \lambda \sum_{j=1}^{\infty} \lambda^{j-1}(\lambda-1)e_j = \lambda f(\lambda) , \end{aligned}$$

as claimed.

We recall that $\text{Alg Lat } \Lambda = \text{Alg Lat } T$ is the algebra of all operators $A \in \mathcal{L}(\mathcal{H})$ such that $A\mathcal{M} \subset \mathcal{M}$ for every invariant subspace $\mathcal{M}$ of T.

LEMMA 5. Fix $A \in \text{Alg Lat } \Lambda$, and define $u : \Lambda \longrightarrow \mathbb{C}$ by $u(\lambda) = (A e_0, f(\bar{\lambda}))$, $\lambda \in \Lambda$. Then u is analytic and bounded on $\text{int}(\lambda)$, and $\lim_{r\uparrow 1} u(r) = u(1)$. Moreover, if $u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n$ is the power series expansion of u, then

$$\begin{aligned} (A e_i, e_j) &= 0 && \text{if } j < i , \\ &= u_{j-i} && \text{if } j \geq i \geq 1 , \\ &= u(1) - \sum_{k=0}^{j-1} u_k && \text{if } j \geq i = 0 . \end{aligned}$$

Proof. The analyticity of u, and the relation $\lim_{r\uparrow 1} u(r) = u(1)$, follow immediately from Lemma 4. To show that u is bounded, we verify that $\overline{u(\bar{\lambda})}$ is an eigenvalue of A^* with eigenvector $f(\lambda)$. Indeed, since $A^* \in \text{Alg Lat } T^*$, each $f(\lambda)$ is an eigenvector of A^*, and the formula for the corresponding eigenvalue follows because $(f(\lambda), e_0) = 1$. In order to determine the matrix entries of A we use now the relations

$$A^* e_0 = A^* f(1) = \overline{u(1)}\, e_0 ,$$

and $A^* f(\lambda) = \overline{u(\bar{\lambda})} f(\lambda)$, $|\lambda| < 1$. The latter equation can be rewritten as

$$\sum_{k=0}^{\infty} \lambda^k (A^* e_k - A^* e_{k+1}) = \left[\sum_{k=0}^{\infty} \bar{u}_k \lambda^k\right]\left[\sum_{k=0}^{\infty} \lambda^k (e_k - e_{k+1})\right], \quad |\lambda| < 1 ,$$

or equivalently,

$$A^* e_k - A^* e_{k+1} = \sum_{j=0}^{k} \overline{u_j} (e_{k-j} - e_{k-j+1}) .$$

These equations yield now

$$A^* e_k = A^* e_0 - \sum_{p=0}^{k-1} (A^* e_p - A^* e_{p+1})$$

$$= \overline{u(1)}\, e_0 - \sum_{p=0}^{k-1} \sum_{j=0}^{p} \overline{u_j} (e_{p-j} - e_{p-j+1})$$

$$= \overline{u(1)}\, e_0 - \sum_{j=0}^{k-1} \overline{u_j} \sum_{p=j}^{k-1} (e_{p-j} - e_{p-j+1})$$

$$= \overline{u(1)}\, e_0 - \sum_{j=0}^{k-1} \overline{u_j} (e_0 - e_{k-j})$$

$$= \left[\overline{u(1)} - \sum_{j=0}^{k-1} \overline{u_j}\right] e_0 + \sum_{j=1}^{k} \overline{u_{k-j}}\, e_j \ .$$

These relations imply immediately the formulas for (Ae_i, e_j) . The lemma is proved.

COROLLARY 6. Let A and u be as in Lemma 5.

(i) If A is compact then $A = 0$.

(ii) $\|A\| \le \sup\{|u(\lambda)| : |\lambda| < 1\} + \left[\sum_{i=0}^{\infty} |u(1) - \sum_{k=0}^{i-1} u_k|^2\right]^{1/2}$.

Proof. (i) If A is compact then we must have $u_k = \lim_{n\to\infty} (A e_n, e_{n+k}) = 0$ for every k . We conclude that $u = 0$, and hence all the entries in the matrix of A are zero.

(ii) We have

$$\|A\| \le \|AP_0\| + \|A(I - P_0)\|$$

$$= \|AP_0\| + \|ASS^*\|$$

$$\le \|AP_0\| + \|AS\| .$$

Clearly AS is a Toeplitz operator with symbol $\lambda u(\lambda)$, so that

$$\|AS\| = \sup\{|\lambda u(\lambda)| : |\lambda| < 1\} = \sup\{|u(\lambda)| : |\lambda| < 1\} ,$$

while AP_0 is a rank-one operator with norm $\left[\sum_{i=0}^{\infty} |u(1) - \sum_{k=0}^{i-1} u_k|^2\right]^{1/2}$. The corollary follows.

LEMMA 7. Every operator in $\mathrm{Alg\,Lat}\, T$ is the weak limit of a sequence of operators of the form $p(T)$, with p a polynomial. In particular, $\mathcal{A}$ is a reflexive algebra.

Proof. Let A and u be as in Lemma 5, and consider the polynomials.

$$u_n(\lambda) = \sum_{k=0}^{n} (1 - \frac{k}{n}) u_k \lambda^k ,$$

and the operators $A_n = u_n(T)$, $n \geq 0$. Clearly

$$
\begin{aligned}
(A_n e_i, e_j) &= 0 && \text{if} \quad j < i, \\
&= u^n_{j-i} && \text{if} \quad j \geq i \geq 1, \\
&= u_n(1) - \sum_{k=0}^{j-1} u^n_k && \text{if} \quad j \geq i = 0,
\end{aligned}
$$

where $u^n_k = (1 - \frac{k}{n})u_k$ if $k \leq n$, and $u^n_k = 0$ if $k > n$. We have $\lim_{n\to\infty} u^n_k = u_k$, $k \geq 0$. Moreover, since $\sum_{i=0}^{\infty} |u(1) - \sum_{k=0}^{i-1} u_k|^2 < \infty$, it follows that $u(1) = \sum_{k=0}^{\infty} u_k$. Consequently the Cesàro sums $u_n(1)$ converge to $u(1)$ as $n \longrightarrow \infty$. Thus we conclude that $\lim_{n\to\infty} (A_n e_i, e_j) = (Ae_i, e_j)$ for all i and j. The lemma will follow once we prove that $\sup_n \|A_n\| < \infty$. First, it is a well-known consequence of the positivity of the Féjer kernel that

$$\sup\{|u_n(\lambda)| : n \geq 0, \ |\lambda| < 1\} \leq \sup\{|u(\lambda)| : |\lambda| < 1\}.$$

Thus, by virtue of Corollary 6.(ii), it suffices to show that

$$\sup\left\{\left[\sum_{i=0}^{\infty} |u_n(1) - \sum_{k=0}^{i-1} u^n_k|^2\right]^{1/2} : n \geq 0\right\} < \infty .$$

Let us set

$$\alpha_i = u(1) - \sum_{k=0}^{i-1} u_k \ , \quad \alpha_i^n = u_n(1) - \sum_{k=0}^{i-1} u_k^n \ , \quad i,n \geq 0 \ .$$

We have then $\alpha_i^n = 0$ for $i \geq n$, and for $i < n$

$$\alpha_i^n = \sum_{k=i}^{n} u_k^n = \sum_{k=i}^{n} (1 - \frac{k}{n})(\alpha_k - \alpha_{k+1})$$

$$= (1 - \frac{i}{n})\alpha_i - \frac{1}{n} \sum_{k=i+1}^{n} \alpha_k \ .$$

A famous result of Hardy (cf. [6]), showing that the Cesàro operator is bounded with norm 2 in ℓ^2 , implies that

$$\left[\sum_{i=0}^{n} \left| \frac{1}{n-i} \sum_{k=i+1}^{n} \alpha_k \right|^2 \right]^{1/2} \leq 2 \left[\sum_{k=0}^{n} |\alpha_k|^2 \right]^{1/2} \ .$$

We deduce that

$$\left[\sum_{i=0}^{\infty} |\alpha_i^n|^2\right]^{1/2} \le \left[\sum_{i=0}^{n} |(1-\tfrac{i}{n})\alpha_i|^2\right]^{1/2} + \left[\sum_{i=0}^{n-1} |\tfrac{1}{n} \sum_{k=i+1}^{n} \alpha_k|^2\right]^{1/2}$$

$$\le \left[\sum_{i=0}^{n} |\alpha_i|^2\right]^{1/2} + \left[\sum_{i=0}^{n-1} |\tfrac{1}{n-i} \sum_{k=i+1}^{n} \alpha_k|^2\right]^{1/2}$$

$$\le 3\left[\sum_{i=0}^{\infty} |\alpha_i|^2\right]^{1/2},$$

and this concludes the proof of the lemma.

Let $\pi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})/\mathcal{K}$ denote the quotient map. The proof of Proposition 3 follows immediately from Lemma 7 and the next observation.

LEMMA 8. The algebra $\mathcal{A}$ contains no nonzero compact operators, and $\pi|\mathcal{A}$ is not bounded below.

Proof. That $\mathcal{A} \cap \mathcal{K} = \{0\}$ follows from Corollary 6.(i). To see that $\pi|\mathcal{A}$ is not bounded below we note that $\|\pi(T^n)\| = \|\pi(S^n)\| = 1$, while $\|T^n\| = \sqrt{n+1}$, $n \ge 0$.

We note that a somewhat more detailed analysis of $\mathcal{A}$ shows that the weak and ultraweak topologies coincide on this algebra.

We proceed now to our example concerning Problem 2. Let $\mathcal{H}$ be, as before, a Hilbert space with orthonormal basis $\{e_n : 0 \le n < \infty\}$, and define operators $R, U_n, R_n \in \mathcal{L}(\mathcal{H})$ such that

$Re_j = 2^{-j}e_j$, $j \geq 0$,

$U_n e_n = e_{n+1}$, $U_n e_{n+1} = e_n$, $U_n e_j = e_j$, $n \neq j \neq n+1$,

and $R_n = U_n^{-1} R U_n$, $n \geq 0$. (Note that $U_n^{-1} = U_n$.) Define three-dimensional algebras $\mathcal{A}, \mathcal{A}_n \subset \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ by

$$\mathcal{A} = \left\{ \begin{bmatrix} \lambda I & \gamma R \\ 0 & \mu I \end{bmatrix} : \lambda, \mu, \gamma \in \mathbb{C} \right\} ,$$

$$\mathcal{A}_n = \left\{ \begin{bmatrix} \lambda I & \gamma R_n \\ 0 & \mu I \end{bmatrix} : \lambda, \mu, \gamma \in \mathbb{C} \right\} , \quad n \geq 0 .$$

We recall that, for two subspace $\mathcal{M}$, $\mathcal{N}$ of a normed space $\mathcal{X}$, we have $\text{dist}(\mathcal{M}, \mathcal{N}) \leq \varepsilon$ if and only if for every vector x in the open unit ball of $\mathcal{M}$ [resp., $\mathcal{N}$] there is a vector y in the open unit ball of $\mathcal{N}$ [resp., $\mathcal{M}$] such that $\|x-y\| < \varepsilon$.

PROPOSITION 9. The algebras $\mathcal{A}_n$ and $\mathcal{A}$ are similar, reflexive, and $\lim_{n\to\infty} \text{dist}(\mathcal{A}_n, \mathcal{A}) = 0$. However, if $X_n \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ are invertible operators such that $\mathcal{A}_n = X_n^{-1} \mathcal{A} X_n$, then $\liminf_{n\to\infty} \|X_n - I\| > 0$.

Proof. Clearly $\mathcal{A}_n = (U_n \oplus U_n)^{-1} \mathcal{A}(U_n \oplus U_n)$ so that $\mathcal{A}_n$ and $\mathcal{A}$ are indeed similar. The equality $\lim_{u\to\infty} \text{dist}(\mathcal{A}_n, \mathcal{A}) = 0$ is an immediate consequence of the fact that $\lim_{n\to\infty} \|R_n - R\| = 0$. The reflexivity of $\mathcal{A}$ (and $\mathcal{A}_n$) follows easily from [7], but is also easy to verify directly. Indeed, if $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Alg Lat } \mathcal{A}$, clearly

$C = 0$ and $A, D \in \mathrm{Alg\,Lat}(I)$ so that $A = \lambda I$, $D = \mu I$ for some scalars λ and μ. Thus $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \in \mathrm{Alg\,Lat}\,\mathcal{A}$. Using invariant subspace of the forms $\{\alpha Rx \oplus \beta x : \alpha, \beta \in \mathbb{C}\}$, we see that for each $x \in \mathcal{H}$ there is $\gamma_x \in \mathbb{C}$ such that $Bx = \gamma_x Rx$. Linearity of B implies now that $\gamma_x = \gamma$ does not depend on x.

We will conclude the proof of the proposition assuming the following result, which we prove later.

LEMMA 10. Assume that $X_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ is an operator such that $X_n \mathcal{A}_n = \mathcal{A} X_n$ and $D_n \neq 0$. Then there exists a scalar γ_n such that $R\, D_n = \gamma_n A_n R_n$.

Assume that there exist operators $X_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ such that $X_n \mathcal{A}_n = \mathcal{A} X_n$ and $\lim_{n\to\infty} \|X_n - I\| = 0$. Clearly then $D_n \neq 0$ eventually, so we can choose γ_n as in Lemma 10. Denote by $[a^n_{ij}]^\infty_{i,j=0}$ and $[d^n_{ij}]^\infty_{i,j=0}$ the matrices of A_n and D_n, respectively, in the basis $\{e_i : i \geq 0\}$. It is immediate that $d^n_{oo} = \gamma_n a^n_{oo}$ and $2^{-n} d^n_{nn} = 2^{-n-1} \gamma_n a^n_{nn}$. Thus $\gamma_n = d^n_{oo}/a^n_{oo} = 2\, d^n_{nn}/a^n_{nn}$, and the last equality implies that

$$1 = \lim_{n\to\infty} \frac{d^n_{oo}}{a^n_{oo}} = 2 \lim_{n\to\infty} \frac{d^n_{nn}}{a^n_{nn}} = 2\,,$$

which is simply not true. This contradiction concludes the proof of the proposition.

We conclude the paper with a proof of Lemma 10. The relation $X_n \mathcal{A}_n = \mathcal{A} X_n$ implies the existence of scalars $\lambda_n, \mu_n, \gamma_n$ such that

$$\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} \lambda_n I & \gamma_n R_n \\ 0 & \mu_n I \end{bmatrix} = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} .$$

Thus we have $\mu_n D_n = 0$ and $\gamma_n A_n R_n + \mu_n B_n = R D_n$. Since $D_n \neq 0$, we deduce that $\mu_n = 0$, and therefore $R D_n = \gamma_n A_n R_n$, as desired.

Let us note that Lemma 10 can also be deduced from a more general result proved in [10].

References

1. W. Arveson, Ten lectures on operator algebras, CBMS Regional Conf. Ser. in Math., No. 55, Amer. Math. Soc., Providence, 1984.

2. M.D. Choi, K.R. Davidson, Perturbations of finite-dimensional operator algebras, preprint, to appear Michigan Math. J.

3. K.R. Davidson, Perturbations of reflexive operator algebras, J. Operator Theory 15(1986), 289-306.

4. K.R. Davidson, C.K. Fong, An operator algebra which is not closed in the Calkin algebra, Pacific J. Math. 72(1977), 57-58.

5. T. Fall, W. Arveson, P. Muhly, Perturbations of nest algebras, J. Operator Theory 1(1979), 137-150.

6. G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, The University Press, Cambridge, 1934.

7. J. Kraus, D.R. Larson, Some applications of a technique for constructing reflexive operator algebras, J. Operator Theory 13(1985), 227-236.

8. E.C. Lance, Cohomology and perturbations of nest algebras, Proc. (Ser. 3) London Math. Soc. 43(1981), 334-356.

9. C. Laurie, On density of compact operators in reflexive algebras, Indiana U. Math. J. 30(1981), 1-16.

10. F. Pop, A remark on a question of M.D. Choi and K.R. Davidson, INCREST preprint No. 41, 1986.