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## 0. Introduction

One usually understands by Lefschetz theory the study of the topology of a smooth projective complex variety $X \subset P^N$ via its hyperplane sections $X_\lambda = X \cap H_\lambda$, where $\{H_\lambda\}_{\lambda \in P^1}$ is a generic pencil of hyperplanes (a so called Lefschetz pencil).

A clear and complete presentation from the topological point of view of this setting can be found in the paper of K. Lamotke [L].

In the present paper we endeavour a similar study for an affine smooth variety $V \subset C^N$. The main new difficulty is that we are no longer dealing with proper mappings and hence we cannot use Ehresmann's fibration Theorem.

There are two possible approaches to overcome this difficulty. The first is to construct and integrate explicit vector fields (to get the local triviality of some mappings). The second approach is to consider the projective closure $X = \overline{V} \subset P^N$ of the affine variety $V \subset C^N \subset P^N$. If we denote by $H_\infty = P^N \setminus C^N$ the hyperplane at infinity, then X may have singularities on its part at the infinity $X_\infty = X \cap H_\infty$. Thus we are forced to consider first a regular stratification of the projective variety X and then apply the Isotopy Lemmas of Thom-Mather to get local triviality (see for instance [Ha - Lê]).

We have chosen to develop here the first approach since it yields a precise description fo the "bad" hyperplane section (or hyperplane pencils) which should be avoided.

Our results contain as a special case (and complete) the work of Broughton [B] on the topology of the fibers of a polynomial mapping $f: C^N \longrightarrow C$ satisfying a certain condition at infinity (f is a tame polynomial in the sense of definition 1.11 below).

The paper is organized as follows. In section 1 we introduce the basic notations and state our main results.

The other sections contain the proofs of these results as well as some technical lemmas, corollaries and further examples.

To complete the picture, the reader may enjoy consulting some other recent approaches to Lefschetz Theory (especially to the Hyperplane Section Theorems) as for instance $[H_1]$, $[H_2]$, [H - Lê].

1. The main results

Let $V \subset C^N$ be a closed connected nonsingular algebraic variety with dim V=n. As above, we let $X=\overline{V}$ denote the projective closure of V in $P^N$. The coordinates $[z_1:\ldots:z_{N+1}]$ are chosen such that $(z_1,\ldots,z_N)$ are coordinates on $C^N \subset P^N$ and the hyperplane at the infinity $H_\infty$ is given by $z_{N+1}=0$. Hence the corresponding point to the hyperplane $H_\infty$ in the dual projective space $\hat{P}^N$ is $\infty=[0:\ldots 0:1]$.

The construction of the dual variety $\hat{X} \subset \hat{P}^N$ is well-known [L] and runs as follows. Let us consider the sets:

$$D_{X,V} = \{(z,w) \in P^N \times \hat{P}^N;\ z \in V \text{ and } H_w \supset T_z V\},$$

$D_X = \overline{D_{X,V}}$ (the closure in $P^N \times \hat{P}^N$) and put

$\hat{X} = pr_2(D_X)$, where $pr_2$ is the second projection.

To obtain the dual of the affine variety $V \subset C^N$ we proceed similarly. The affine hyperplanes in $C^N$ are parametrized by the open set:

$$A = \hat{P}^N - \{\infty\} = \{[w:c] = [w_1:\ldots:w_N:c];\ w_i \neq 0 \text{ for some } i\}$$

in the obvious way

$$H_{[w:c]} = \{z \in C^N,\ \langle z,w\rangle + \bar{c} = 0\}, \text{ where}$$

$\langle a,b\rangle = \sum_{i=1,N} a_i \bar{b}_i$ stands for the usual inner product on $C^N$.

The projection $p : A \to P^{N-1}$, $[w:c] \mapsto [w]$ associates to a hyperplane its direction and a fiber $p^{-1}([w])$ will be regarded as an affine pencil of hyperplanes in $C^N$.

We define

$$D_V = \{(z,[w:c]) \in V \times A\ ;\ H_{[w:c]} \supset T_z V\} \text{ and take}$$

$\hat{V} = pr_2(D_V) \subset A$ be the dual of the affine variety $V$. To describe the behaviour at the infinity of the affine variety $V$, we have to consider in addition some other subvarieties in $A$ and $P^{N-1}$. Namely, we define

$$V'_\infty = \{(z,[w:c]) \in D_X \subset P^N \times \hat{P}^N\ ;\ z_{N+1} = 0\}$$

$$V_\infty = pr_2(V'_\infty) \subset \hat{X},\ W_1 = p(V_\infty \setminus \{\infty\}) \text{ and } V_1 = p^{-1}(W_1)$$

One can think of $V_\infty$ as the set of limits of tangent hyperplanes $H_{[w:c]} \supset T_z V$, where $z$ goes to infinity.

In order to handle the limits of asymptotic hyperplanes, we introduce also the following subvarieties $V_2=C_\infty(\hat{X})-\{\infty\}\subset A$ and $W_2=p(V_2)$, where $C_\infty(\hat{X})$ denote the projective tangent cone t the dual variety $\hat{X}$ at the point $\infty$. By convention, when $\infty\notin X$ we take $V_2=W_2=\phi$.

If Z is a constructible subset in A, then $\bar{Z}$ will denote its closure (in Zariski/complex strong topology) in A.

The first result gives some basic properties of the vari varieties introduced above to parametrize the "bad" hyperplan sections of the variety V.

<u>Proposition 1.1.</u>

a) $\dim \hat{V}=\dim \hat{X}\leq N-1$, $\dim V_i\leq N-1$, $\dim W_i\leq N-2$ for $i=1,2$.

b) $\bar{\hat{V}} = \hat{X} -\{\infty\} =\hat{V}\cup(V_\infty-\{\infty\})\subset\hat{V}\cup V_1$.

c) $\bar{V}_1\subset V_1\cup V_2$.

d) $[w:c]\notin\hat{V} \Rightarrow V\cap H_{[w:c]}$ is nonsingular.

e) $[w:c]\in\hat{V}\setminus V_\infty \Rightarrow V\cap H_{[w:c]}$ has only isolated singularities

We state now the results concerning the fibrations obtai ned by taking hyperplane sections of the variety V. There are two possibilities to get such fibrations. The first one is to consider only the sections belonging to an affine pencil of hyperplanes.

More precisely, for a point $[w]\in P^{N-1}$ we chose a represe tative $w\in C^N$ with $\|w\|=1$. Then we identify C with $p^{-1}([w])$ v the map $i_w:C\to p^{-1}([w])$, $c\mapsto[w:c]$, and we put $\Lambda_w=i_w^{-1}(\bar{\hat{V}})$. Note that $\Lambda_w$ is a closed constructible subset in C and hence eithe $\Lambda_w=C$ or $\Lambda_w$ is a finite set of points (and the latter case ho for [w] in some dense Zariski open subset in $P^{N-1}$).

The function $P_w:V\to C$, $P_w(z)=-\langle w,z\rangle$ has as fibers $P_w^{-1}$(c exactly the hyperplane sections $V\cap H_{[w:c]}$.

<u>Theorem 1.2.</u> Assume that $\Lambda_W \neq C$. Then

i) $P_W | V \setminus P_W^{-1}(\Lambda_W) : V \setminus P_W^{-1}(\Lambda_W) \to C \setminus \Lambda_W$

is a $C^\infty$ locally trivial fibration.

ii) If $[w:c] \in \hat{V} \setminus V_\infty$, then for any sufficiently small disc $D_c$ centered at the point $c$ there is a deformation retract $r : P_W^{-1}(D_c) \to P_W^{-1}(c)$.

<u>Remark 1.3.</u>

When $[w] \notin W_1 \cup W_2$ and $c_1, c_2 \notin \Lambda_W$, we get in the proof of this Theorem a diffeomorphism $\varphi : P_W^{-1}(c_1) \to P_W^{-1}(c_2)$ which extends naturally to a homeomorphism of the projective closures $\overline{P_W^{-1}(c_1)} \to \overline{P_W^{-1}(c_2)}$.

A second possibility to get fibration theorems is to consider the family of all "good" hyperplane sections. Namely, consider the algebraic variety $U = \{(z, [w:c]) \in V \times A;\ z \in V \cap H_{[w:c]}\}$ and note that $U$ is a $P^{N-1}$ - bundle over $V$ via the first projection. The second projection $P_2 : U \to A$ gives rise to the following commutative diagram

$$(1.4) \qquad \begin{array}{ccccc} V & \xrightarrow[\sim]{j_W} & P_2^{-1}(P^{-1}([w])) & \hookrightarrow & U \\ \downarrow P_W & & \downarrow P_2| & & \downarrow P_2 \\ C & \xrightarrow[\sim]{i_W} & P^{-1}([w]) & \hookrightarrow & A \end{array}$$

where $i_W(c) = [w:c]$, $j_W(c) = (z, [w:P_W(z)])$. This shows that the maps $P_W$ corresponding to various pencils embed all in the map $P_2$. Note that the dual variety $\hat{V}$ is precisely the set of critical values for the map $P_2$.

<u>Theorem 1.5.</u> The mapping

$$P_2|_{U \setminus P_2^{-1}(\hat{V} \cup V_1 \cup V_2)} : U \setminus P_2^{-1}(\hat{V} \cup V_1 \cup V_2) \longrightarrow A \setminus (\hat{V} \cup V_1 \cup V_2)$$

is a $C^\infty$ locally trivial fibration.

<u>Remark 1.6.</u>

When $[w] \notin W_1 \cup W_2$, the firbation $P_w|V \setminus P_w^{-1}(\Lambda_w)$ from (1.2) is the restriction of the fibration $P_2|U \setminus P_2^{-1}(\hat{V} \cup V_1 \cup V_2)$ over the line $P^{-1}([w])$. In particular, the corresponding fibers (i.e. hyperplane sections) are diffeomorphic. When $[w] \in W_1 \cup W_2$, it is possible that even the fibers of these two fibrations are different (see Example 3.1).

Both fibration theorems above have a relative version, which we state now.

<u>Theorem 1.7.</u> Suppose that $[v:d] \notin \bar{\hat{V}}$, $\dim V'_\infty \cap Z(\langle z, v\rangle) < \dim V'_\infty$, $[w] \neq [v]$ and assume that $\Lambda_{w,[v:d]} = i_w^{-1}(\bar{\hat{V}} \cup \overline{(V \cap H_{[v:d]})}^{\wedge})$ is a finite subset in $C$. Then

$$P_w : (V \setminus P_w^{-1}(\Lambda_{w,[v:d]}), (V \cap H_{[v:d]}) \setminus P_w^{-1}(\Lambda_{w,[v:d]})) \longrightarrow C \setminus \Lambda_{w,[v:d]}$$

is a $C^\infty$ locally trivial fibration of pairs of spaces.

We denote by $\langle W_2, [v] \rangle$ the convex hull of the sets $W_2$ and $[v]$.

<u>Theorem 1.8.</u> Suppose that $[v:d] \notin \bar{\hat{V}}$ and $\dim V'_\infty \cap Z(\langle z, v \rangle) < \dim V'_\infty$.

Define $A_0 = A \setminus (\hat{V} \cup V_1 \cup P^{-1}(\langle W_2, [v] \rangle) \cup \widehat{(V \cap H_{[v:d]})} \cup (V \cap H_{[v:d]})_1 \cup (V \cap H_{[v:d]})_2)$.

Then $P_2:(P_2^{-1}(A_o), P_2^{-1}(A_o)\cap H_{[v:d]}) \longrightarrow A_o$ is a $C^\infty$ locally trivial fibration of pairs of spaces.

We want to look now more closely at the (or around the) special fibers corresponding to the points in $\Lambda_w$. Consider first the case $[w]\notin W_1$. Then by (1.1) we have $p^{-1}([w]) \cap \bar{\hat{V}} = p^{-1}([w]) \cap \hat{V}$ and hence $\Lambda_w$ is in this case precisely the set of critical values of the map $P_w : V \longrightarrow C$.

In particular $\Lambda_w$ is a finite set of points $\{c_1, \ldots, c_k\}$. In addition we shall show that any special fiber $P_w^{-1}(c_i)$ has only isolated singularities, say at the points $z_{ij}$ for $j=1,\ldots,k_i$. Let $B_\varepsilon^{ij}$ denote the closed ball of radius $\varepsilon$ centered at the point $z_{ij}$. The next theorem relates the global fibration induced by $P_w$ with the local Milnor fibrations (induced also by $P_w$) of the isolated hypersurface singularities $z_{ij}$.

<u>Theorem 1.9.</u> If $[w]\notin W_1$, then there exists $\varepsilon_o>0$ so that for each $0<\varepsilon<\varepsilon_o$ there exists a disc $B_{c_i}$ centered at $c_i$ with the following properties:

a) $P_w : P_w^{-1}(B_{c_i}) \setminus \bigcup_j B_\varepsilon^{ij} \longrightarrow B_{c_i}$ is a $C^\infty$ trivial firbation for any $i=1,\ldots,k$.

b) $P_w:(P_w^{-1}(B_{c_i}-\{c_i\}), P_w^{-1}(B_{c_i}-\{c_i\})\cap \bigcup_j B_\varepsilon^{ij}) \longrightarrow B_{c_i} - \{c_i\}$ is a $C^\infty$ locally trivial fibration of pairs of spaces for any $i=1,\ldots,k$.

When $[w]\in W_1$ the behaviour of the fibers $P_w^{-1}(c_i)$ is more complicated and it will be described in Proposition 4.1.

Coming back to the assumption $[w]\notin W_1$ we can get the following analog of the Lefschetz Hyperplane Sections Theorem. We denote by $\mu(z_{ij})$ the Milnor numbers of the isolated hypersurfac singularity $(P_w^{-1}(c_i), z_{ij})$.

<u>Theorem 1.10.</u>

a) For $c_0 \in C \setminus \Lambda_w$ one has

$$H_q(V, V\cap H_{[w:c_0]}) = \begin{cases} Z^{\sum_{i,j}\mu(z_{ij})} & \text{if } q=n \\ 0 & \text{if } q\neq n. \end{cases}$$

b) For $c_{i_0} \in \Lambda_w$ one has

$$H_q(V, V\cap H_{[w:c_{i_0}]}) = \begin{cases} Z^{\sum_{j, i\neq i_0}\mu(z_{ij})} & \text{if } q=n \\ 0 & \text{if } q\neq n. \end{cases}$$

The sums of Milnor numbers which occur here can be interpreted in terms of the position of the projective line $l=P^{-1}([w])$ with respect to the dual variety $\hat{X}$ in the following way. Because $\sum_j \mu(z_{ij}) = m_{[w:c_i]}\hat{\bar{V}}$ (see Proposition 5.4) and $m_{[w:c_i]}\hat{\bar{V}} = m_{[w:c_i]}(\hat{\bar{V}}, l)$ (see Remark 2.5) we have:

$$\sum_{i,j}\mu(z_{ij}) = \sum_i m_{[w:c_i]}(\hat{\bar{V}}, l) = \text{degree}(\hat{\bar{V}}) - m_\infty(\hat{\bar{V}}, l).$$

Here $m_{[w:c_i]}\hat{\bar{V}}$ denotes the multiplicity of the point $[w:c_i]$ on $\hat{\bar{V}}$ and $m_{[w:c_i]}(\hat{\bar{V}}, l)$ denotes the intersection multiplicity of $l$ and $\hat{\bar{V}}$ at the point $[w:c_i]$.

The various fibrations described above give rise to corresponding monodromy actions. In particular we get an action of the fundamental group $G=\pi_1(A\setminus(\hat{V}\cup V_1\cup V_2), [w:c])$ on the exact homology sequence

$$0 \to H_n(V) \to H_n(V, V \cap H_{[w:c]}) \xrightarrow{\partial_*} H_{n-1}(V \cap H_{[w:c]}) \to H_{n-1}(V) \to 0$$

We also consider an analog of Hard Lefschetz Theorem i.e. an orthogonal decomposition for $H_{n-1}(V \cap H_{[w:c]})$ as the sum of the vanishing and the invariant cycles subspaces (with coefficients in a field). But exactly as in [L], the pure topological tools are unable to yield the complete result.

In the end we consider the special problem investigated in [B], [Hà-Lê], i.e. the topology of the fibers of a polynomial function $f: \mathbb{C}^N \to \mathbb{C}$. If we take $V = \{(z, z_{N+1}) \in \mathbb{C}^{N+1};\ f(z) - z_{N+1} = 0\}$ to be the graph of $f$, and $w_f = [0 \ldots 0:1] \in \mathbb{P}^N$, then we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow[\sim]{(1,f)} & V \\ \downarrow f & & \downarrow p_{w_f} \\ \mathbb{C} & \xrightarrow{c \mapsto -\bar{c}} & \mathbb{C} \end{array}$$

This shows that our setting contains indeed this special problem. Let us introduce the sets: $\Sigma_f = \{c \in \mathbb{C} \mid c = f(z) \text{ and } \partial f(z) = 0 \text{ for some } z \in \mathbb{C}^N\}$.

$$\Lambda_f = \{c \in \mathbb{C} \mid c = \lim_{k \to \infty} (f(z_k) - \langle z_k, \overline{\partial f(z_k)} \rangle) \text{ for some sequence } \{z_k\}_k \in \mathbb{C}^N \text{ with } \lim_{k \to \infty} \partial f(z_k) = 0\}$$

Then it is clear that $\Sigma_f \subset \Lambda_f$ and $\Lambda_{w_f} \simeq \Lambda_f$ via the isomorphism in the above diagram. By Theorem (1.2) $f$ is a $C^\infty$ locally trivial fibration over $\mathbb{C} \setminus \Lambda_f$. In general $\Sigma_f \neq \Lambda_f$ (a simple example is $f(z_1, z_2) = z_1^2 z_2 + z_1$). Broughton has introduced a class of polynomials for which this equality holds and which have other

Definition - Lemma 1.11.

A polynomial $f\in C[z_1,\dots,z_N]$ is called tame if it satisfies (one of) the following equivalent proprieties

1) There exists no sequence $\{z_k\}_k\in C^N$ with

$$\lim_{k\to\infty}\|z_k\|=\infty \quad \text{and} \quad \lim_{k\to\infty}\partial f(z_k)=0.$$

2) $w_f\notin W_1\cup W_2$

where $W_1$ and $W_2$ are the "bad" sets corresponding to $V$=graph $f$.

Since as we have seen above the main obstruction to the fibration theorems lies in the "bad" set $W_1$, it is natural to consider the following larger class of polynomials.

Definition-Lemma 1.12.

A polynomial $f\in C[z_1,\dots,z_N]$ is called quasitame if it satisfies (one of) the following equivalent proprieties.

1) There exists no sequence $\{z_k\}_k\in C^N$ with $\lim_{k\to\infty}\|z_k\|=\infty$,

$$\lim_{k\to\infty}\partial f(z_k)=0 \quad \text{and} \quad \{f(z_k)-\langle z_k,\overline{\partial f(z_k)}\rangle\}_k \text{ convergent.}$$

2) $w_f\notin W_1$.

We show by an example that there exist polynomials which are quasitame but not tame. (see Example 7.1).

Our above results specialize in this case to the following

Theorem 1.13. Let $f:C^N\to C$ be a quasitame polynomial. Th

a) $\Sigma_f=\Lambda_f=\{c_1,\dots,c_p\}$ and $f^{-1}(c_i)$ has only isolated singularities (say at the points $z_{ij}$ for $j=1,\dots,k_i$) for all

$c_i \in \Sigma_f$.

b) For any $c_i \in \Sigma_f$ there exists a closed disc $Bc_i$ centered at $c_i$, closed balls $B_\varepsilon^{ij}$ and a deformation retract $r: f^{-1}(Bc_i) \longrightarrow f^{-1}(c_i)$. In addition $f: f^{-1}(Bc_i) \setminus \bigcup_j B_\varepsilon^{ij} \longrightarrow Bc_i$ is a $C^\infty$ trivial fibration and $f: (f^{-1}(Bc_i - \{c_i\}), f^{-1}(Bc_i - \{c_i\} \cap \bigcup_j B_\varepsilon^{ij}) \longrightarrow Bc_i - \{c_i\}$ is a $C^\infty$ locally trivial fibration of pairs of spaces.
The fibers in the balls are exactly the local Milnor fibers of the isolated singularities $(f^{-1}(c_i), z_{ij})$.

c) When $c_k \notin \Sigma_f$, the nonsingular fiber $f^{-1}(c)$ has the homotopy type of a bouquet of $\sum_{i,j} \mu(z_{ij}) = \text{class } \bar{V} - m_{[0:0:1]}(p^{-1}([0:\ldots]), \hat{\bar{V}})$ spheres of dimension $(N-1)$.

When $c_{i_0} \in \Sigma_f$, the singular fiber $f^{-1}(c_{i_0})$ has the homotopy type of a bouquet of $\sum_{j, i \neq i_0} \mu(z_{ij})$ spheres of dimension $(N-1)$.

. If the polynomial $f$ is tame, then we have the following <u>supplementary proprieties</u>.

d) When $c \notin \Sigma_f$, $f^{-1}(c)$ is diffeomorphic with

$$\left\{ z \in C^N \,\middle|\, \langle z, w \rangle + f(z) \cdot \bar{w}' + \bar{c} = 0 \right\} \text{ where } [w : w' : c] \notin \hat{V} \cup V_1 \cup V_2$$

e) When $c_1, c_2 \notin \Sigma_f$, the natural diffeomorphism $\varphi: f^{-1}(c_1) \longrightarrow f^{-1}(c_2)$ extends to a homeomorphism of the projective closure

$$\bar{\varphi}: \overline{f^{-1}(c_1)} \longrightarrow \overline{f^{-1}(c_2)} \quad \text{with } \bar{\varphi} \big| \overline{f^{-1}(c_1)} \setminus f^{-1}(c_1) = \text{identity}.$$

In fact, this problem can be generalized. More precisely, we obtain in the final section similar results for a polynomial function $f|_Z : Z \longrightarrow C$ where $Z \subset C^N$ is any affine smooth variety.

## 2. The dual variety

Proof of (1.1. a,b,c). $D_V$ is equal with $D_{X,V}$ via the inclusion $V\times A\subset P^N\times\hat{P}^N$, hence $\hat{X}-\{\infty\}=\bar{\hat{V}}$. In particular $\dim\hat{X}=\dim\hat{V}$. Because $D_X=D_V\cup V'_\infty$, we have $\hat{X}=pr(D_X)=pr(D_V)\cup pr(V'_\infty)=\hat{V}\cup V_\infty$. The first projection $D_V\xrightarrow{\pi_1}V$ fibres $D_V$ locally trivially with fibers isomorphic to (N-n-1)-dimensional projective subspaces of $\hat{P}^N$, hence $\dim D_V=N-1$ and $\dim\hat{V}\leq N-1$. In particular $\dim V'_\infty=\dim D_X-1=N-2$, hence $\dim W_1\leq\dim V_\infty\leq\dim V'_\infty=N-2$ and $\dim W_2=\dim V_2-1\leq N-2$. (1.1.c) is a consequence of the inclusions $\bar{V}_1\subset V_1\cup C_\infty V_\infty\subset V_1\cup V_2$. □

For the proof of local triviality of some mappings we will use some vector fields. The integrability of these vector fields depends on the behaviour of the norm $\|pr_{T_zV}(w)\|$, where $pr_{T_zV}(w)$ is the orthogonal projection of w on the tangent space at z. The following lemmas relate the norm $\|pr_{T_zV}(w)\|$ to the dual variety $\hat{V}$ and the "bad" sets $V_1$ and $V_2$.

Lemma 2.1.

a) If $[w:c]\notin\bar{\hat{V}}$, then there exists a neighbourhood $B_c\subset P^{-1}($

of c and $\delta>0$ such that

$$\inf_{z\in V\cap H_{[w:c']}}\left\{\|pr_{T_zV}(w)\|\cdot\frac{1}{\|w\|}+|\langle z,pr_{T_zV}(w)\rangle|\frac{1}{\|w\|}\right\}>\delta$$

for all $c'\in B_c$.

b) If $[w:c]\notin\hat{V}\cup V_1\cup V_2$, then there exists a neighbourhood $B_{[w:c]}\subset A$ of $[w:c]$ and $\delta>0$ such that

$$\inf_{z\in V\cap H_{[w':c']}}\left\{\|pr_{T_zV}(w')\|\cdot\frac{1}{\|w'\|}\right\}>\delta \quad \text{for all } [w':c']\in B_{[w:c]}$$

c) If [w:c] [illegible] then there exists a neighbourhood

$B_c \subset P^{-1}([w])$ of $c$ and $\delta > 0$ such that the following set

$$\left\{ z\in V \,\middle|\, \inf_{z\in V\cap H_{[w:c']}} \left\{ \|pr_{T_zV} w\| \cdot \frac{1}{\|w\|} + |\langle z, pr_{T_zV} w\rangle| \cdot \frac{1}{\|w\|} \right\} \leq \delta \text{ for some } [w:c'] \in B_c \right.$$

is compact.

d) If $[w:c] \in \hat{V} \setminus (V_1 \cup V_2)$, then there exists a neighbourhood $B_{[w:c]} \subset A$ of $[w:c]$ and $\delta > 0$ such that the set

$$\left\{ z\in V \,\middle|\, \inf_{z\in V\cap H_{[w':c']}} \left\{ \|pr_{T_zV} w'\| \cdot \frac{1}{\|w'\|} \right\} \leq \delta \text{ for some } [w':c'] \in B_{[w:c]} \right\}$$

is compact.

Proof. We prove only the (2.1.b), the rest can be proved with similar techniques.

We suppose that there exists a sequence $\{z_n\}_n$ $z_n \in V \cap H_{[w_n':c_n']}$ such that $\lim\limits_{n\to\infty} w_n' = w$, $\lim\limits_{n\to\infty} c_n' = c$ and $\lim\limits_{n\to\infty} pr_{T_{z_n}V} w_n' = 0$. If $w_n = w_n' - pr_{T_{z_n}V} w_n'$ and

$c_n = -\langle w_n, z_n\rangle$, then $(z_n, [w_n:c_n]) \in D_V$ and $[w_n:c_n] \in \hat{V}$.

If $\{z_n\}_n$ has a convergent subsequence $\{z_{n_k}\}_k$ with $\lim z_{n_k} = z \in V$, then $pr_{T_zV} w = 0$, $z \in V \cap H_{[w:c]}$ and $[w:c] = \lim [w_{n_k} : c_{n_k}]$, hence $[w:c] \in \overline{\hat{V}}$.

If $\lim \|z_n\| = \infty$ and $\{c_n\}_n$ has a convergent subsequence $\{c_{n_k}\}_k$ with $\lim c_{n_k} = c'$; then there exists $z \in X \cap H_\infty$ with $(z, [w:c']) = V_\infty'$, hence $[w:c] \in V_1$.

If $\lim \|z_n\| = \infty$ and $\lim |c_n| = \infty$, then $\lim [w_n:c_n] = [0:1]$ hence $[w] \in V_2$.

It follows that $[w:c] \in \hat{V} \cup V_1 \cup V_2$ which is in contradiction with the hypothesis. □

If we use the inequality $|\langle z, pr_{T_zV} w\rangle| \leq \|z\| \cdot \|pr_{T_zV} w\|$ then the Lemma 2.1 has the following.

Corollary 2.2.

a) If $[w:c] \notin \bar{\hat{V}}$, then there exists a neighbourhood $B_c$ of c and $\delta > 0$ such that $\| pr_{T_zV} w\| \geq \frac{\delta \cdot \|w\|}{1+\|z\|}$ for all $z \in V \cap H_{[w:c']}$, $c' \in B_c$.

b) If $[w:c] \in \hat{V} \setminus V_\infty$, then there exists a neighbourhood $B_c$ of c and $\delta > 0$ such that for a some compact set $K \subset V$ $\|pr_{T_zV} w\| \geq \frac{\delta \cdot \|w\|}{1+\|z\|}$ for all $z \in (V \setminus K) \cap H_{[w:c']}$ and $c' \in B_c$.

In particular we obtain the proof of the (1.1.e) because the singular set $\sum(V \cap H_{[w:c]})$ is a compact, hence a finite set.

If $[v:d] \notin \bar{\hat{V}}$ then $V \cap H_{[v:d']}$ is a smooth variety for all d' in a sufficiently small neighbourhood $B_d$ of d. Thus we can use Lemma 2.1 for all varieties $V \cap H_{[w:d']}$, $d' \in B_d$. The following lemma asserts that the neighbourhoods $B_c$ and $B_{[w:c]}$ given by the Lemma 2.1 can be choosen independently on the numbers d'.

Lemma 2.3. We suppose that $[v:d] \notin \bar{\hat{V}}$ and

$$\dim(V'_\infty \cap Z(\langle z, v\rangle)) < \dim V'_\infty$$

a) If $[w:c] \notin \overline{(V \cap H_{[v:d]})}^{\wedge} \cup P^{-1}([v])$ then there exist a neighbourhood $B_d \subset P^{-1}([v])$ of d, a neighbourhood $B_c \subset P^{-1}([w])$ of c and $\delta > 0$ such that

$$\inf_{z \in V \cap H_{[v:d']} \cap H_{[w:c']}} \left\{ \|pr_{T_z(V \cap H_{[v:d']})} w\| \cdot \frac{1}{\|w\|} + \right.$$

$$\left. + |\langle z, pr_{T_z(V \cap H_{[v:d']})} w\rangle| \cdot \frac{1}{\|w\|} \right\} > \delta$$

b) If $[w:c]\notin(V\cap H_{[v:d]})^{\wedge}\cup(V\cap H_{[v:d]})_1\cup P^{-1}(\langle W_2,[v]\rangle)$

then there exist a neighbourhood $B_d\subset P^{-1}([v])$ of $d$, a neighbourhood $B_{[w:c]}\subset A$ of $[w:c]$ and $\delta>0$ such that

$$\inf_{z\in V\, H_{[v:d']}\cap H_{[w':c']}}\left\{\left\|pr_{T_z(V\cap H_{[v:d']})}w'\right\|\cdot\frac{1}{\|w'\|}\right\}>\delta \quad \text{for all}$$

$d'\in B_d$ and $[w':c']\in B_{[w:c]}$.

Proof. Suppose that there exists a sequence $\{z_n\}_n$ $z_n\in V\cap H_{[w:d_n]}\cap H_{[w:c'_n]}$ with $\lim d_n=d$, $\lim c'_n=c$, $\lim pr_{T_{z_n}(V\cap H_{[v:d_n]})}w=0$ and $\lim\langle z_n,pr(w)\rangle=0$.

If we take $w_n=w-pr\, w$, $c_n=-\langle w_n, z_n\rangle$ then $(z_n,[w_n:c_n])\in$ $\in D_{V\cap H_{[v:d_n]}}$. Because $[v:d]\notin\hat{V}$, for sufficiently large n, $H_{[v:d_n]}$ intersects V transversally, hence there is a sequence $(z_n,[u_n:t_n])\in D_V$ such that $z_n\in H_{[v:d_n]}$ and $[u_n:t_n]$, $[w_n:c_n]$, $[v:d_n]$ are collinear points. If $\{z_n\}_n$ has a convergent subsequence then $[w:c]\in(V\cap H_{[v:d]})^{\wedge}$. If $\lim\|z_n\|=\infty$, then we have a convergent subsequence such that

$$\lim(z_{n_k},[u_{n_k}:t_{n_k}])=(z,[u:t])\in D_x\cap H_{[v:d]}\cap H_\infty$$

From the collinearity of the points $[u:t]$, $[w:c]$, $[v:d]$ and from $[w:c]\notin P^{-1}([v])$ we obtain that $[u:t]\neq[0:1]$. On the other hand $D_x\cap H_{[v:d]}=\overline{D_V\cap H_{[v:d]}}$ because of the dimensional condition. Thus $(z,[u:t])\in(V\cap H_{[v:d]})'_\infty$ hence $[u:t]\in(V\cap H_{[v:d]})_\infty$. But the set $(V\cap H_{[v:d]})_\infty$ is a cone with vertex at $[v:d]$. Therefore $[w:c]\in(V\cap H_{[v:d]})_\infty$.

The part (b) can be proved in a similar way. In this case it is also possible that $\lim\,[w_n:c_n]=[w:c']$ (then $[w:c]\in$ $\in(V\cap H_{[v:d]})_1$) or $\lim\,[u_n:t_n]=\lim\,[w_n:c_n]=\infty$. In this final case $\lim\,[u_n]=[u]\in W_2$ and $[u],[w],[v]$ are collinear points. Thus $[w]\in\langle W_2,[v]\rangle$.

Remark 2.4.

Like in Corollary 2.2 we can obtain that

$$\| pr_{T_z(V\cap H_{[v:d']})}\, w \| \geq \frac{\delta\cdot\|w\|}{1+\|z\|} \quad \text{for all } z\in V\cap H_{[v:d']}\cap H_{[w:c']}$$

$d'\in B_d$, $c'\in B_c$.

Remark 2.5.

It is interesting to study the mutual position of the projective line $P^{-1}([w])$ and the projective tangent cones to the dual variety $\overline{\hat{V}}$ at the points $P^{-1}([w])\cap\overline{\hat{V}}$. From definition $[w]\in W_2$ when $P^{-1}([w])\in C_\infty\overline{\hat{V}}$. We have the following result:

If $P^{-1}([w])\subset C_{[w:c]}\overline{\hat{V}}$ for some $[w:c]\in\overline{\hat{V}}\cap P^{-1}([w])$ then $[w]\in W_1$.

Indeed, if $P^{-1}([w])\subset C_{[w:c]}\overline{\hat{V}}$, then there exists a hyperplane $H=\hat{z}\in\hat{P}^N$ such that $P^{-1}([w])\subset H$ and $([w:c],H)\in D_{\overline{\hat{V}}}[W]$. From the Duality Theorem [L] we have $D_{\hat{X}}=D_X$, hence $(z,[w:c])\in D_X$ and $z\in P^{-1}([w])$. In particular $z\in\hat{\infty}=H_\infty$. Thus $[w:c]\in V_\infty$.

Therefore, if $[w]\notin W_1$, then $m_{[w:c_i]}(P^{-1}([w]),\overline{\hat{V}})=$ $=m_{[w:c_i]}\overline{\hat{V}}$ for all $[w:c_i]\in P^{-1}([w])\cap\overline{\hat{V}}$.

3. The global fibration theorems

Proof of Theorem 1.2.

a) Let $c\notin\Lambda$. Then by the Corollary 2.2.a there exists a neighbourhood $B_c \subset C \setminus \Lambda_w$ of c and $\delta > 0$ such that

$\| pr_{T_z V} w \| > \frac{\delta}{1+\|z\|}$ for all $z \in P_w^{-1}(B_c)$. For any vector field $\tilde{v}$ on $B_c$ with $\|\tilde{v}\| \leq \delta . k$ we define a vector field on $P_w^{-1}(B_c)$ by

$$v(z) = - \frac{pr_{T_z V} w}{\|pr_{T_z V} w\|^2} . \overline{\tilde{v}(P_w(z))}$$ with the following

proprieties:

(*) $(dP_w)_z v(z) = \tilde{v}(P_w(z))$

(**) $\|v(z)\| \leq k(1+\|z\|)$.

Let $\ell_z(t) = 1+\|\gamma_z(t)\|^2$ where $\gamma_z(t): P_w^{-1}(B_c) \to P_w^{-1}(B_c)$ is the local flow defined by v with $\gamma_z(o)=z$ . Then $\ell_z'(t) \leq 4k . \ell(t)$ hence $\ell(t) \leq \ell(0) . e^{4kt}$ or $1+\|\gamma_z(t)\|^2 \leq (1+\|z\|^2) . e^{4kt}$. Thus the maximal domain of definition for $\gamma_z$ is all the real axis. The existence of such vector field v for any $\tilde{v}$ gives us a trivialisation of $P_w^{-1}(B_c)$.

b) Let $\tilde{v}(c') = c-c'$ on $B_c - \{c\}$. By Corollary 2.2.b. the domain of definition for $\gamma_z$ is again all the real axis. We define the deformation retract $r: P_w^{-1}(B_c) \to P_w^{-1}(c)$ by $r(z) = \lim_{t\to\infty} \gamma_z(t)$. Because $P_w(\gamma_z(t)) = c + e^{-t}(P_w(z)-c)$ we have $P_w(r(z)) = c$.

<u>Proof of Remark 1.3.</u> When $[w] \notin W_1 \cup W_2$ , then by Lemma 2.1. $\|pr_{T_z V} w\| > \delta \|w\|$ for all $z \in P_w^{-1}(B_c)$, hence $\|v(z)\| \leq k$. Thus for any $c_1$ and $c_2$ from $B_c$ we have a diffeomorphism $\gamma: P_w^{-1}(c_1) \longrightarrow P_w^{-1}(c_2)$ with $\|z - \gamma(z)\| \leq$ constant. In this case $\gamma$ can be extended to a homeomorphism of the projective closures $\overline{\gamma}: \overline{P_w^{-1}(c_1)} \to \overline{P_w^{-1}(c_2)}$ such that $\overline{\gamma}|P_w^{-1}(c_1) = \gamma$ and

$\overline{\gamma}|\overline{P_w^{-1}(c_1)} \setminus P_w^{-1}(c_1) =$ identity.

Proof of Theorem 1.5.

If $[w:c]\in\hat{V}\cup V_1\cup V_2$ , then by Lemma 2.1.b. there exists a neighbourhood $B_{[w:c]}\subset A$ of $[w:c]$ and $\delta>0$ such that $\|pr_{T_zV}\, w'\|>\delta\cdot\|w'\|$ for all $(z,[w':c'])\in P_2^{-1}(B_{[w:c]})$. We can choose $B_{[w:c]}$ in $A\setminus(\hat{V}\cup V_1\cup V_2)$ as a product $B_w\times B_c$ by a parametrization $(w_1,\ldots,w_{N-1};c)\mapsto[w_1:\ldots:1:\ldots:w_{N-1}:c]$ . For any vector field $\tilde{v}(w',c')=(\tilde{v}_1(w',c'),\ \tilde{v}_2(w',c'))$ on $B_{[w:c]}=B_wxB_c$ we define a vector field on $P_2^{-1}(B_{[w:c]})$ by

$$v(z,w',c')=\left(-(\langle z,\tilde{v}_1(w',c')\rangle+\overline{\tilde{v}_2(w',c')})\cdot\frac{pr_{T_zV}w'}{\|pr_{T_zV}\,w'\|^2},\tilde{v}(w',c')\right)$$

It is easy to show that $v(z,w',c')\in T_{(z,w',c')}U$.

If $(\gamma(t),w(t),c(t))$ is a integral curve of $v$, then $\frac{d}{dt}(\langle\gamma(t),\ w(t)\rangle+\overline{c(t)})=0$. Thus when $\gamma(0)\in H_{[w(0):c(0)]}$ then $\gamma(t)\in H_{[w(t):c(t)]}$. If we choose the neighbourhood $B_{[w:c]}$ and the vector field $\tilde{v}$ such that $\|w'\|>\varepsilon$ for all $[w':c']\in B_{[w:c]}$ (recall that $\|w\|=1$) and $\|\tilde{v}\|<\varepsilon\cdot\delta\cdot k$, then $\|\gamma'(t)\|\leq k(1+\|\gamma(t)\|)$. Therefore the maximal domain of definition for $\gamma$ is all the real axis. Thus $\mathfrak{P}_2$ is trivial on $B_{[w:c]}$.

Examples 3.1.

Let $X=\{[x:y:z:t]\in P^3:\ xy-z^2-t^2=0\}$ be a nonsingular projective hyperquadric. The dual variety is

$$\hat{X}=\{[w_1:w_2:w_3:w_4]\in\hat{P}^3:\ 4w_1w_2-w_3^2-w_4^2=0\}.$$

(A). In the first case we choose $H_\infty=Z(z)=H_{[0:0:1:0]}$.

The affine variety $V$ is $Z(xy-t^2-1)$ in $C^3$. A simple calculation

$X \cap H_\infty$ is smooth (which is equivalent with $[0:0:1:0] \notin \hat{X}$) the "bad" set $V_2 = \emptyset$.

A hyperplane section $V \cap H_{[w:c]}$ with $[w:c] \notin \hat{V} \cup V_1 \cup V_2$ is diffeomorphic with the section $V \cap \{t=c\}_{c \neq \pm i} = Z(xy-c^2-1) \approx S^2 - \{$two points$\}$. If we consider the hyperplanes $[0:1:c:0]_{c\neq 0} \in V_1 \setminus V_\infty$ then the corresponding sections are diffeomorphic with the section $V \cap \{y=c\}_{c\neq 0} \approx S^2 - \{$one point$\}$, hence the fibration $P_2$ cannot be extended over the line $P^{-1}([0:1:0])$. The special section $V \cap \{y=0\} = P^{-1}_{[0:1:0]}(0) = Z(t^2-1)$ is a nonconnected (smooth) variety, but $P^{-1}_{[0:1:0]}$ (disc centered at zero)$=\{(x,t): xc-t^2-1=0$ with $|c|<\varepsilon\}$ is homeomorphic with $S^2 - \{$two points$\}$, hence is connected. Thus there exists no deformation retract $r: P^{-1}_{[0:1:0]}(D_0) \longrightarrow$

$\longrightarrow P^{-1}_{[0:1:0]}(0)$. Therefore in the Theorem (1.2.ii) the condition $[w:c] \in \hat{V} \setminus V_\infty$ is essential.

(B). If we choose $H_\infty = Z(X) = H_{[1:0:0:0]}$, then $V = Z(y-z^2-t^2) \subset C^3$, $V_\infty = V_1 = Z(w_2, w_3^2+w_4^2)$ and $V_2 = Z(w_2)$. In particular $V_2 \setminus V_1 \neq \emptyset$. The general sections $V \cap H_{[w:c]}$ with $[w:c] \notin \hat{V} \cup V_1 \cup V_2$ are diffeomorphic with $Z(z^2+t^2=c)_{c\neq 0} \approx S^2 - \{$two points$\}$. If we choose $[0:1:1] \in W_2 \setminus W_1$ then the corresponding fibers $P^{-1}_{[0:1:1]}(c) = V \cap H_{[c:1:1:0]}$ are diffeomorphic with $C$.

<u>Proof of Theorem 1.7.</u>

Let $c \in C \setminus \Lambda_{w,[v:d]}$. By Corollary 2.2 and Remark 2.4 there exist a disc $B_d = \{d' : |d'-d| < \varepsilon\}$, a neighbourhood $B_c$ of $c$ and $\delta > 0$ such that:

(*) $\| pr_{T_z(V \cap H_{[v:d']})} w \| \geq \frac{\delta}{1+\|z\|}$ for all $z \in V \cap H_{[v:d']} \cap H_{[w:c']}$

$d' \in B_d$, $c' \in B_c$.

(**) $\| pr_{T_z V} w \| \geq \frac{\delta}{1+\|z\|}$ for all $z \in V \cap H_{[w:c']}$, $c' \in B_c$

(***) $\|pr_{T_zV}\ v\| \geq \frac{\delta \cdot \|v\|}{1+\|z\|}$ for all $z \in V \cap H_{[v:d']}$, $d' \in B_d$.

By (***) for all $z \in V \cap H_{[v:d']}$ ($d' \in B_d$) the tangent space at $z$ has an orthogonal decomposition

$T_zV = T_z(V \cap H_{[v:d']}) \perp N_z$. Thus $pr_{T_zV}\ w = w_1(z) + w_2(z)$

where $w_1(z) \in N_z$ and $w_2(z) \in T_z(V \cap H_{[v:d']})$.

Let $\varphi: [0,\infty) \to [0,1]$ be a $C^\infty$ function such that

$$\varphi = \begin{cases} 0 & \text{in some neighbourhood of zero} \\ 1 & \text{in } (\frac{\varepsilon}{2}, \infty) \end{cases}$$

For any vector field $\tilde{v}$ on $B_c$ we define the following vector field on $P_w^{-1}(B_c)$

$$v(z) = \begin{cases} -\dfrac{pr_{T_zV}\ w}{\|pr_{T_zV}\ w\|^2} \cdot \overline{\tilde{v}(P_w(z))} & \text{if } -\langle v,z\rangle \notin B_d \\ -\dfrac{\varphi(|\langle z,v\rangle + \bar{d}|) \cdot w_1(z) + w_2(z)}{\langle \varphi(|\langle z,v\rangle + \bar{d}|)\, w_1(z) + w_2(z), w\rangle} \cdot \overline{\tilde{v}(P_w(z))} & \text{if } -\langle v,z\rangle \in B_c \end{cases}$$

The vector field $v$ has the following proprieties:

1. $v(z) \in T_zV$ for all $z \in P_w^{-1}(B_c)$

2. $v(z) \in T_z(V \cap H_{[v:d]})$ for all $z \in P_w^{-1}(B_c) \cap H_{[v:d]}$

3. $\|v(z)\| \leq \dfrac{|\tilde{v}|}{\sqrt{\varphi \cdot \|w_1\|^2 + \|w_2\|^2}}$

If $-\langle v,z\rangle \in B_d$, then by (*) $\sqrt{\varphi \cdot \|w_1\|^2 + \|w_2\|^2} \geq \|w_2\| \geq \frac{\delta}{1+\|z\|}$

If $-\langle v,z\rangle \notin B_d$, then by (**) $\|pr_{T_zV} w\| \geq \frac{\delta}{1+\|z\|}$

Thus for any vector field $\tilde{v}$ with $\|\tilde{v}\| \leq k \cdot \delta$ we have $\|v(z)\| \leq k(1+\|z\|)$.

4. $(dP_z)v(z) = \tilde{v}(P_w(z))$.

The integral curves of the vector field $v$ give us the trivialization of the $P_w$ over $B_c$.

Proof of Theorem 1.8. We use the assertions of Corollary 2.2.b. and Lemma 2.3.b. The proof is a combination of the proofs of Theorem 1.5 and Theorem 1.7.

## 4. Fibration theorems around the singular fibers.

Proof of Theorem 1.9.

There exists $\varepsilon_j>0$ such that the two vectors $pr_{T_zV}\,w$ and $pr_{T_zV}(z-z_{ij})$ are C-linearly independent on $V\cap H_{[w:c_i]}\cap B^{ij}_{\varepsilon_j}\setminus\{z_{ij}\}$. Indeed, if we suppose that there are points $z\in V\cap H_{[w:c_i]}$ arbitrarily close to the singularities $z_{ij}$, such that these two vectors are C-linearly dependent, then by the curve selection lemma there must exist a real analytic path $\rho:[0,\varepsilon)\to V\cap H_{[w:c_i]}$ with $\rho(0)=z_{ij}$ and $\rho(t)\neq z_{ij}$ for $t\neq 0$ such that

$$pr_{T_{\rho(t)}V}(\rho(t)-z_{ij})=\lambda(t).pr_{T_{\rho(t)}V}\,w \quad \text{for all } t\neq 0.$$

But a small calculation shows that in this case $\|\rho(t)-z_{ij}\|>0$ is constant when $t\in(0,\varepsilon)$, which is in contradiction with $\rho(0)=z_{ij}$.

For any $0<\varepsilon_j'<\varepsilon_j$ there exists a neighbourhood $B_{c_i}$ of $c_i$ such that the two vectors $pr_{T_zV}\,w$ and $pr_{T_zV}(z-z_{ij})$ are C-linearly independent on $V\cap P_w^{-1}(B_c)\cap(B^{ij}_{\varepsilon_j}\setminus B^{ij}_{\varepsilon_j'})$.

Thus there is a vector field $u_j(z)$ on $V\cap P_w^{-1}(B_c)\cap(B^{ij}_{\varepsilon_j}\setminus B^{ij}_{\varepsilon_j'})$ with $u_j(z)\in T_zV$, $\langle u_j(z),\, pr_{T_zV}\,w\rangle=0$ and $\langle u_j(z), pr_{T_zV}(z-z_{ij})\rangle=1$.

We change the vector field $v$ from the proof of Theorem

1.2. in the following way:

For any vector field $\tilde{v}$ on $B_{c_i}$ we define the vector field $v$ on $P_W^{-1}(B_{c_i}) \setminus \bigcup_j \{z_{ij}\}$ by:

$$v(z) = - \frac{pr_{T_zV}\, w + \sum_j \varphi_j(\|z-z_{ij}\|) . g_j(z) . u_j(z)}{\|pr_{T_zV}\, w\|^2} . \overline{\tilde{v}(P_W(z))}$$

where $g_j(z) = i . \tilde{v}(P_W(z)) - \langle pr_{T_zV}\, w,\ pr_{T_zV}(z-z_{ij}) \rangle$ and $\varphi_j : [0,\infty) \to [0,1]$ are $C^\infty$- functions with

$$\varphi_j(X) = \begin{cases} 0 & \text{if } X \notin (\varepsilon_j', \varepsilon_j) \\ 1 & \text{if } X \in \left(\varepsilon_j' + \frac{\varepsilon_j - \varepsilon_j'}{3}, \varepsilon_j' + 2\frac{\varepsilon_j - \varepsilon_j'}{3}\right) = I_j \end{cases}$$

This vector field has a very important supplementary propriety: $\mathrm{Re}\langle v(z), pr(z-z_{ij})\rangle = 0$ for all $z$ with $\|z-z_{ij}\| \in I_j$. Therefore the sphere of radius $\frac{\varepsilon_j' + \varepsilon_j}{2}$ centered at the point $z_{ij}$ is invariant for any $j=1,\dots,k_i$.

With similar arguments we can proof the following.

<u>Proposition 4.1.</u> We suppose that $[w] \in W_1$ and $\Lambda_W \neq C$.

a) If $[w:c] \in \hat{V} \setminus V_\infty$, then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_j$ there is a disc $B_c$ centered at $c$ such that

$P_W : P_W^{-1}(B_c) \setminus \bigcup_j B_\varepsilon^j \to B_c$ is a $C^\infty$ trivial fibration, and

$P_W : (P_W^{-1}(B_c - \{c\}),\ P_W^{-1}(B_c - \{c\}) \cap (\bigcup_j B_\varepsilon^j)) \to B_c - \{c\}$ is a $C^\infty$ locally trivial fibration of pairs of spaces.

(Here $B_\varepsilon^j$ denote the closed balls of radius $\varepsilon$ centered at the isolated singularities of the $P_W^{-1}(c)$).

b) If $[w:c] \in V_\infty \setminus \hat{V}$ then there exists $R_0 > 0$ such that for any $R > R_0$ there is a disc B

$p_w: p_w^{-1}(B_c)\cap B_R \to B_c$ is a $C^\infty$ trivial fibration, and

$p_w: (p_w^{-1}(B_c-\{c\}), p_w^{-1}(B_c-\{c\})\cap \complement B_R) \to B_c-\{c\}$ is a $C^\infty$ locall trivial fibration of pairs of spaces.

($B_R$ denote the open ball of radius R centered at the origin, $\complement B_R$ is the complement of $B_R$).

## 5. The homology of hyperplane sections

Proof of Theorem 1.10

Let $B_i$ be a sufficiently small closed disc of radius $\rho$ centered at $c_i$. By Theorem 1.2, Theorem 1.9 and the excision propriety we have

$$H_q(V, V\cap H_{[w:c_o]}) \approx \bigoplus_i H_q(p_w^{-1}(B_{c_i})\cap(\bigcup_j B_\varepsilon^{ij}), p_w^{-1}(c_i+\rho)\cap(\bigcup_j B_\varepsilon^{ij})) .$$

But $p_w^{-1}(B_{c_i})\cap B_\varepsilon^{ij}$ is a contractible space, hence $H_q(V, V\cap H_{[w:c_o]}) \approx \bigoplus_i \tilde{H}_{q-1}$ (Milnor fibre of the singularity $z_{ij}$). □

Recall that $\Lambda_w=\{c_1,\ldots,c_k\}$. Let $B_{c_i}$ as in the previous proof. If we choose $c_o\in B_{c_i}\setminus\{c_i\}$, then we have the inclusion $p_w^{-1}(c_o)\overset{i}{\hookrightarrow} p_w^{-1}(B_{c_i})$ and the deformation retract $r: p_w^{-1}(B_{c_i}) \longrightarrow p_w^{-1}(c_i)$. The diagram

$$\begin{array}{ccc} V\cap H_{[w:c_o]} & \xrightarrow{r\circ i} & V\cap H_{[w:c_i]} \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

is commutative up to homotopy.

Corollary 5.1.

a) If $q\leq n-2$ then $H_q(V)=H_q(V\cap H_{[w:c_i]})=H_q(V\cap H_{[w:c_o]})$ for all $i=1,\ldots,k$

b) We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to H_n(V) & \to & H_n(V, V\cap H_{[w:c_0]}) & \to & H_{n-1}(V\cap H_{[w:c_0]}) & \to & H_{n-1}(V) & \to & 0 \\ \downarrow \text{id.} & & \downarrow \alpha & & \downarrow \beta & & \downarrow \text{id.} & & \\ 0 \to H_n(V) & \to & H_n(V, V\cap H_{[w:c_i]}) & \to & H_{n-1}(V\cap H_{[w:c_i]}) & \to & H_{n-1}(V) & \to & 0 \end{array}$$

(where $\alpha$ and $\beta$ are induced by the map $r \circ i$).

In particular the sequence

$$0 \to Z^{\sum_j \mu(z_{ij})} \xrightarrow{\varphi} H_{n-1}(V\cap H_{[w:c_0]}) \xrightarrow{\beta} H_{n-1}(V\cap H_{[w:c_i]}) \to 0$$

is exact for all $i=1,\ldots,k$.

Remark 5.2. In the last exact row $\varphi$ is a morphism of lattices. In fact we have

$$0 \to \bigoplus_j L_{ij} \xrightarrow{\varphi} H_{n-1}(V\cap H_{[w:c_0]}) \to H_{n-1}(V\cap H_{[w:c_i]}) \to 0$$

where $L_{ij}$ is the Milnor lattice of the singularity at $z_{ij}$ and $H_{n-1}(V\cap H_{[w:c_0]})$ is considered with its intersection form. This exact sequence can be used to relate the topology of a smooth hyperplane section $X\cap H_0$ with the topology of a section $X\cap H_1$ having only isolated singularities ($X$ being here a smooth projective variety). For details in the case when $X$ is a smooth complete intersection see [Di].

Corollary 5.3.

$$\chi(V) - \chi(V\cap H_{[w:c_0]}) = (-1)^n \sum_{i,j} \mu(z_{ij})$$

$$\chi(V) - \chi(V\cap H_{[w:c_{i_0}]}) = (-1)^n \sum_{i\neq i_0, j} \mu(z_{ij})$$

(where $\chi$ denotes the Euler-Poincaré characteristic).

The sums $\sum_j \mu(z_{ij})$ can be calculated in the following way:

Proposition 5.4.

$$\sum_j \mu(z_{ij}) = m_{[w:c_i]} \hat{\hat{V}}$$

Proof.

By Remark 2.5 $\overline{P^{-1}([w])} \not\subset C_{[w:c_i]} \hat{\hat{V}}$.

Therefore we can choose the balls $B^{ij}_{\varepsilon}$ and $[w']$ sufficientl close to $[w]$ such that the line $P^{-1}([w'])$ intersects $\hat{\hat{V}}$ only in simple points and transversally with the following propriety: the critical points of the map $P_{w'}$ are inside $\bigcup_j B^{ij}_{\varepsilon}$ ( If $P^{-1}([w]) \subset C_{[w:c]} V$, then it is possible that critical points appear in the neighbourhood of the hyperplane at infinity). The map $P_{w'}$ is a generic approximation (morsification) of $P_w$. It has exactly $m_{[w:c_i]} \hat{\hat{V}}$ critical values and its critical points are nondegenerate.

Remark 5.5.

If $X$ is a nonsingular projective variety, then the position of $\infty \in \hat{X}$ is not special. Thus for any $[w] \in \hat{X}$ such that $X \cap H_{[w]}$ has only isolated singularities with Milnor numbers $\mu(z)$ we have $m_{[w]} \hat{X} = \sum_{z \in sing(X \cap H_{[w]})} \mu(z)$.

Remark 5.6.

If we use Proposition 5.4 and Remark 2.5 we obtain

$$\chi(V\cap H_{[w_1:c_0]})_{[w_1]\notin W_1\cup W_2} - \chi(V\cap H_{[w_2:c_0]})_{[w_2]\in W_1\setminus W_2} =$$

$$=(-1)^n\left[m_\infty \hat{V} - m_\infty(\hat{V}, \overline{P^{-1}([w_2])})\right] .$$

In particular this difference is nonzero. Therefore a general section $V\cap H_{[w:c]}$ (with $[w]\notin W_1\cup W_2$, $[w:c]\notin\hat{V}$) is not diffeomorphic with a section $V\cap H_{[w:c]}$ with $[w]\in W_2\setminus W_1$, $[w:c]\notin\hat{V}$.

Remark 5.7. (The homotopy of hyperplane sections).

Let $[w]\notin W_1\cup W_2$ be a direction such that the line $P^{-1}([w])$ intersects $\hat{V}$ transversally at the (smooth) points $[w:c_i]$, $i=1,\ldots,r$. This is equivalent with the fact, that $k_i=1$ and every singular point $z_i$ is nondegenerate. Let $c_0\notin\Lambda_w$ and $B_{c_0}$ a sufficiently small open disc of radius $\rho$ centered at $c_0$. Then by Theorem 1.2 there is a homotopy equivalence between $P_W^{-1}(c_0)$ and $P_W^{-1}(B_{c_0})$. Let $g:V\setminus P_W^{-1}(B_{c_0})\to[\rho^2,\infty)$ defined by $g(z)=|P_W(z)-c_0|^2$.

The critical points of $g$ are exactly the points $z_i$ and all of them are nondegenerate. The Morse index of $g$ at these points is $n$, precisely one half the real dimension of $V$. By studying the vector field arguments in [Mi$_1$] and the fibration theorems from sections 2. and 3. one sees that the homotopy type of $V$ is obtained from the homotopy type of the section $V\cap H_{[w:c_0]}$ by attaching $r$ cells of dimension $n$. (Recall that $r$ = degree $\hat{V}-m_\infty\hat{V}$.)

(Compare to [H$_1$]).

6. The monodromy

Let $[w:c]\notin\hat{V}\cup V_1\cup V_2$. We have a monodromy action of the fundamental group $G=\pi_1(\Lambda\setminus(\hat{V}\cup V_1\cup V_2), [w:c])$ on the homology group $H_q(V\cap H_{[w:c]})$. For all $u\in G$, we denote by $u_q$ the corresponding is morphism of this homology group.

Proposition 6.1.

a) The action of G on $H_q(V\cap H_{[w:c]})$ for $q\leq n-2$ is trivial

b) There exists an action of the group G on $H_n(V, V\cap H_{[w:c]})$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 \to H_n(V) & \to & H_n(V, V\cap H_{[w:c]}) & \xrightarrow{\partial_*} & H_{n-1}(V\cap H_{[w:c]}) & \xrightarrow{i_*} & H_{n-1}(V) \to 0 \\ \downarrow id. & & \downarrow u'_n & & \downarrow u_{n-1} & & \downarrow id. \\ 0 \to H_n(V) & \to & H_n(V, V\cap H_{[w:c]}) & \xrightarrow{\partial_*} & H_{n-1}(V\cap H_{[w:c]}) & \xrightarrow{i_*} & H_{n-1}(V) \to 0. \end{array}$$

Proof. Let Cyl V be the cylinder of V in $C^{N+1}$. We have a natural inclusion $A(V) \hookrightarrow A(\mathrm{Cyl}\ V)$ given by $[z_1:\ldots:z_N:z_{N+2}] \mapsto$ $\mapsto [z_1:\ldots:z_N:z_{N+1}:z_{N+2}]$. We apply the Theorem 1.8 for Cyl V and $[v:d]=[0:1:0]$. The "bad" sets are the followings

$$(\mathrm{Cyl}\ V)^\wedge = \hat{V}\ (\text{in } A(V)),\quad (\mathrm{Cyl}\ V)_\infty = \bar{\hat{V}},$$

$(\mathrm{Cyl}\ V)_1 = P^{-1}(P(\bar{\hat{V}}))$, $(\mathrm{Cyl}\ V)_2 = V_2$, all sets lying in $P^N = \{z_{N+1}=0\}$. Evidently $[v:d]=[0:1:0]$ satisfies the hypotheses of the Theorem 1.8.

Let $V_H = \mathrm{Cyl}V\cap H_{[0:1:0]}$. Then $\hat{V}_H = \langle \hat{V}, [0:1:0]\rangle$,

$$(V_H)_1 = \langle V_1, [0:1:0]\rangle,\quad (V_H)_2 = \langle V_2, [0,1:0]\rangle = P^{-1}(\langle W_2, [v]\rangle).$$

Thus we have a monodromy action of the group

$$G_1 = \pi_1(A(\mathrm{Cyl}\ V)\setminus(P^N \cup \langle \hat{V}\cup V_1 \cup V_2, [0:1:0]\rangle), [w:w_{N+1}:c])$$

on the pairs of spaces $(\mathrm{Cyl}\ V\cap H_{[w:w_{N+1}:c]}, \mathrm{Cyl}\ V\cap H_{[0:1:0]}\cap H_{[w:w_{N+1}:c]})$

(where $w_{N+1} \neq 0$). By Theorem 1.5 the action on $\mathrm{Cyl}\ V \cap H_{[w:w_{N+1}:c]}$ can be extended to an action of $\pi_1(A(\mathrm{Cyl}\ V) \setminus P[w:w_{N+1}:c]) = \pi_1(P^{N+1} \setminus P^N) = 0$, hence this action is trivial.

$\mathrm{Cyl}\ V \cap H_{[w:w_{N+1}:\ c]}$ is isomorphic with $V$ by the projection $z_{N+1} \mapsto 0$.

The action of $G_1$ on $H_{n-1}(V \cap H_{[0:1:0]} \cap H_{[w:w_{N+1}:c]})$ can be extended to an action of

$\pi_1(A(\mathrm{Cyl}\ V) \setminus \langle \hat{V} \cup V_1 \cup V_2, [0:1:0] \rangle, [w:w_{N+1}:c])$ which is isomorphic with $G$.

$\square$

With similar arguments we have similar results for the group $\pi_1(C \setminus \Lambda_w, c)$.

If $[w] \notin W_1 \cup W_2$ is chosen such that $p^{-1}([w])$ intersects $\hat{\bar{V}}$ transversally at the (smooth) points $[w:c_i]$ $(i=1,\ldots,r)$, then

$H_n(V, V \cap H_{[w:c]}) = \bigoplus_{i=1,r} H_{n-1}$ (Milnor fiber of a nondegenerate singularity)

is generate by the corresponding "thimbles" $\Delta_i$ $i=1,\ldots,r$. Let $\partial_*(\Delta_i) = \delta_i$ be the corresponding vanishing cycles. We denote the group of vanishing cycles $\mathrm{im}\,\partial_* \subset H_{n-1}(V \cap H_{[w:c]})$ by $\mathcal{V}$, its kernel with respect to the intersection form by $\ker \mathcal{V}$, the lattice $(H_{n-1}(V \cap H_{[w:c]})$, intersection form) by $L$ and its kernel (radical) by $\ker L$.

<u>Corollary 6.2.</u>

a) $\mathcal{V}, \mathcal{V}^{\perp}$, $\ker L$ are $G$-submodules

b) $\mathrm{im}(u_{n-1} - 1) \subset \mathcal{V}$ for all $u \in G$.

Our main problem is to find the position of these submodules in L and also their mutual position. The proofs of the following results are not given here. They are simple and more or less similar to the proofs in [L] (§7).

Lemma 6.3.

a) If there is a vanishing cycles such that $\delta_{i_0} \in \ker L$, then for all $i=1,\dots,r$ $\delta_i \in \ker L$.

b) If there is no vanishing cycles such that $\delta_i \in \ker L$ then for any two vansihing cycles $\delta_i$ and $\delta_j$ there is a path $u \in G$ with $u_{n-1}(\delta_i) = \pm\delta_j$.

(For n-1 even one has $\delta_i^2 = \pm 2$ for all $\delta_i$ and hence the hypothesis (b) is automatically fulfilled).

Corollary 6.4. If the coefficients are taken in a field, then

a) $\mathcal{V}/_{\ker \mathcal{V}}$ is a simple G-module.

b) $\mathcal{V}^{\perp}/_{\ker \mathcal{V}}$ is a trivial G-module.

If $\mathcal{V} \not\subset \ker L$, then

$$\mathcal{V}^{\perp}/_{\ker \mathcal{V}} = \{\hat{x} \in L/_{\ker \mathcal{V}} : u_{n-1}(\hat{x}) = \hat{x} \text{ for all } u \in G\} =$$

$$= \text{the maximal trivial G-submodule in } L/_{\ker \mathcal{V}}.$$

Proposition 6.5.

For coefficients in a field the following properties are equivalent:

a) $\ker \mathcal{V} = \mathcal{V} \cap \ker L$

b) $\mathcal{V} + \mathcal{V}^{\perp} = L$

c) $\mathcal{V}/_{\ker \mathcal{V}} \oplus \mathcal{V}^{\perp}/_{\ker \mathcal{V}} = L/_{\ker \mathcal{V}}$

d) $L/_{\ker L}$ is a semi-simple G-module

e) $L/_{\ker L} = \mathcal{V}/_{\ker \mathcal{V}} \oplus \mathcal{V}^{\perp}/_{\ker L}$ where

$\mathcal{V}/_{\ker \mathcal{V}}$ is a simple G-module and $\mathcal{V}^{\perp}/_{\ker L}$ is a trivial G-module. (We denote by + the sum, and by $\oplus$ the orthogonal direct sum).

Remark 6.6. We assume that the coefficients are taken in a field.

If X is smooth, then $H_{n-1}(X \cap H_{[w:c]})$ is a semi-simple $\pi_1(\hat{P}^N \setminus \hat{X}, [w:c])$-module. (Hard Lefschetz Theorem), and via the group epimorphism

$$G = \pi_1(A \setminus (\hat{V} \cup V_1 \cup V_2), [w:c]) \xrightarrow{incl_*} \pi_1(\hat{P}^N \setminus \hat{X}, [w:c])$$ is a semi-simple G-modul. If $X \cap H_{[w:c]} \cap H_\infty$ is smooth, by the Ehresmann's fibration theorem the inclusion $i: V \cap H \hookrightarrow X \cap H$ induces a G-morphism $i_*: H_{n-1}(V \cap H) \longrightarrow H_{n-1}(X \cap H)$. Evidently, the intersection form remains invariant, hence $\ker i_* \subset \ker L$. Thus in this special case $L/_{\ker L} \approx im\, i_*/ \frac{\ker L}{\ker i_*}$ is a semi-simple G - modul.

(In fact $\ker i_* = \ker L$. The case X=smooth complete intersection see [Di]).

7. The topology of polynomial hypersurfaces

The proofs of (1.11) and (1.12) are not complicated, they are given implicitly in the proofs of the lemmas in section 2.

Example 7.1. The polynomial $f(x,y) = x^2y + xy^2 + x^5y^3 + x^3y^5$ is an example of a quasitame polynomial which is not tame. It has the following critical points:

$$(0,0), \left(\sqrt[5]{\tfrac{3}{8}}\,\varepsilon_i, \sqrt[5]{\tfrac{3}{8}}\,\varepsilon_i\right)_{i=\overline{1,5}}, \left(\frac{(3 \pm 2\sqrt{2})\varepsilon_i}{\sqrt[5]{232 \pm 164\sqrt{2}}}, \frac{\varepsilon_i}{\sqrt[5]{232 \pm 164\sqrt{2}}}\right)_{i=\overline{1,5}}$$

The corresponding Milnor numbers are : $\mu(0,0)=4$ and the other $\mu_{i=\overline{1,15}} = 1$. Therefore $\sum\mu=19$.

The nonsingular fiber $F=f^{-1}(c)$ is obtained from its closure $\overline{F}$ (which is an irreductible curve of degree 8 in $P^2$ having two singular points with Milnor number 13) by taking out four points:the two singularities mentionned above and two simple points. This fiber F has the homotopy type of a bouquet of 19-spheres of dimension 1.

The singular fiber $f^{-1}(0)$ is obtained from its closure (which is the union of two lines and an irreducible curve of degree 6 with two singularities $A_4$ such that these irreducible components have the origin as common point) by taking out four points: the two singularities and two simple point from the two lines. It has the homotopy type of a bouquet of 15-circles. The other singular fibers have $\chi=-18$.

Finally we will consider the following more general case: Let Z be an affine smooth variety in $C^N$ and let $f:C^N\longrightarrow C$ be a polynomial map. If we take $V=\{(z,z_{N+1})\in C^{N+1}:\ z\in Z,\ f(z)-z_{N+1}=0\}$ and $w_f=[0:1]\in P^N$, then we have a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow[\sim]{(1,f)} & V \\ \downarrow f|_Z & & \downarrow p_{w_f} \\ C & \xrightarrow[\sim]{c\to -\bar{c}} & C \end{array}$$

We define $\Lambda_{f|Z}=\{c\in C \mid c=\lim\limits_{k\to\infty}(f(z_k)-\langle z_k,\overline{\partial f(z_k)+v_k}\rangle)$ for some sequence $\{z_k\}_k\in Z$ and $v_k$ in the normal space $N_{z_k}Z$ at $z_k$ with $\lim\limits_{k\to\infty}(\overline{\partial f(z_k)}+v_k)=0\}$.

Because $\Lambda_{f|Z} = -\bar{\Lambda}_{w_f}$, $f:Z\setminus f^{-1}(\Lambda_{f|Z})\longrightarrow C\setminus\Lambda_{f|Z}$ is a $C^\infty$ locally trivial fibration (by Theorem 1.2).

lowing equivalences:

Definition - Lemma 7.2.

A polynomial map $f:C^N \longrightarrow C$ is called Z-tame (respectively Z-quasitame) if it satisfies the equivalent proprieties 1) and 2) (respectively 3) and 4)).

1) There exists no sequence $\{z_k\}_k \in Z$ and $v_k \in N_{z_k}$ Z such that $\lim \|z_k\| = \infty$ and $\lim(\overline{\partial f(z_k)} + v_k) = 0$.

2) $w_f \notin W_1 \cup W_2$.

3) There exist no sequence $\{z_k\}_k \in Z$ and $v_k \in N_{z_k}$ Z such that $\lim \|z_k\| = \infty$, $\lim(\overline{\partial f(z_k)} + v_k) = 0$ and $\{f(z_k) - \langle z_k, \overline{\partial f(z_k)} + v_k\rangle\}_k$ convergent.

4) $w_f \notin W_1$.

All the results of the general case can be specialized for this case similar with Theorem 1.3. The formulations are simple and are not given here.

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