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ISSN 0250 3638

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Nicolae MANOLACHE

PREPRINT SERIES IN MATHEMATICS
No.68/1986

BUCURESTI

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Nicolae MANOLACHE*)

December 1986

*) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.*

Syzygies of abelian surfaces embedded in $\mathbb{P}^4(\mathbb{C})$

Nicolae Manolache

Introduction

In this note we compute the shape of the minimal free resolution of the ideals of abelian surfaces in $\mathbb{P}^4$ (see Theorem 1). A natural idea is to use the symmetries of such a surface. But the computations become more efficient using the Horrocks-Mumford bundle E, which is strongly related to all abelian surfaces in $\mathbb{P}^4$ and has more symmetries. In fact we compute also a minimal resolution of E, i.e. of $\Gamma_*(E) = \bigoplus_n \Gamma(E(n))$ as an $S = \mathbb{C}[X_0, X_1, X_2, X_3, X_4]$-graded module (see Theorem 1') and this provides us with minimal resolutions for all $X = V(s), s \in \Gamma(E)$. Modulo automorphisms of $\mathbb{P}^4$ these give all locally complete intersection subschemes of $\mathbb{P}^4$ of dimension 2, degree 10 and $\omega_X \simeq O_X$. To prove this fact one uses the description of nilpotent structures given in [7] and the "uniqueness" of the Horrocks-Mumford bundle, shown in [2]. Some nonsingular such X's are described in [4] and there are also nilpotent schemes among them (cf. [5]). One obtains also minimal resolutions for all "surfaces" Y (algebraic schemes of dimension 2) which are zero sets of sections of E(n), any $n \geq 1$. They are locally complete intersections of degree $n^2+5n+10$, with $\omega_Y \simeq O_Y(2n)$ (see corollaries 3,4).

Preliminaries

By [4], any abelian surface X in $\mathbb{P}^4 = \mathbb{P}^4(\mathbb{C})$ is projectively equivalent to the zero set of a certain section s_X of a fixed vector bundle E on $\mathbb{P}^4$. From here, the moduli space of abelian surfaces in $\mathbb{P}^4$ has dimension 3 (cf. [4], 6.1.), hence the family of abelian surfaces in $\mathbb{P}^4$ has dimension 27. By the very construction of E, as given in [4], E has a group of symmetries of the form $N = H \rtimes SL_2(Z_5)$. Namely, if $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$, $V = Map(Z_5, C)$, $\varepsilon = \exp(2\pi i/5)$ and $\sigma, \tau \in SL_5(\mathbb{C})$ are given in $Aut_{\mathbb{C}}(V)$ by $(\sigma x)(k) = x(k+1)$, $(\tau x)(k) = \varepsilon^k x(k)$ for any $x \in V, k \in \mathbb{Z}_5$, then H is the group generated by σ and τ and it is realised as an extension

$$1 \longrightarrow \mu_5 \longrightarrow H \longrightarrow \mathbb{Z}_5 \times \mathbb{Z}_5 \longrightarrow 1$$

where μ_5 is the group of 5^{th} roots of 1. Then the normalizer N of H in $SL_5(\mathbb{C})$ is a semidirect product like above. If θ is the generator of the Galois group of $\mathbb{Q}(\varepsilon)$ over $\mathbb{Q}$, then we can take also $\theta : H \longrightarrow H$ given by $\theta(\varepsilon) = \theta^2$. The compositions $H \xrightarrow{\theta^i} H \longrightarrow Aut(V)$ give the representations $\theta^i V = V_i$ (i=o,1,2,3) of H. These, together with the 25 representations of $\mathbb{Z}_5 \times \mathbb{Z}_5$ give all irreducible representations of H. The table of characters for $SL_2(\mathbb{Z}_5)$ and the table of multiplication for them are given in the appendix, for easy reference. The first one, very well known is mentioned also in [4], the second one can be obtained as a standard exercise. The Horrocks-Mumford bundle E is obtained as the object of homology of a monad

$$V_1 \otimes O(2) \xrightarrow{p} W \otimes \Lambda^2 T \xrightarrow{q} V_3 \otimes O(3)$$

(i.e. p,q are locally split and qp=o), where W is a certain representation of degree 2 of $N/H = SL_2(\mathbb{Z}_5)$. The morphisms p,q being compatible with the action of N, the bundle E admits N as a group of symmetries. Remind also that here $\mathbb{P}^4$ is identified with the space representing the lines (through origin) of V and the action of N on O(1) is inherited from the natural action of $SL_5(\mathbb{C}) > N$. Because $\Gamma(O(1))$ = dual of $V = V_2$, one identifies $S := \mathbb{C}[X_o, X_1, X_2, X_3, X_4]$ with the symmetric algebra of V_2. In fact, by [1], N gives all the symmetries of E, in the sense that the stabilizer of E in $SL_5(\mathbb{C})$ is N.

The cohomology groups of E, as N-modules are computed in [4], and we recall them :

$H^o(E(k))=o$ for $k \leq -1, H^o(E)=T, H^o(E(n))=W' \otimes S^n V_2 - W \otimes V \otimes S^{n+1}V_2 +$
$+ U \otimes V_1 \otimes S^{n+2}V_2 - V_3 \otimes S^{n+3}V_2$, for $n \geq 1$;

$H^1(E(-3)) = V_3, H^1(E(-2)) = W' \otimes V_1$, $H^1(E(-1)) = W' \otimes V$,
$H^1(E) = W'$, $H^1(E(k)) = o$ for all other k ;

$H^2(E(-5)) = W$, $H^2(E(k)) = o$ for $k \neq -5$;

H^3 and H^4 of $E(k)$ are obtainable by duality.

One considers the element $\iota \in N$, given by $(\iota x)(k) = x(-k)$, whose image in $SL_2(\mathbb{Z}_5)$ is (-identity) and denote by G the group generated by H and ι . Then $G = H \rtimes \mathbb{Z}_2$. Again, it is standard to compute the character table of G (see [6] and the appendix). We have $H^o(E) = 4I$ as a G-module, so that, for any $s \in \Gamma(E)$ the zero set $X = V(s)$ is G-invariant. If I_X is the ideal of such an X then the cohomology of I_X follows easily from the cohomology of E (cf. [6]), via the exact sequence :

$$o \longrightarrow O \longrightarrow E \longrightarrow I_X(5) \longrightarrow o$$

$H^o(I_X(n)) = o$ for $n \leq 4$, $H^o(I_X(5)) = 3I$, $H^o(I_X(6)) = 6V_2$,
$H^o(I_X(7) = 13V_3 + 4V_3^{\#}$, $H^o(I_X(8)) = 23V_1 + 12V_1^{\#}$, $H^o(I_X(9)) =$
$= 38V + 24V^{\#}$, $H^o(I_X(1o)) = 19I + 2S + 2oZ$,etc.

$H^1(I_X(2)) = V_3, H^1(I_X(3)) = 2V_1^{\#}$, $H^1(I_X(4)) = 2V^{\#}$, $H^1(I_X(5)) =$
$=2S$ and all other H^1 are zero.

$H^2(I_X) = 2S$ and the others H^2 are zero.

$H^3(I_X(-5)) = 3I + 2S + 5Z$, $H^3(I_X(-4)) = 1oV_2 + 6V_2^{\#}$,
$H^3(I_X(-3)) = 5V_3 + 4V_3^{\#}$, $H^3(I_X(-2)) = 4V$, $H^3(I_X(-1)) = V$,
$H^3(I_X) = I$, $H^3(I_X(n)) = o$ for $n \geq 1$.

$H^4(I_X(-5)) = I$ and $H^4(I_X(n)) = o$ for $n \geq -4$.

Recall also that a surface X like above must have degree 1o and $\omega_X \simeq O_X$, as E has Chern classes $c_1=5, c_2=1o$.

Syzygies

For any subscheme X of $\mathbb{P}^4$ we denote by I(X) the homogeneous ideal $I(X) = \bigoplus_n H^o(I_X(n))$, and by S(X) the graded algebra $S(X) = \mathbb{C}[X_o,\ldots,X_4]/I(X) = S/I(X)$.

Theorem 1. The notations being those from above, for any $s \in \Gamma(E)$, if $X=V(s)$, then S(X) has a G-invariant minimal resolution over S, which, sheafified, is of the form :

$$o \longrightarrow 2S\otimes O(-1o) \longrightarrow 4V_1^{\#}\otimes O(-8) \longrightarrow (5V_3+2V_3^{\#})\otimes O(-7) \longrightarrow$$

$$\longrightarrow (3V_2\otimes O(-6))\oplus 3O(-5) \longrightarrow O \longrightarrow O_X \longrightarrow o$$

or, if we do not take into account the symmetry of X :

$$o \longrightarrow 2O(-1o) \longrightarrow 2oO(-8) \longrightarrow 35O(-7) \longrightarrow 15O(-6)\oplus 3O(-5) \longrightarrow O \rightarrow O_X \rightarrow o$$

Proof. E is 2-regular in the sense of Castelnuovo, since $H^i(E(j)) = o$ for $i \geqslant 1, i+j=2$. Then I_X is 7-regular and so $I_X(7)$ is generated by its global sections (cf. [8], lecture 14). We want to show that the elements of $\Gamma(I_X(5))$ are not subjected to linear relations (i.e. $\Gamma(O(1))\otimes\Gamma(I_X(5)) \longrightarrow \Gamma(I_X(6))$ is injective). As $\Gamma(I_X(5)) = 3I \subset \Gamma_H(O(5))$:= the space of H-invariant quintics, this will follow from the following lemma :

Lemma 1. There are no relations of degree 1 or 2 among the H invariant quintics in $S = \mathbb{C}[X_o,\ldots,X_4]$.

Proof. Simple checking, using the explicit form for a basis of $\Gamma_H(O(5))$ as given in [4]: $A = \sum X_o^5$, $B = \sum X_o^3X_1X_2$, $B' = \sum X_o^3X_2X_3$, $C = \sum X_o^2X_2^2X_1$, $C' = \sum X_o^2X_1^2X_3$, $D = \sum X_oX_1X_2X_3X_4$, the sum being done over the powers of σ . □

Taking succesively minimal surjections and using a "certain order"-regularity of their kernels, one obtains that I_X has a G-invariant minimal resolution of the form :

$$(*)\quad o \to L_4 = \begin{matrix} 2SO(-1o) \\ \oplus \\ AO(-9)\end{matrix} \xrightarrow{a} L_3 = \begin{matrix} AO(-9) \\ \oplus \\ (4V_1^{\#}+B)O(-8)\end{matrix} \xrightarrow{b} L_2 =$$

$$= \begin{matrix} BO(-8) \\ \oplus \\ (5V_3+2V_3^{\#}+C)O(-7)\end{matrix} \xrightarrow{c} L_1 = \begin{matrix} CO(-7) \\ \oplus \\ 3V_2O(-6) \\ \oplus \\ 3O(-5)\end{matrix} \longrightarrow I_X \longrightarrow o$$

Here we use notations like $V_2O(-6)$ instead of $V_2 \otimes O(-6)$, etc., and A, B, C are undetermined representations of G. We shall show A=B==C=o.

By the minimality of the resolution, a, b are of the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$ and c is of the form $\begin{pmatrix} * & 0 \\ * & * \\ * & * \end{pmatrix}$. Indeed, we have the following easy and well-known lemma :

Lemma 2. If X is a scheme and $A : P \longrightarrow M$, $B : P \longrightarrow N$, $C : N \longrightarrow M$, $U : N \longrightarrow N$ are O_X-modules homomorphisms, with U an isomorphism, then the exact sequence :

$$o \to \mathrm{Ker}\lambda = K \longrightarrow P \oplus N \xrightarrow{\lambda} M \oplus N \longrightarrow Q = \mathrm{Coker}\lambda \longrightarrow o,$$

where $\lambda = \begin{pmatrix} A & C \\ B & U \end{pmatrix}$, is the direct sum of the exact sequences :

$$o \longrightarrow K \longrightarrow P \xrightarrow{\mu} M \longrightarrow Q \longrightarrow o$$

$$o \longrightarrow o \longrightarrow N \xrightarrow{U} N \longrightarrow o \longrightarrow o$$

where $\mu = A - CU^{-1}B$.

Proof. All follows from the commutative diagram

$$\begin{array}{ccc} P \oplus N & \xrightarrow{\lambda} & M \oplus N \\ \rho \downarrow & & \downarrow \sigma \\ P \oplus N & \xrightarrow{\nu} & M \oplus N \end{array}$$

where $\nu = \begin{pmatrix} \mu & 0 \\ 0 & U \end{pmatrix}$, $\rho = \begin{pmatrix} id & 0 \\ U^{-1}B & id \end{pmatrix}$, $\sigma = \begin{pmatrix} id & -CU^{-1} \\ 0 & id \end{pmatrix}$ □

The resolution (*) of I_X gives, via the exact sequence (1), a minimal free resolution of E of the form :

$$(*)'\quad o \to L_4 \xrightarrow{a} L_3 \xrightarrow{b} L_2 \xrightarrow{c'} L_1' = \begin{matrix} CO(-7) \\ \oplus \\ 3V_2O(-6) \\ \oplus \\ 4O(-5)\end{matrix} \to E(-5) \to o$$

This resolution must be in fact N-invariant. $H^0(E) = T, H^0(E(1)) =$

$=(T+U)V_2$ shows that $L_1^{\circ} = CO(-7) \oplus UV_2O(-6) \oplus TO(-5)$.

Lemma 3. $H^0(E(2)) = (L+2T+T^{\#}+U) \otimes V_3$

Proof. We have : $H^0(E(2)) = W \otimes S^2V_2 - W \otimes V \otimes S^3V_2 + U \otimes V_1 \otimes S^4V_2 - V_3 \otimes S^5V_2$. One uses the formulas from the appedix, with the remark that $V_3 \otimes Z$ can be computed with the substitution $Z = V \otimes V_2 - I$. □

Using $H^0(E(2))$ we get $L_2 = BO(-8) \oplus (LV_3+WV_3+C)O(-7)$. In order to determine the representation of N which restricted to G gives the term $4V_1^{\#}$ in L_3 , observe that this $4V_1^{\#}$ is in fact the excedent of the part $H^0((5V_3+2V_3^{\#}+C)O(1))$ of $H^0(L_2(8))$ compared with $H^0(L_1(8))$. As N-modules this means the excedent of $H^0((LV_3+WV_3+C)O(1))$ compared to $H^0(L_1(8))$. Using the appropriate formulas from the appendix, one sees that $4V_1^{\#}$ comes from $T^{\#}V_1$. If we separate the sequence $(\ast)'$ into three short exact sequences, one obtains taking their cohomology $H^4(L_4(5)) \simeq H^1(E) = W'$.

Thus, the resolution $(\ast)'$, twisted by $O(5)$, becomes :

$$(\;)'' \quad o \to \begin{matrix} W'O(-5) \\ \oplus \\ AO(-4) \end{matrix} \longrightarrow \begin{matrix} AO(-4) \\ \oplus \\ (T^{\#}V_1+B)O(-3) \end{matrix} \longrightarrow \begin{matrix} BO(-3) \\ \oplus \\ (LV_3+WV_3+C)O(-2) \end{matrix} \longrightarrow$$

$$\longrightarrow \begin{matrix} CO(-2) \\ \oplus \\ UV_2O(-1) \\ \oplus \\ TO \end{matrix} \longrightarrow E \longrightarrow o$$

We want to show A=B=C=o. It is suffucient to show A = o. Indeed, A=o and the minimality of $(\ast)''$ implies that the kernel of $L_1 \longrightarrow E$ contains a factor $BO(-3)$, namely we have an exact sequence

$$o \longrightarrow BO(-3) \oplus Q \longrightarrow L_1(5) \longrightarrow E \longrightarrow o$$

where from : $o = H^3(E(-2)) \cong H^4(BO(-5) \oplus Q(-2)) \geqslant B$. Then B=o and also C=o.

Assume $A \neq o$. A limitation for C can be obtained using again Lemma 1. Indeed, the injectivity of the natural map $\Gamma(I_X(5)) \otimes \Gamma(O(2)) \to \Gamma(O(7))$ and the exact sequence (1) shows the injectivity of $\Gamma(E) \otimes \Gamma(O(2)) \longrightarrow \Gamma(E(2))$. As $\Gamma(E(2)) = (L+2T+T^{\#}+U)V_3$ and $\Gamma(E) \otimes \Gamma(O(2)) = T \otimes S^2 V_2 = TU'V_3 = (L+T+U)V_3$, follows $C \leqslant (T+T^{\#})V_3$

We separate the exact sequence $(\ast)$" twisted by $O(-1)$ into short exact sequences and take the cohomology of them. With the information obtained from the others (one uses the cohomology of E), the cohomology of the first gives the exact sequence :

$$o \to W'V \to W'V \oplus A \xrightarrow{\alpha_1} A \to o$$

where $\alpha_1 = H^4(a(-1))$ is of the form $(\ast \ \ o)$. Then $A = W'V$.

A similar computation for $(\ast)$" twisted by $O(-2)$ gives the exact sequence :

$$o \to W'V_1 \to \begin{matrix} (T^{\#}V_1+W'V_1) \\ \oplus \\ (T^{\#}V_1+W'V_1+TV_1) \end{matrix} \xrightarrow{\alpha_2} \begin{matrix} (T^{\#}V_1+W'V_1+TV_1) \\ \oplus \\ (T^{\#}V_1+B) \end{matrix} \xrightarrow{\beta_2} B \to o$$

where $\alpha_2 = H^4(a(-2))$ is of the form $\begin{pmatrix} \ast & o \\ \ast & \ast \end{pmatrix}$ and β_2 being $H^4(b(-2))$ corestricted to its image is of the form $\beta_2 = (\ast \ \ o)$. One obtains from here $TV_1 \leqslant B \leqslant TV_1 + W'V_1 + T^{\#}V_1$.

Playing the same game with $(\ast)$" twisted by $O(-3)$, one obtains firstly $C \leqslant BV$ and secondly an exact sequence :

$$o \to V_3 \to \begin{matrix} W'(L+W')V_3 \\ \oplus \\ W'U'VV_1 \end{matrix} \xrightarrow{\alpha_3} \begin{matrix} W'U'VV_1 \\ \oplus \\ (T^{\#}VV_1+BV) \end{matrix} \xrightarrow{\beta_3} \begin{matrix} (BV-C) \\ \oplus \\ (LV_3+WV_3+C) \end{matrix} \to o$$

where $\alpha_3 = \begin{pmatrix} \ast & o \\ \ast & \ast \end{pmatrix}$, $\beta_3 = \begin{pmatrix} \ast & o \\ \ast & \ast \end{pmatrix}$. The special form of α_3, β_3 gives a decomposition of this long exact sequence into short ones, which, introducing new representations $K_o, \ldots, K_4$, can be written as equalities :

$$K_o+K_1+\delta V_3 = W'(L+W')V_3$$
$$K_2+(1-\delta)V_3 = W'U'VV_1$$
$$K_o+K_3+BV-C = W'U'(U+W')V_3$$
$$K_1+K_2+K_4 = T^{\#}(U+W')V_3+BV$$
$$K_3+K_4 = LV_3+WV_3+C$$

with $\delta = o$ or 1 .

As we have seen, $B = (T+\varepsilon W'+\varphi T^{\#})V_1$, ε, φ being also o or 1 . Using this and the multiplication table from the appendix, the third relation becomes :

$$K_o+K_3+TV_3+\varphi(L+M+U'+W+T^{\#})V_3+\varepsilon(I+M+U')V_3 = (I+M+T^{\#}+U')V_3 + C$$

From here $\varphi = o$, as the right term contains no L. Then $C = (T+\alpha T^{\#})V_3$,with $\alpha = o$ or 1. The inequality $C \leqslant BV = (T+\varepsilon W')(U+W')V_3$, in which the second term has no $T^{\#}V$ gives C=T.

We continue the game with (*)" twisted by O(-4). Taking into account that $H^j(E(-4)) = o$ for all j , one obtains a surjection :

$$H^4(BO(-7))\oplus H^4((LV_3+MV_3+C)O(-6)) \xrightarrow{\gamma} H^4(CO(-6))\oplus H^4(UV_3O(-5))$$

with $\gamma = \begin{pmatrix} x & o \\ * & * \end{pmatrix}$. Then we must have $H^4(BO(-7)) \geqslant H^4(CO(-6))$, which means $B(H^o(O(2)))^\vee \geqslant C(H^o(O(1)))^\vee$, or,equivalently : $U'(T+\varepsilon W')(U+W')V_2 \geqslant T(U'+W)V_2$. This relation is impossible,because the second member contains a factor $W'V_2$ and the fiest does not.

By this we have proved Theorem 1 and also :

Theorem 1'. The Horrocks-Mumford bundle has an N-invariant minimal resolution of the form :

$$o \to W'O(-5) \to T^{\#}V_1O(-3) \to (L+W)V_3O(-2) \to UV_2O(-1)\oplus TO \to E \to o$$

Corollary 1. If E is the Horrocks-Mumford bundle,then E(1) is generated by its global sections.

Remark. By [4] , E is generated by its global sections outside the set of 25 lines whose ideal is generated by $\Gamma_H(O(5))$.

Corollary 2. If X is a locally complete intersection closed subscheme of $\mathbb{P}^4$, of dimension 2, degree 1o and $\omega_X \simeq O_X$, then the syzygies of X look like :

$$0 \to 2O(-10) \to 20O(-8) \to 35O(-7) \to 15O(-6) \oplus 3O(-5) \to O \to O_X \to 0$$

Proof. By the correspondence between l.c.i. subschemes of codimension 2 in $\mathbb{P}^4$ and vector bundles of rank 2 on $\mathbb{P}^4$ (see [9]), we have a rank 2 vector bundle F with $c_1(F) = 5$, $c_2(F) = 10$ and an exact sequence :

$$0 \to O \to F \to I_X(5) \to 0 .$$

We want to show that F is stable, i.e. $H^0(F(-3)) = 0$. In any case, since a X like above cannot be contained in a hyperplane, $H^0(F(-4)) = 0$. Assume $H^0(F(-3)) \neq 0$. Then any section of F(-3) would vanish in codimension 2 and its scheme of zeroes Y would have degree = $= c_2(F(-3)) = 4$ and $\omega_Y \simeq O_Y(-6)$.

Lemma 3. In $\mathbb{P}^4$ there is no l.c.i. subscheme Y of dimension 2, degree 4 and $\omega_Y \simeq O_Y(-6)$.

This lemma settles the corollary 2. Indeed, by it F is stable and all stable rank 2 vector bundles on $\mathbb{P}^4$ with $c_1 = 5$, $c_2 = 10$ are projectively equivalent with the Horrocks-Mumford bundle E, by a theorem of Decker and Schreyer (cf. [2]). Then the minimal resolution of E gives the minimal resolution of I_X.

Proof of Lemma 3. Any irreducible component of Y must cut the other components along a curve, because otherwise Y would be disconnected removing a point, contradicting the Cohen-Macaulayness of Y (see [3], 3.9.). The section of Y with a generic hyperplane must be a curve $C \subset P^3$ of degree 4, with $\omega_C \simeq O_C(-5)$, hence with the Hilbert polynomial $\chi_C(n) = 4n+10$. This shows that C, and also Y, must have a nilpotent structure. Observe that C cannot contain a line or a conic as an irreducible component C_1 such that in the points of C_1 not in the other components of C to have $C = C_1$. Indeed we should have an exact sequence :

$$o \to I_C/I_{C_1}I_C \to I_{C_1}/I_{C_1}^2 \to I_{C_1}/(I_{C_1}^2+I_C) = P \to o$$

where P is concentrated in a finite set if points;then $\omega_C|_{C_1} \simeq \omega_{P^3}\otimes \det^{-1}(I_{C_1}/I_{C_1}^2)\otimes \det P \simeq \omega_{C_1}\otimes \det P \not\simeq \omega_{C_1}(-5)$. This shows that Y cannot contain a plane or a quadric as an irreducible component Y_1 such that in the points of Y_1 not in other components to have $Y = Y_1$. It follows that Y is a l.c.i. structure of degree 4 on a plane or on a quadric (may be degenerated or singular). By [7], Remarks 3,4(p. 564),if the support is a quadric X,the structure Y on it can be obtained by the so-called "Ferrand doubling", i.e. there is a line bundle L on X such that to exist an exact sequence :

$$o \to I_Y/I_X^2 \to I_X/I_X^2 \to \omega_X\otimes L \to o$$

where $L\simeq \omega_Y^{-1}\Big|_X = O_X(6)$. Taking into account that $I_X/I_X^2 \simeq$ $\simeq O_X(-1)\oplus O_X(-2)$ and $\omega_X \simeq O_X(-3)$, one sees that the exact sequence from above is not possible. (In fact the only double structures on globally complete intersection surfaces in $\mathbb{P}^4$ are globally complete intersections.) If the support of Y is a plane X,then by [7] Theorem 1, the structure Y is obtainable in a process described by exact sequences :

$$o \to I_{Y_1}/I_X^2 \to I_X/I_X^2 \to E_o \to o$$

$$o \to I_{Y_{r+1}}/I_XI_{Y_r} \to I_{Y_r}/I_XI_{Y_r} \to E_r \to o$$

where E_o, E_r $(r = 1,2,\ldots,t)$ are vector bundles on X, $Y_{t+1} = Y$ and $\mathrm{rank}E_o+\ldots+\mathrm{rank}E_r = 3$. Using the constructions described in [7] , one sees that all l.c.i. structures of degree 4 on a plane in $\mathbb{P}^4$ are globally complete intersections.Lemma is proved. □

Corollary 3. If Y is a locally complete intersection subscheme of $\mathbb{P}^4$ of dimension 2, degree 16 with $\omega_Y \simeq O_Y(2)$ and $\Gamma(I_Y(2)) = o$, then O_Y has a minimal resolution of the form :

$$o \to 2O(-1o) \to 2oO(-8) \to 35O(-7) \to 14O(-6) \oplus 4O(-5) \to O \to O_Y \to o$$

Proof. A surface with the above invariants gives rise to an exact sequence

$$o \to O \to E(1) \to I_Y(7) \to o$$

with E a vector bundle with Chern classes $c_1=5, c_2=1o$. One has $H^o(E(-4)) = H^o(I_Y(2)) = o$ and then $H^o(E(-3)) = o$ by lemma 3. This shows that E is stable and one applies again [2] .

Remark. In fact the condition $\Gamma(I_Y(2)) = o$ is superfluous. One can prove this observing that, if $H^o(I_Y(2)) \neq o$, then $H^o(E(-4)) \neq o$ and $H^o(E(-5)) = H^o(I_Y(1)) = o$, so that any section of E(-4) would vanish in codimension 2. This would give an extension

$$o \to O \to E(-4) \to I_Z(-3) \to o$$

with Z a surface of degree 6 with $\omega_Z \simeq O_Z(-8)$. The fact that E(1) has sections vanishing in codimension 2 implies $H^o(I_Z(2)) \neq o$. One shows that there is no surface Z with the above properties. Firstly, one shows like in lemma 3 that Z has no irreducible component Z_1 such that $Z = Z_1$ in the points outside the other irreducible components. This shows that Z is a multiple structure, and one analyses the various possibilities depending on the irreducible components of Y_{red} . Some of this are directly excluded using [7] and the others cutting Z with a generic hyperplane H and showing that the multiple curve $Z \cap H \subset H$, of degree 6, with $\omega \simeq O(-7)$ has contradictory properties.

Corollary 4. If Y is a locally complete intersection surface in $\mathbb{P}^4$ of degree $n^2+5n+1o$, with $\omega_Y \simeq O_Y(2n)$ and $\Gamma(I_Y(n+1)) = o$, then O_Y has a minimal resolution of the form :

$$o \to 2O(-n-1o) \to 2oO(-n-8) \to O(-2n-5) \oplus 35O(-n-7) \to 15O(-n-6) \oplus 4O(-n-5) \to O \to O_Y \to o$$

Proof. Like above, we may assume the exact sequence

$$o \longrightarrow \mathcal{O} \longrightarrow E(n) \longrightarrow I_Y(2n+5) \longrightarrow o$$

where E is the Horrocks-Mumford bundle. If $q : 15\mathcal{O}(-n-6) \oplus 4\mathcal{O}(-n-5) \longrightarrow E(-n-5)$ is the minimal surjection from Theorem 1' twisted by $\mathcal{O}(-n-5)$, we have a minimal surjection $p: 15\mathcal{O}(-n-6) \oplus 4\mathcal{O}(-n-5) \longrightarrow I_Y$ and an exact sequence :

$$o \longrightarrow Ker(q) \longrightarrow Ker(p) \longrightarrow \mathcal{O}(-2n-5) \longrightarrow o$$

which splits. From here one obtains the resolution.

APPENDIX

I The character table of $SL_2(Z_5)$

C_i	$\begin{pmatrix}1&o\\o&1\end{pmatrix}$	$\begin{pmatrix}-1&o\\o&-1\end{pmatrix}$	$\begin{pmatrix}2&o\\o&3\end{pmatrix}$	$\begin{pmatrix}2&2\\-1&2\end{pmatrix}$	$\begin{pmatrix}3&2\\-1&3\end{pmatrix}$	$\begin{pmatrix}1&1\\o&1\end{pmatrix}$	$\begin{pmatrix}1&2\\o&1\end{pmatrix}$	$\begin{pmatrix}-1&1\\o&-1\end{pmatrix}$	$\begin{pmatrix}-1&2\\o&-1\end{pmatrix}$	
$\#C_i$	1	1	3o	2o	2o	12	12	12	12	
χ_1	1	1	1	1	1	1	1	1	1	I
χ_5	5	5	1	-1	-1	o	o	o	o	L
χ_6	6	-6	o	o	o	1	1	-1	-1	M
χ_4	4	4	o	1	1	-1	-1	-1	-1	T
$\chi_4^{\#}$	4	-4	o	1	-1	-1	-1	1	1	T
χ_3	3	3	-1	o	o	$-\eta$	$-\eta'$	$-\eta$	$-\eta'$	U
χ_3'	3	3	-1	o	o	$-\eta'$	$-\eta$	$-\eta'$	$-\eta$	U'
χ_2	2	-2	o	-1	1	η	η'	$-\eta$	$-\eta'$	W
χ_2'	2	-2	o	-1	1	η'	η	$-\eta'$	$-\eta$	W'

Here $\varepsilon = \exp(2\pi i/5)$, $\eta = \varepsilon + \varepsilon^4$, $\eta' = \varepsilon^2 + \varepsilon^3$. In computing some formulas from this appendix one uses also the explicit form of the elements in $SL_5 C$ inducing $\bar{\lambda} = \begin{pmatrix}-1&1\\o&-1\end{pmatrix}$, $\bar{\mu} = \begin{pmatrix}2&o\\o&3\end{pmatrix}$, $\bar{\nu} = \begin{pmatrix}1&2\\o&1\end{pmatrix}$ $\bar{\iota} = \begin{pmatrix}-1&o\\o&-1\end{pmatrix}$ in $SL_2(Z_5) \simeq N/H$, namely $\lambda x(i) = \varepsilon^{3i^2+3i} x(i)$, $\mu x(i) = -x(2i)$, $\nu x(i) = \varepsilon^{i^2} x(i)$, $\iota x(i) = x(-i)$.

II Multiplication table for irreducible representations of $SL_2(Z_5)$

$L\otimes L = I+2L+2T+U+U'$, $L\otimes M = 3M+2T^{\#}+W+W'$, $L\otimes T = 2L+T+U+U'$, $L\otimes T^{\#} = 2M+T^{\#}+W+W'$, $L\otimes U = L+T+U+U'$, $L\otimes U' = L+T+U+U'$, $L\otimes W = M+T^{\#}$, $L\otimes W' = M+T^{\#}$;

$M\otimes M = I+3L+2T+2U+2U'$, $M\otimes T = 2M+2T^{\#}+W+W'$; $M\otimes T^{\#} = 2L+2T+U+U'$, $M\otimes U = 2M+T^{\#}+W'$, $M\otimes U' = 2M+T^{\#}+W$, $M\otimes W = L+T+U'$, $M\otimes W' = L+T+U$;

$T\otimes T = I+L+T+U+U'$, $T\otimes T^{\#} = 2M+T^{\#}$, $T\otimes U = L+T+U'$, $T\otimes U' = L+T+U$, $T\otimes W = M+W$, $T\otimes W' = M+W$;

$T^{\#}\otimes T^{\#} = I+L+T+U+U'$, $T^{\#}\otimes U = M+T^{\#}+W$, $T^{\#}\otimes U' = M+T^{\#}+W'$, $T^{\#}\otimes W = L+U$, $T^{\#}\otimes W' = L+U'$;

$U\otimes U = I+L+U$, $U\otimes U' = L+T$, $U\otimes W = T^{\#}+W$, $U\otimes W' = M$;

$U'\otimes U' = I+L+U'$, $U'\otimes W = M$, $U'\otimes W' = T^{\#}+W'$;

$W\otimes W = I+U$, $W\otimes W' = T$;

$W'\otimes W' = I+U'$.

III Some formulas over the normalizer N of H in $SL_5(C)$.

The formulas given here involve the irreducible representations of $SL_2(Z_5) = N/H$, the irreducible representations V_i , and $Z = V\otimes V_2 - I$, (which is irreducible over N and decomposes over H in the sum of all 24 nontrivial characters of $Z_5\times Z_5$):

$V\otimes V = (U'+W)V_1$, hence : $V_1\otimes V_1 = (U+W')V_2$, $V_2\otimes V_2 = (U'+W)V_3$, $V_3\otimes V_3 = (U+W')V$;

$V\otimes V_1 = (U+W')V_3$, hence $V_1\otimes V_2 = (U'+W)V$; $V_2\otimes V_3 = (U+W')V_1$, $V_3\otimes V = (U'+W)V_2$;

$\Lambda^2 V_i = W\otimes V_{i+1}$, $\Lambda^3 V_i = W\otimes V_{i+3}$, $\Lambda^4 V_i = V_{i+2}$;

$S^2 V_i = U'\otimes V_{i+1}$, $S^3 V = (L+W')\otimes V_{i+3}$, $S^4 V_i = (T+T^{\#}+U+U')\otimes V_{i+2}$,

$S^5 V_i = U+U'+Z\otimes(U+U'-I)$.

IV The character table of G

$\{\alpha\}$	$C_{m,n}$	C_α	
1	1	1	I
$5\theta^i(\ )$	o	$\theta^i(\alpha)$	V_i
1	1	-1	S
2	$\varepsilon^{sn+tm}+\varepsilon^{-sn-tm}$	o	$Z_{s,t}$
$5\theta^i(\alpha)$	o	$-\theta^i(\alpha)$	$V_i^{\#}$

If we write $(\alpha,m,n)\iota^k$ for $\alpha\varepsilon^{2mn}\sigma^m\tau^n\iota^k$, ($\alpha\in\mu_5$, $m,n\in\mathbb{Z}_5$, $k = o,1$), then $\{\alpha\}$ is the class containing only the central element $\alpha\in\mu_5$, there are 12 classes $C_{m,n} = \{(\alpha,m,n),(\alpha,-m,-n)\mid\alpha\in\mu_5\}$ and 5 classes $C_\alpha = \{(\alpha,m,n)\iota\mid m,n\in\mathbb{Z}_5\}$.

We have the following formulas :

$V_i\otimes V_i = 3V_{i+1}\oplus 2V_{i+1}^{\#}$, $V_i\otimes V_{i+1} = 3V_{i+3}\oplus 2V_{i+3}^{\#}$; $V_i\otimes V_{i+2} =$
$= I\oplus Z$, $V_i\otimes S = V_i^{\#}$, $V_i\otimes Z = 12V_i\oplus 12V_i^{\#}$;

$S\otimes S = I$, $S\otimes Z = Z$

$Z\otimes Z = 12I\oplus 12S\oplus 23Z$

$\Lambda^2V_i = 2V_i^{\#}$, $\Lambda^3V_i = 2V_{i+3}^{\#}$, $\Lambda^4V_i = V_{i+2}$

$S^2V_i = 3V_{i+1}$, $S^3V_i = 5V_{i+3}\oplus 2V_{i+3}^{\#}$, $S^4V_i = 1oV_{i+2}\oplus 4V_{i+2}^{\#}$,
$S^5V_i = 6I\oplus 5Z$, $S^6V_i = 26V_i\oplus 16V_i^{\#}$, $S^7V_i = 38V_i\oplus 28V_i^{\#}$, etc.

(Note that Z decomposes over G into the sum of all $Z_{s,t}$.)

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Department of Mathematics, INCREST, Bd.Păcii 220,
79622 Bucharest, Romania