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by

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In [B1], [B2] one has proved a duality theorem with "two arguments", of type hyperext-hyperext, for complexes with coherent cohomology, which extends the usual duality for cohomology of Ramis and Ruget [R-R]. In the case of a complex manifold and two coherent sheaves on it, the result is:

THEOREM. Let X be an n-dimensional manifold with countable topology and $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$. Then $\mathrm{Ext}^q(X;\mathcal{E},\mathcal{F})$ has a natural topology of type QFS, $\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$ has a natural topology of type QDFS and there exists a natural pairing (Yoneda followed by the trace morphism) which induces a topological duality between the associated separated vector spaces. Moreover, $\mathrm{Ext}^q(X;\mathcal{E},\mathcal{F})$ is separated iff $\mathrm{Ext}_c^{n-q+1}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$ is. A similar statement holds for the pair $(\mathrm{Ext}_c^q, \mathrm{Ext}^{n-q})$.

For the case of an algebraic manifold, the result was pointed out in [D-P]. That paper and [B3] were at the origin of our work on duality for hyperext. Golovin found, independently, the result of the theorem under the supplementary condition that the sheaves $\mathcal{E}, \mathcal{F}$ admit, globally on X, finite length resolutions with locally free sheaves of finite rank ([G2]). We stress on the fact that in general one does not have such a property and this makes the interest of the theorem in analytic geometry.

The topologies mentioned in the theorem are the natural topologies introduced in [B2], proposition 4.4, and represent extensions of the Čech topologies on the cohomology of coherent sheaves.

The theorem is most interesting when the invariants are already separated and the pairing gives a genuine duality. This is always the case if X is holomorphically convex (see [B3]). If X is compact, the invariants are finite dimensional and one gets an algebraic duality.

COROLLARY. Let X be an n-dimensional, holomorphically convex manifold with countable topology and $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$. Then $\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$ has a natural topology of type FS, $\mathrm{Ext}_c^{n-q}(X; \mathcal{F}, \mathcal{E} \otimes \omega_X)$ has a natural topology of type DFS and there exists a natural pairing which induces a topological duality between them. A similar statement holds for the pair $(\mathrm{Ext}_c^q, \mathrm{Ext}^{n-q})$.

COROLLARY. Let X be an n-dimensional compact manifold and $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$. Then there exists a natural pairing which induces an algebraic duality between the finite dimensional vector spaces $\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$ and $\mathrm{Ext}^{n-q}(X; \mathcal{F}, \mathcal{E} \otimes \omega_X)$.

In section 3 of the paper, several other consequences of the theorem shall be mentioned.

The theorem given above is a particular case of theorem 5.3 in [B2]. However, the proof of this latter theorem uses duality for the hypercohomology of complexes of sheaves even when one starts with two coherent sheaves. This paper contains a proof of the theorem, which avoids the use of hypercohomology and the derived category. The idea of this proof is the systematic use of semisimplicial systems of sheaves (s.s.s.) and of free s.s.s. (see [F-K], [V], [F]).

In the paper, we also show that if X is Gorenstein (in particular a locally complete intersection), the statement of the theorem is still true (in this case ω_X is, of course, the duality sheaf, and one uses essentially the fact that it is invertible).

The same technique gives another proof of the general result in [B1] or [B2] which does not use the Ramis-Ruget duality for complexes in $D_{coh}(X)$. We also remark that, in the context of [R-R], the above approach gives another proof for the absolute duality theorems, as well.

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1. THE NATURAL TOPOLOGY

We shall recall briefly the facts on semi-simplicial systems of sheaves and on the natural topologies that we shall need. For more details see [B1] or [B2].

1.1. DEFINITIONS. Let $(X,\mathcal{O}_X)$ be a finite dimensional complex space with countable topology. Let $\mathcal{U}=(U_i)_{i\in I}$ be a locally finite covering of X with Stein open sets and let $\mathcal{N}$ be the nerve of $\mathcal{U}$ (one considers only alternated simplexes). A semi-simplicial system of sheaves (s.s.s.) relative to $\mathcal{U}$ consists of a family of sheaves $(\mathcal{A}_\alpha)_{\alpha\in\mathcal{N}}$, where $\mathcal{A}_\alpha \in \mathrm{Mod}(U_\alpha)$, and a family of connecting morphisms $(\rho_{\beta\alpha})_{\alpha\subset\beta}$, $\rho_{\beta\alpha}: \mathcal{A}_\alpha|U_\beta \longrightarrow \mathcal{A}_\beta$, such that for every α, $\rho_{\alpha\alpha}$ =id and for $\alpha\subset\beta\subset\gamma$, $\rho_{\gamma\beta}\circ(\rho_{\beta\alpha}|U_\gamma)=\rho_{\gamma\alpha}$

A morphism between two such s.s.s. relative to $\mathcal{U}$, $\varphi : \mathcal{A} \to \mathcal{A}'$, consists of a family of morphisms $(\varphi_\alpha)_{\alpha\in\mathcal{N}}$, $\varphi_\alpha : \mathcal{A}_\alpha \longrightarrow \mathcal{A}'_\alpha$, which commute with the connecting morphisms. Denote by $\mathrm{Mod}(\mathcal{U})$ the category of s.s.s. relative to $\mathcal{U}$.

In a similar way, one defines s.s.s. relative to a covering of X with closed sets, replacing sheaves and morphisms by germs of sheaves and germs of morphisms on the closed sets concerned.

If $U \subset X$ is an open set then every $\mathcal{A} \in \mathrm{Mod}(\mathcal{U})$ induces, in an obvious way, a s.s.s. relative to $\mathcal{U} \cap U$, denoted by $\mathcal{A}|U$. For every $\mathcal{A}$, $\mathcal{B} \in \mathrm{Mod}(\mathcal{U})$, the presheaf $U \longmapsto \mathrm{Hom}(\mathcal{A}|U, \mathcal{B}|U)$ is actually a sheaf that we denote by $\mathcal{H}om(\mathcal{A}, \mathcal{B})$.

1.2. <u>EXAMPLES</u>. a) Every $\mathcal{F} \in \mathrm{Mod}(X)$ defines the s.s.s. $\mathcal{F}|\mathcal{U} = (\mathcal{F}|U_\alpha)_{\alpha\in\mathcal{N}}$, with connecting morphisms the identities. If no confusion is likely, we shall write sometimes simply $\mathcal{F}$ for $\mathcal{F}|\mathcal{U}$.

b) Every $\mathcal{G} \in \mathrm{Mod}(U_\alpha)$, with $\alpha \in \mathcal{N}$, defines the s.s.s. $\tilde{\mathcal{G}} \in \mathrm{Mod}(\mathcal{U})$ with $\tilde{\mathcal{G}}_\beta = \mathcal{G}|U_\beta$ if $\beta \supset \alpha$ and $\tilde{\mathcal{G}}_\beta = 0$ otherwise.

1.3. <u>THE SHEAF ASSOCIATED TO A s.s.s.</u> For every $U \subset X$ open set let $\alpha(U) \in \mathcal{N}$ be the largest simplex such that $U \subset U_{\alpha(U)}$. If $\mathcal{A} \in \mathrm{Mod}(\mathcal{U})$ we denote by $\hat{\mathcal{A}}$ the sheaf associated to the presheaf $U \longmapsto \mathcal{A}_{\alpha(U)}(U)$. $\hat{\mathcal{A}}$ is the sheaf associated to the s.s.s. $\mathcal{A}$ and was introduced by Belkilani [B].

If $\mathcal{F} \in \mathrm{Mod}(X)$ and $\mathcal{A} \in \mathrm{Mod}(\mathcal{U})$, it is easy to verify that one has the isomorphisms:

$$\mathcal{F} \longrightarrow (\mathcal{F}|\mathcal{U})^\wedge$$

$$\mathrm{Hom}(\mathcal{A}, \mathcal{F}) \longrightarrow \mathrm{Hom}(\hat{\mathcal{A}}, \mathcal{F})$$

$\mathcal{H}om\,(\mathcal{A}, \mathcal{F}) \longrightarrow \mathcal{H}om\,(\hat{\mathcal{A}}, \mathcal{F})$.

REMARK 1. Let $\mathcal{F}^{\bullet}$ be a complex of $\mathcal{O}_X$-modules. Every quasi-isomorphism $\mathcal{A}^{\bullet} \longrightarrow \mathcal{F}^{\bullet}|\mathcal{U}$ of complexes of objects in Mod($\mathcal{U}$) induces a quasiisomorphism $\hat{\mathcal{A}}^{\bullet} \longrightarrow \mathcal{F}^{\bullet}$.

1.4. FREE s.s.s. A free s.s.s. is a direct sum $\mathcal{L} = \bigoplus_{\alpha \in \mathcal{N}} \widetilde{\mathcal{P}_\alpha}$ where each $\mathcal{P}_\alpha$ is $\mathcal{O}_{U_\alpha}$-free of finite rank (see [F-K] or [F]).

REMARK 2. The sheaf associated to the free s.s.s. $\mathcal{L}$ is $\hat{\mathcal{L}} = \bigoplus_{\alpha \in \mathcal{N}} \mathcal{P}_\alpha|^X$ where $\mathcal{P}_\alpha|^X$ is the extension with 0 to X of $\mathcal{P}_\alpha \in$ Mod(U_α). If $\mathcal{B} \in$ Mod($\mathcal{U}$) then one verifies easily that Hom($\mathcal{L}$, $\mathcal{B}$) = $\prod_{\alpha \in \mathcal{N}} \Gamma\,(U_\alpha, \mathcal{H}om\,(\mathcal{P}_\alpha, \mathcal{B}_\alpha))$.

Suppose now that $\mathcal{U} = (U_i)_{i \in I}$ has in addition the property that $\overline{\mathcal{U}} = (\overline{U}_i)_{i \in I}$ is a covering of X with Stein compact sets.

LEMMA 3. Every $\mathcal{E} \in$ Coh(X) admits a resolution with free s.s.s. in Mod($\mathcal{U}$), $\mathcal{L}^{\bullet} \longrightarrow \mathcal{E}|\mathcal{U} \longrightarrow 0$

Proof

As in [F-K], one shows first that every s.s.s. in Mod($\overline{\mathcal{U}}$) with coherent components is the quotient of a free s.s.s. from Mod($\overline{\mathcal{U}}$). Then one can construct inductively a resolution for $\mathcal{E}|\overline{\mathcal{U}}$ with free s.s.s. in Mod($\overline{\mathcal{U}}$), $\mathcal{L}^{\bullet} \longrightarrow \mathcal{E}|\overline{\mathcal{U}} \longrightarrow 0$, which yields, simply by restricting all the data from the compact sets $\overline{U}_\alpha$ to the open sets U_α, a resolution of $\mathcal{E}|\mathcal{U}$ with free s.s.s. from Mod($\mathcal{U}$).

REMARK 4. Lemma 3 is still true if $\mathcal{U}$ is a covering of X with Stein open sets, dominated by a covering of X with Stein compact sets.

REMARK 5. Suppose $\mathcal{E} \in \mathrm{Coh}(X)$ has finite tor-dimension (f.t.d.) (for instance $\mathcal{E}$ is locally free of finite rank or X is a complex manifold and $\mathcal{E}$ is any coherent sheaf). Then for $\mathcal{U}$ sufficiently small, the free resolution in lemma 3 can be chosen to have finite length (see [B2], proposition 3.3).

1.5. Free resolutions with s.s.s. can be used to compute $\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$ if $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$. To this end the following "projectivity" property of free s.s.s. is essential:

If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an epimorphism in $\mathrm{Mod}(\mathcal{U})$ and if $\ker \varphi$ has components acyclic on Stein open sets (i.e. with trivial cohomology on Stein open sets), then for every free s.s.s. $\mathcal{L} \in \mathrm{Mod}(\mathcal{U})$, the morphism $\mathrm{Hom}(\mathcal{L}, \mathcal{A}) \longrightarrow \mathrm{Hom}(\mathcal{L}, \mathcal{B})$ induced by φ is surjective.

LEMMA 6. Let $\mathcal{L}^\bullet$ be a complex of free s.s.s. in $\mathrm{Mod}(\mathcal{U})$, bounded above (e.g. the resolution of a coherent $\mathcal{O}_X$-module) and $\mathcal{B}^\bullet$ an exact complex of s.s.s. of $\mathrm{Mod}(\mathcal{U})$, having components with trivial cohomology on Stein open sets. Then $\mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{B}^\bullet)$ is exact.

Proof

Using an exact sequence argument, one proves that the sheaves of cocycles of $\mathcal{B}^\bullet$ have also trivial cohomology on Stein open sets. Now, according to the "projectivity" property of free s.s.s., it follows that any morphism of given degree from $\mathcal{L}^\bullet$ to $\mathcal{B}^\bullet$ is homotopic to 0 (the proof goes exactly as if $\mathcal{L}^\bullet$ would have projective components).

Let $\mathcal{E} \in \mathrm{Coh}(X)$, $\mathcal{F}$ be an $\mathcal{O}_X$-module acyclic on Stein open sets, $\mathcal{L}^\bullet \longrightarrow \mathcal{E}$ be a resolution of $\mathcal{E}$ with free s.s.s. in $\mathrm{Mod}(\mathcal{U})$ and $\mathcal{F} \longrightarrow \mathcal{J}^\bullet$ a resolution of $\mathcal{F}$ with injective sheaves. Then all the morphisms in the diagram below are quasiisomorphisms:

$$\mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{F}) \xrightarrow{\alpha} \mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet) \xrightarrow{\beta} \mathrm{Hom}^\bullet(\hat{\mathcal{L}}^\bullet, \mathcal{J}^\bullet) \xleftarrow{\gamma} \mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{J}^\bullet)$$

(let $\mathcal{C}^\bullet$ be the mapping cone of the morphism $\mathcal{F} \longrightarrow \mathcal{J}^\bullet$; according to lemma 6, $\mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{C}^\bullet)$ is exact and consequently α is a quasi-isomorphism; from section 1.3 it follows that β is an isomorphism; remark 1 and the fact that $\mathcal{J}^\bullet$ is an injective resolution imply that γ is a quasiisomorphism). Since the cohomology groups of $\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{J}^\bullet)$ are, by definition, $\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$, it follows that one can also compute these groups from the complex $\mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{F})$.

If $\mathcal{F} \in \mathrm{Coh}(X)$, let $\mathcal{M}^\bullet \longrightarrow \mathcal{F}$ be a resolution of $\mathcal{F}$ with free s.s.s. in $\mathrm{Mod}(\mathcal{U})$. One can prove easily, using lemma 6, that the cohomology groups of $\mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$ also compute $\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$.

Replacing $\mathrm{Hom}^\bullet$ with $\mathcal{H}om^\bullet$ in the diagram above, one obtains, much in the same way, that the cohomology sheaves of the complex $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{F})$ are $\mathcal{E}xt^q(\mathcal{E}, \mathcal{F})$.

One can prove (see [B2], lemma 2.11) that the components of $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{F})$ are $\Gamma(X, \cdot)$ and $\Gamma_c(X, \cdot)$-acyclic. Consequently, the cohomology groups of $\Gamma_c(X, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{F}))$ are $\mathrm{Ext}^q_c(X; \mathcal{E}, \mathcal{F})$.

1.6. NATURAL TOPOLOGY. If $\mathcal{A}, \mathcal{B} \in \mathrm{Mod}(\mathcal{U})$ have coherent components then $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ carries a natural topology of type FS (Fréchet-Schwarz) (it is a closed subspace of $\prod_{\alpha \in \mathcal{N}} \Gamma(U_\alpha, \mathcal{H}om(\mathcal{A}_\alpha, \mathcal{B}_\alpha))$).

In particular, if $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$, keeping the notations of the previous section, $\mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{F})$ is a complex of FS spaces. Hence one gets a topology of type QFS on $\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$ called the natural topology (see [B2], proposition 4.4). This topology is independent of the covering $\mathcal{U}$ and the resolution $\mathcal{L}^\bullet$, it is functorial in $\mathcal{E}$ and $\mathcal{F}$, and is compatible with the restriction morphisms.

Let $\mathcal{K}$ be a locally finite covering of X with Stein compact sets. If $\mathcal{A}$, $\mathcal{B} \in \mathrm{Mod}(\mathcal{K})$ have coherent components, then $\Gamma_c(X, \mathcal{H}om(\mathcal{A}, \mathcal{B}))$ carries a natural topology of type DFS (strong dual of a FS space) (it is a closed subspace in $\bigoplus_{\alpha \in W} \Gamma(K_\alpha, \mathcal{H}om(\mathcal{A}_\alpha, \mathcal{B}_\alpha)))$.

If $\mathcal{N}^\bullet \longrightarrow \mathcal{E}|\mathcal{K}$ is a resolution of $\mathcal{E}$ with free s.s.s. in $\mathrm{Mod}(\mathcal{K})$ then one proves as in section 1.5 that the cohomology groups of $\Gamma_c(X, \mathcal{H}om^\bullet(\mathcal{N}^\bullet, \mathcal{F}))$ are $\mathrm{Ext}^q_c(X; \mathcal{E}, \mathcal{F})$. Since $\Gamma_c(X, \mathcal{H}om^\bullet(\mathcal{N}^\bullet, \mathcal{F}))$ is a complex of DFS spaces one gets a topology of type QDFS on $\mathrm{Ext}^q_c(X; \mathcal{E}, \mathcal{F})$, called the natural topology. It is independent of the covering $\mathcal{K}$ and the resolution $\mathcal{N}^\bullet$, it is functorial in $\mathcal{E}$ and $\mathcal{F}$, and is compatible with the extension morphisms. Moreover, one can prove, by squeezing $\mathcal{U}$ between two locally finite coverings of X with Stein compact sets that $\Gamma_c(X, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{F}))$ also induces on $\mathrm{Ext}^q_c(X; \mathcal{E}, \mathcal{F})$ the natural topology (see [B2], remark 4.3).

2. PROOF OF THE THEOREM

First we must describe the pairing. Since we do not use derived categroy, the description will be rather awkward and a bit long.

Let $\mathcal{U} = (U_i)_{i \in I}$ be a locally finite covering of X with relatively compact Stein open sets, such that $\mathcal{E}$, $\mathcal{F}$ admit resolutions of finite length, $\mathcal{L}^\bullet \longrightarrow \mathcal{E}$, $\mathcal{M}^\bullet \longrightarrow \mathcal{F}$, with free s.s.s. in $\mathrm{Mod}(\mathcal{U})$. (see remark 1.5). Let $\omega_X \longrightarrow \mathcal{J}^\bullet$ be a resolution of ω_X with injective sheaves.

The composition of morphisms yields a pairing:

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \times \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \hat{\mathcal{L}}^\bullet \otimes \omega_X) \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \hat{\mathcal{L}}^\bullet \otimes \omega_X)$$

Taking cohomology, respectively cohomology with compact supports, of degrees q and n-q, in the two factors of the left hand side term, one gets a pairing, called the Yoneda pairing (see [B-S] chapter 7, §1):

$$\mathrm{Ext}^q(X;\mathcal{E},\mathcal{F})\times\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)\longrightarrow\mathrm{Ext}_c^n(X;\mathcal{E},\mathcal{E}\otimes\omega_X).$$

Consider now the diagram:

$$\begin{tikzcd}
\mathcal{H}om^\bullet(\mathcal{L}^\bullet;\omega_X)\otimes\hat{\mathcal{L}}^\bullet \arrow[r] \arrow[d] & \mathcal{H}om^\bullet(\hat{\mathcal{L}}^\bullet,\hat{\mathcal{L}}^\bullet\otimes\omega_X). \arrow[d] \\
\mathrm{Hom}^\bullet(\hat{\mathcal{L}}^\bullet_x,\omega_{X,x})\otimes\hat{\mathcal{L}}_x \arrow[r] & \mathrm{Hom}^\bullet(\hat{\mathcal{L}}^\bullet_x,\hat{\mathcal{L}}^\bullet_x\otimes\omega_{X,x})
\end{tikzcd}$$

Since $\mathcal{E}xt^q(\mathcal{E},\omega_X)_x \xrightarrow{\sim} \mathrm{Ext}^q(\mathcal{E}_x,\omega_{X,x})$ and $\mathcal{E}xt^q(\mathcal{E};\mathcal{E}\otimes\omega_X)_x \xrightarrow{\sim} \mathrm{Ext}^q(\mathcal{E}_x,\mathcal{E}_x\otimes\omega_{X,x})$ it follows that the vertical morphisms above are quasiisomorphisms. Moreover, since $\mathcal{L}^\bullet$ has finite length or since ω_X is locally free, one verifies easily that the lower morphism in the diagram is an isomorphism. Hence the upper morphisms is a quasi-isomorphism.

One has the following sequence of morphisms:

$$\mathcal{H}om^\bullet(\hat{\mathcal{L}}^\bullet,\hat{\mathcal{L}}^\bullet\otimes\omega_X)\longleftarrow\mathcal{H}om^\bullet(\hat{\mathcal{L}}^\bullet,\omega_X)\otimes\hat{\mathcal{L}}^\bullet\longrightarrow$$
$$\longrightarrow\mathcal{H}om^\bullet(\hat{\mathcal{L}}^\bullet,\mathcal{J}^\bullet)\otimes\hat{\mathcal{L}}^\bullet\longrightarrow\mathcal{J}^\bullet$$

Since the first morphism is a quasiisomorhism, taking cohomology with compact supports we obtain a natural morphism:

$$\mathrm{Ext}_c^n(X;\mathcal{E},\mathcal{E}\otimes\omega_X)\xrightarrow{\alpha}H_c^n(X,\omega_X).$$

The pairing in the theorem is obtained by composing the Yoneda pairing, the morphism α and the trace morphism

$T_X : H^n_c(X, \omega_X) \longrightarrow \mathbb{C}$ (see [R-R]; here, since X is a manifold, T_X is induced by integration).

The pairing for the couple (Ext^q_c, Ext^{n-q}) is defined in the same way.

Step 1) There exists a topology of type QDFS on $\mathrm{Ext}^{n-q}_c(X; \mathcal{F}, \mathcal{E} \otimes \omega_X)$, (not necessarily the natural one), such that the statement of the theorem is true when on $\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$ one considers the natural topology.

Let $(\mathcal{P}^i_\alpha)_{\alpha \in W}$ be the family of free sheaves that defines the free s.s.s. $\mathcal{L}^i$ in the resolution $\mathcal{L}^\bullet \longrightarrow \mathcal{E}$.

$\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{F})$ is computed (natural topology included) from the FS complex $K^\bullet = \mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{F})$ having components

$$K^q = \mathrm{Hom}^q(\mathcal{L}^\bullet, \mathcal{F}) = \mathrm{Hom}(\mathcal{L}^{-q}, \mathcal{F}) = \prod_{\alpha \in W} \Gamma(U_\alpha, \mathcal{H}om(\mathcal{P}^{-q}_\alpha, \mathcal{F})).$$

Using the duality theorem for Stein manifolds, one obtains, by dualizing $K^\bullet$, the complex of DFS spaces $L^\bullet$ with

$$L^q = \bigoplus_{\alpha \in W} \mathrm{Ext}^n_c(U_\alpha; \mathcal{H}om(\mathcal{P}^q_\alpha, \mathcal{F}), \omega_X) \simeq$$

$$\simeq \bigoplus_{\alpha \in W} \mathrm{Ext}^n_c(U_\alpha; \mathcal{F}, \mathcal{P}^q_\alpha \otimes \omega_X)$$

(L^q is in duality with K^{-q}).

To prove step 1 it is enough to show that for every q the cohomology of $L^\bullet$ in degree q is $\mathrm{Ext}^{n+q}(X, \mathcal{F}, \mathcal{E} \otimes \omega_X)$. To this end, consider the double complex

$$N^{qr} = \bigoplus_\alpha \Gamma_c(U_\alpha, \mathcal{H}om(\mathcal{F}, \mathcal{P}^q_\alpha \otimes \mathcal{J}^r)) =$$

$$= \bigoplus_\alpha \Gamma_c(U_\alpha, \mathcal{H}om(\mathcal{H}om(\mathcal{P}^q_\alpha, \mathcal{F}), \mathcal{J}^r))$$

with differentials in the r-direction deduced from those of $\mathcal{J}^{\bullet}$ and differentials in the q-direction defined as those in $L^{\bullet}$.

The cohomology of N^{qr} along the r-direction is

$$\begin{cases} 0 & \text{for } r \neq n \\ \bigoplus_{\alpha} \mathrm{Ext}_c^n(U_\alpha ; \mathcal{F}, \mathcal{P}_\alpha^q \otimes \omega_X) \simeq L^q & \text{for } r=n . \end{cases}$$

as follows immediately from the usual duality theorem for a Stein manifold. Hence $N^{\bullet}$, the simple complex associated to the double complex (N^{qr}) is quasiisomorphic with $L^{\bullet}[n]$. (i.e. $L^{\bullet}$ shifted to the left with n positions). Consequently we have to verify that the cohomology of $N^{\bullet}$ in degree q is $\mathrm{Ext}_c^q(X; \mathcal{F}, \mathcal{E} \otimes \omega_X)$.

Since $(U_\alpha)_\alpha$ is a locally finite family of relatively compact open sets, $(\mathcal{H}om(\mathcal{F}, \mathcal{P}_\alpha^p \otimes \mathcal{J}^r))_\alpha$ is a locally finite family of sheaves and one has the equalities:

$$N^{pr} = \bigoplus_{\alpha} \Gamma_c(U_\alpha, \mathcal{H}om(\mathcal{F}, \mathcal{P}_\alpha^p \otimes \mathcal{J}^r)) =$$

$$= \bigoplus_{\alpha} \Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{P}_\alpha^p \otimes \mathcal{J}^r)|^X) =$$

$$= \Gamma_c(X, \prod_{\alpha} \mathcal{H}om(\mathcal{F}, \mathcal{P}_\alpha^p \otimes \mathcal{J}^r)|^X)$$

where $|^X$ means extension with 0 to the whole of X.

Now, since $\mathcal{J}^r$ is injective and $\mathcal{P}_\alpha^p$ is free of finite rank, $\mathcal{P}_\alpha^p \otimes \mathcal{J}^r$ is also injective and $\mathcal{H}om(\mathcal{F}, \mathcal{P}_\alpha^p \otimes \mathcal{J}^r)$ is flabby and, in particular, soft. According to [Go], theorem 3.5.5.c., $\mathcal{H}om(\mathcal{F}, \mathcal{P}_\alpha^p \otimes \mathcal{J}^r)|^X$ is also soft and so is $\prod_{\alpha} \mathcal{H}om(\mathcal{F}, \mathcal{P}_\alpha^p \otimes \mathcal{J}^r)|^X$. Consequently, the components of N^{pr} are $\Gamma_c(X, \cdot)$-acyclic.

Since $\mathcal{F}$ is coherent and $\mathcal{H}om(\mathcal{F}, \cdot)$ commutes with taking fibers, one verifies easily the equalities:

$$\prod_\alpha \mathcal{H}om(\mathcal{F}, \mathcal{P}^r_\alpha \otimes \mathcal{J}^n)|^X = \prod_\alpha \mathcal{H}om(\mathcal{F}, (\mathcal{P}^r_\alpha \otimes \mathcal{J}^n)|^X) =$$

$$= \prod_\alpha \mathcal{H}om(\mathcal{F}, \mathcal{P}^r_\alpha|^X \otimes \mathcal{J}^n) = \mathcal{H}om(\mathcal{F}, \prod_\alpha (\mathcal{P}^r_\alpha|^X \otimes \mathcal{J}^n)) =$$

$$= \mathcal{H}om(\mathcal{F}, \bigoplus_\alpha (\mathcal{P}^r_\alpha|^X \otimes \mathcal{J}^n)) = \mathcal{H}om(\mathcal{F}, (\bigoplus_\alpha \mathcal{P}^r_\alpha|^X) \otimes \mathcal{J}^n) =$$

$$= \mathcal{H}om(\mathcal{F}, \hat{\mathcal{L}}^r \otimes \mathcal{J}^n).$$

(for the last equality see remark 1.2).

Since $\mathcal{L}^\bullet$ has finite length, the following direct sums are finite and one gets:

$$N^q = \bigoplus_{p+r=q} N^{pr} = \bigoplus_{p+r=q} \Gamma_c(X, \mathcal{H}om(\mathcal{F}, \hat{\mathcal{L}}^p \otimes \mathcal{J}^r)) =$$

$$= \Gamma_c(X, \mathcal{H}om(\mathcal{F}, \bigoplus_{p+r=q} (\hat{\mathcal{L}}^p \otimes \mathcal{J}^r)) =$$

$$= \Gamma_c(X, \mathcal{H}om^q(\mathcal{F}, \hat{\mathcal{L}}^\bullet \otimes \mathcal{J}^\bullet))$$

Now take $\hat{\mathcal{L}}^\bullet \otimes \mathcal{J}^\bullet$. It has obviously the same cohomology as $\mathcal{E} \otimes \omega_X$ and, since $\hat{\mathcal{L}}^\bullet$ has free fibers, it has injective fibers. Since $\mathcal{F}$ is coherent it follows that $\mathcal{H}om^\bullet(\mathcal{F}, \hat{\mathcal{L}}^\bullet \otimes \mathcal{J}^\bullet)$ can be used to compute $\mathcal{E}xt^\bullet(\mathcal{F}, \mathcal{E} \otimes \omega_X)$ (is a representative for $R\mathcal{H}om(\mathcal{F}, \mathcal{E} \otimes \omega_X)$). Taking into account that the terms of $\mathcal{H}om^\bullet(\mathcal{F}, \hat{\mathcal{L}}^\bullet \otimes \mathcal{J}^\bullet)$ are $\Gamma_c(X, \cdot)$-acyclic we deduce that the cohomology of $N^\bullet$ is $\mathrm{Ext}^q_c(X; \mathcal{F}, \mathcal{E} \otimes \omega_X)$ which ends the proof of step 1.

In a similar way one can prove:

Step 2) There exists a topology of type QFS on $\mathrm{Ext}^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$ (not necessarily the natural one) such that the statement of the theorem is true when on $\mathrm{Ext}_c^q(X;\mathcal{E},\mathcal{F})$ one takes the natural topology.

Step 3) The "duality" topologies introduced at steps 1 and 2 on $\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$ and $\mathrm{Ext}^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$ coincide with the natural topologies.

According to step 1 one has a duality modulo separation:

$$\mathrm{Ext}^q(X;\mathcal{E},\mathcal{F})\times\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)\longrightarrow\mathrm{Ext}_c^n(X;\mathcal{E},\mathcal{E}\otimes\omega_X)\longrightarrow\mathbb{C}$$

and according to step 2 one has a duality modulo separation

$$\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)\times\mathrm{Ext}^q(X;\mathcal{E}\otimes\omega_X,\mathcal{F}\otimes\omega_X)\longrightarrow\mathrm{Ext}_c^n(X;\mathcal{F},\mathcal{F}\otimes\omega_X)\longrightarrow\mathbb{C}$$

where the left hand side factors carry the natural topology and the right hand side ones, the "duality" topology.

Let $u:\mathrm{Ext}^q(X;\mathcal{E},\mathcal{F})\longrightarrow\mathrm{Ext}^q(X;\mathcal{E}\otimes\omega_X,\mathcal{F}\otimes\omega_X)$ be the natural isomorphism (since X is a manifold, ω_X is invertible) and let v be the identity map of $\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$. One verifies that u and v are each the transposed of the other with respect to the Yoneda pairings above. According to lemma 1.4 in [R-R], this implies that v is continuous and consequently, that the "duality" topology on $\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$ is weaker than the natural one. On the other hand, since u is an isomorphism, it follows that v maps the closure of $\{0\}$ in one topology bijectively on the closure of $\{0\}$ in the other; this implies that v is a topological isomorphism and so that the "duality" topology and the natural topology coincide on $\mathrm{Ext}_c^{n-q}(X;\mathcal{F},\mathcal{E}\otimes\omega_X)$.

The statement on $\mathrm{Ext}^{n-q}(X; \mathcal{F}, \mathcal{E} \otimes \omega_X)$ can be verified in the same way and this ends the proof of the theorem.

Proof of the Gorenstein case

Let X be a pure n-dimensional Gorenstein space. In this case we consider the dualizing sheaf ω_X which is an invertible sheaf and coincides with the sheaf of holomorphic n-forms in the smooth case.

Since X is no longer smooth, not every coherent sheaf on X admits a resolution of finite length with free s.s.s. in $\mathrm{Mod}(\mathcal{U})$. This condition was used only in the proof of steps 1 and 2; so we have to give alternative proofs for these steps. As above, we shall verify only step 1 .

Let $\mathcal{L}^\bullet \longrightarrow \mathcal{E}$, $\mathcal{M}^\bullet \longrightarrow \mathcal{F}$ be resolutions of $\mathcal{E}$ and $\mathcal{F}$ with free s.s.s. in $\mathrm{Mod}(\mathcal{U})$. (see lemma 1.3).

Resuming the notations and the arguments in the proof of the theorem, let:

$$L^p = \bigoplus_{\alpha \in N} \mathrm{Ext}^n_c(U_\alpha ; \mathcal{F}, \mathcal{P}^p_\alpha \otimes \omega_X) \qquad \text{and}$$

$$N^{pr} = \bigoplus_{\alpha \in N} \Gamma_c(U_\alpha , \mathcal{H}om(\mathcal{F}, \mathcal{P}^p_\alpha \otimes \mathcal{J}^r))$$

The simple complex associated to the double complex N^{pr} has the same cohomology as $L^\bullet[n]$, and we shall verify that the cohomology of $N^\bullet$ in degree q is $\mathrm{Ext}^q_c(X; \mathcal{F}, \mathcal{E} \otimes \omega_X)$.

As we have seen

$$N^{pr} \simeq \Gamma_c(X, \mathcal{H}om(\mathcal{F}, \hat{\mathcal{L}}^p \otimes \mathcal{J}^r))$$

Consider now the complexes:

$$M^{pqr} = \Gamma_c(X, \mathcal{H}om\ (\mathcal{M}^q, \hat{\mathcal{L}}^p \otimes \mathcal{J}^r))$$

$$P^{pq} = \Gamma_c(X,\ \mathcal{H}om(\mathcal{M}^q, \hat{\mathcal{L}}^p \otimes \omega_X))$$

$$R^q = \Gamma_c(X, \mathcal{H}om(\mathcal{M}^q, \mathcal{E} \otimes \omega_X))$$

and let $M^i = \bigoplus_{p-q+r=i} M^{pqr}$, $P^i = \bigoplus_{p-q=i} P^{pq}$.

The cohomology of M^{pqr} along the q-direction is (see 1.5):

$$\begin{cases} 0 & \text{for } q \neq 0 \\ \Gamma_c(X, \mathcal{H}om(\mathcal{F}, \hat{\mathcal{L}}^p \otimes \mathcal{J}^r)) = N^{pr} & \text{for } q = 0 . \end{cases}$$

Hence $M^\bullet$ and $N^\bullet$ have the same cohomology.

The quasiisomorphism $\omega_X \longrightarrow \mathcal{J}^\bullet$ induces, since $\hat{\mathcal{L}}^p$ has flat fibers and $\mathcal{M}^q$ is a fres s.s.s., a quasiisomorphism

$$\mathcal{H}om\,(\mathcal{M}^q, \hat{\mathcal{L}}^p \otimes \omega_X) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^q, \hat{\mathcal{L}}^p \otimes \mathcal{J}^\bullet\,)$$

Since both sheaves are $\Gamma_c(X, \cdot\,)$-acyclic, one gets a quasiisomorphism $P^\bullet \longrightarrow M^\bullet$.

In the same way one shows that the quasiisomorphism $\hat{\mathcal{L}}^\bullet \longrightarrow \mathcal{E}$ induces a quasiisomorphism $P^\bullet \longrightarrow R^\bullet$. Since the cohomology of $R^\bullet$ in degree q is $\mathrm{Ext}^q(X, \mathcal{F}, \mathcal{E} \otimes \omega_X)$ (see 1.5) we are done.

3. SOME CONSEQUENCES

3.1. The Malgrange, Ramis-Ruget-Verdier separation criterion ([R-R-V]) holds for the invariants $\mathrm{Ext}_c^\bullet(X; \mathcal{E}, \mathcal{F})$ considered with their natural topologies. In the same way as in the Appendix of

[R-R-V], we get from our duality:

COROLLARY. Let X be an n-dimensional manifold with countable topology, connected and non-compact. If $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$ and $\mathcal{E}$ is without torsion then $\mathrm{Ext}^n(X; \mathcal{E}, \mathcal{F}) = 0$.

3.2. COROLLARY. Let X be a Stein manifold and $U \subset X$ a Stein open set such that $\mathcal{O}(X) \longrightarrow \mathcal{O}(U)$ has dense image. Then, for every $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$, the natural morphisms:

$$\mathrm{Ext}_c^q(U; \mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Ext}_c^q(X; \mathcal{E}, \mathcal{F})$$

are injective.

Indeed, using duality, it is sufficient to show that the restriction morphisms:

$$\mathrm{Ext}^i(X; \mathcal{F}, \mathcal{E} \otimes \omega_X) \longrightarrow \mathrm{Ext}^i(U; \mathcal{F}, \mathcal{E} \otimes \omega_X)$$

have dense image. Since X and U are Stein, $\mathrm{Ext}^i(X; \mathcal{F}, \mathcal{E} \otimes \omega_X) \simeq$ $\simeq H^\circ(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{E} \otimes \omega_X))$ and $\mathrm{Ext}^i(U; \mathcal{F}, \mathcal{E} \otimes \omega_X) \simeq$ $\simeq H^\circ(U; \mathcal{E}xt^i(\mathcal{F}, \mathcal{E} \otimes \omega_X))$. Now the statement follows from the fact that for every $\mathcal{H} \in \mathrm{Coh}(X)$, the restriction morphism $\mathcal{H}(X) \longrightarrow \mathcal{H}(U)$ has dense image.

3.3. Let X be complex space and $A \subset X$ an analytic subset. For every $\mathcal{F} \in \mathrm{Coh}(X)$ one defines:

$$\mathrm{prof}_A(\mathcal{F}) = \inf_{x \in A} \mathrm{prof}_{\mathcal{J}_x} \mathcal{F}_x$$

where $\mathcal{J}$ is any coherent ideal such that $\mathrm{Supp}(\mathcal{O}/\mathcal{J}) = A$ (see, for instance, [B-S] p.73).

COROLLARY. Let X be an n-dimensional Stein manifold, $\mathcal{E}$, $\mathcal{F} \in \mathrm{Coh}(X)$ and let $A = \mathrm{Supp}\,\mathcal{F}$. Then $\mathrm{Ext}_c^q(X; \mathcal{E}, \mathcal{F}) = 0$ for $q > n - \mathrm{prof}_A(\mathcal{E})$.

Proof

Using duality it is enough to show that $\mathrm{Ext}^i(X; \mathcal{F}, \mathcal{E} \otimes \omega_X) = 0$ for $i < \mathrm{prof}_A(\mathcal{E})$. Since $\mathrm{Ext}^i(X; \mathcal{F}, \mathcal{E} \otimes \omega_X) \simeq H^\circ(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{E} \otimes \omega_X))$ one has to verify that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{E} \otimes \omega_X) = 0$ for $i < \mathrm{prof}_A(\mathcal{E})$. Taking fibers, this reduces to a known result of local algebra. (see for instance [B-S] theorem 2.1.17).

In particular one obtains:

COROLLARY. Let X be an n-dimensional complex manifold, $U \subset X$ a relatively compact Stein open set, $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$ and let $A = \mathrm{supp}\,\mathcal{F}$. If $\mathrm{prof}_A(\mathcal{E}) \geq n-1$ on U, then every extension $0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{E} \to 0$ on $X \setminus U$ can be continued to an extension on X.

Indeed, the statement follows by applying the previous corollary in the exact sequence:

$$\mathrm{Ext}^1(X; \mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(X \setminus U; \mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Ext}_c^2(U; \mathcal{E}, \mathcal{F}) .$$

REFERENCES

[B-S] C. Banica, O. Stanasila - Méthodes algébriques dans la théorie globale des espaces complexes, Editura Academiei et Gauthier-Villars, 1977.

[B1] A. Baran - Dualité pour l'hyperext sur les espaces complexes, C.R. Acad. Sc. Paris, 302, série 1, 17, 1986, p.627-630.

[B2] A. Baran - Duality for the hyperext on complex spaces, Rev. Roum. de Math. Pures et Appl., 31, no.2, 1986, p.95-120.

[B3] A. Baran - Un théorème de séparation pour l'hyperext, C.R. Acad. Sc. PAris, 301, série 1, 15, 1985, p. 747-750.

[B] A. Belkilani - Sur le complexe cotangent dans la géométrie analytique, Bull. Soc. Math. Roumanie, 3, 1985, p.195-214.

[D-P] J.M. Drezet, J. Le Potier - Fibrés stables et fibrés exceptionnels sur P_2, Ann. Ec. Norm. Sup. 18, 1985, p. 193-244.

[F] H. Flenner: Über Deformationen holomorpher Abbildungen, Habilitationsschrift, Osnabrück, 1978.

[F-K] O. Forster, K.Knorr - Ein Beweis des Grauertschen Bildgarbensatzes nach Ideen von B. Malgrange, Manuscripta Math. 5, 1971, p. 19-44.

[Go] R. Godement - Topologie algébrique et théorie des faisceaux, Hermann, Paris, 1964.

[G] V.D. Golovin - Local Duality and Injective Dimension, Rev. Roum. Math. Pures et Appl., 31, no.8, 1986, p. 95-120.

[H] R. Hartshorne - Residue and Duality, Lecture Notes 20, Springer Verlag, 1966.

[R-R] J.P. Ramis, G. Ruget - Complexes dualisants et théorèmes de dualité en géométrie analytique complexe, Publ. IHES no.38, 1970, p.77-91.

[R-R-V] J.P. Ramis, G.Ruget, J.-L.Verdier - Dualité rélative en géométrie analytique complexe, Inventiones Math., 13, 1971, p. 261-283.

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