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Serban A. PASARAB

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Serban A. BASARAB<sup>\*)</sup>

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<sup>\*)</sup> The National Institute for Scientific and Technical Creation  
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INTRODUCTION

In a well known series of papers, Ax-Kochen [2]-[4] and Ershov [17]-[20] initiated a metamathematical approach to some basic problems in the theory of Henselian valued fields. These papers were followed by other works of Ax [1], Cohen [14], Kochen [21], MacIntyre [23], Robinson [25], [26] which continued the investigation of Henselian fields using methods of model theory, recursive function theory and nonstandard arithmetic. Further refinements of these results were obtained by Weispfenning [30], Ziegler [33], Delon [15] and the author [5]-[9]. An account of model theoretic and algorithmic results in the elementary theory of valued fields, in an approach that uses explicit, primitive recursive quantifier elimination procedures as a unifying principle is given by Weispfenning [32].

For  $p$ -adic fields  $\mathbb{Q}_p$  and the power series fields  $F((t))$  over a decidable field  $F$  of characteristic zero, Ax-Kochen [4] proved by model theoretic methods decidability and relative quantifier elimination, when a cross-section is included in the language of valued fields. A quantifier elimination with cross-section and a decision method for  $\mathbb{Q}_p$  were also given by Cohen [14] using primitive recursive methods. Later MacIntyre [23] showed that quantifier elimination for  $\mathbb{Q}_p$  can be obtained without cross-section, when more natural root-predicates are included in the language. In a more algebraic approach, this result was extended by Prestel-Roquette

[24] to  $p$ -adically-closed fields. Extensions of MacIntyre's result are also contained in Cherlin-Dickmann [13] and Delon [15]. Some general results concerning the transfer of model completion for Henselian fields with finite absolute ramification index were obtained by Ziegler [33] and the author [7].

On the other hand, Cohen's ideas were strongly generalized by Weispfenning [30], [32] in order to get primitive recursive relative quantifier elimination procedures for Henselian fields of characteristic zero subject to a condition of moderate generality on the value group.

In the last time the quantifier elimination problem for valued fields gained popularity thanks to its interest for computer scientists under the aspect of feasibility as well as for deep applications in diophantine questions. Concerning the diophantine applications, let us mention here Cantor-Roquette [11], [28] and Rumely work on Hilbert's tenth problem for the ring of algebraic integers, Denef's paper [16] on the rationality of certain Poincaré series over  $\mathbb{Q}_p$  and Weispfenning's result [31] on the primitive recursive decidability of the adèle ring and idèle group of an algebraic number field.

The present paper is devoted to the proof by algebraic and basic model theoretic methods of a general result on relative quantifier elimination for Henselian fields of characteristic zero.

Given a valued field  $\underline{K} = (K, v)$  let us denote by  $\mathcal{O}_{\underline{K}}$  the valuation ring, by  $\bar{K}$  the residue field and by  $vK$  the value group. Assume that the characteristic of  $K$  is zero and let  $p$  be the characteristic exponent of  $\bar{K}$ . For  $k \in \mathbb{N}$ , let  $\mathfrak{m}_{\underline{K}, k}$  be the ideal  $\{a \in \mathcal{O}_{\underline{K}} : v a \geq k v p\}$  of  $\mathcal{O}_{\underline{K}}$ . In particular,  $\mathfrak{m}_{\underline{K}, 0} = \mathfrak{m}_{\underline{K}}$  is the maximal ideal of the valuation ring  $\mathcal{O}_{\underline{K}}$ . Denote by  $\mathcal{O}_{\underline{K}, k}$  the factor ring  $\mathcal{O}_{\underline{K}} / \mathfrak{m}_{\underline{K}, k}$ ;  $\mathcal{O}_{\underline{K}, k}$  is a local ring with maximal ideal  $\mathfrak{m}_{\underline{K}} / \mathfrak{m}_{\underline{K}, k}$ . In particular, for



$p=1$ ,  $O_{\underline{K},k} = \underline{K}$  for all  $k \in \mathbb{N}$ .

On the other hand, consider the multiplicative groups  $G_{\underline{K},k} = \underline{K}^{\times} / 1 + \underline{m}_{\underline{K},k}$  for  $k \in \mathbb{N}$ . If  $p \neq 1$  then  $G_{\underline{K},k} = G_{\underline{K},0} = \underline{K}^{\times} / 1 + \underline{m}_{\underline{K}}$  for all  $k \in \mathbb{N}$ . Given  $k \in \mathbb{N}$ , the ring  $O_{\underline{K},2k}$  and the group  $G_{\underline{K},k}$  are related through a natural map  $\theta_k$  defined on the subset  $\{a \in O_{\underline{K},2k} : a/p^k\} = O_{\underline{K},2k} \setminus \underline{m}_{\underline{K},k} / \underline{m}_{\underline{K},2k}$  of  $O_{\underline{K},2k}$  with values in  $G_{\underline{K},k}$ :  $\theta_k(a + \underline{m}_{\underline{K},2k}) = a(1 + \underline{m}_{\underline{K},k})$  for  $a \in O_{\underline{K}}$  subject to  $va \in kvp$ .

For  $k \in \mathbb{N}$ , the valuation  $v$  induces a map  $v_k: O_{\underline{K},2k} \cup G_{\underline{K},k} \rightarrow vK \cup \{\infty\}$ ; the image  $v_k(O_{\underline{K},2k} \setminus \{0\})$  is the convex subset  $vK_{2k} = \{\alpha \in vK : 0 \leq \alpha \leq 2kvp\}$  of the ordered group  $vK$  and the restriction  $v_k|_{G_{\underline{K},k}}: G_{\underline{K},k} \rightarrow vK$  is a group epimorphism. Among other properties, the map  $\theta_k$  as defined above satisfies the following valuation theoretic condition of compatibility: the diagram

$$\begin{array}{ccc} O_{\underline{K},2k} \setminus \{0\} & \xrightarrow{v_k} & vK_{2k} \hookrightarrow vK \\ \uparrow & & \uparrow v_k \\ O_{\underline{K},2k} \setminus \underline{m}_{\underline{K},k} / \underline{m}_{\underline{K},2k} & \xrightarrow{\theta_k} & G_{\underline{K},k} \end{array}$$

commutes.

For  $k \in \mathbb{N}$ , consider the system  $\underline{K}_k = (O_{\underline{K},2k}, G_{\underline{K},k}, vK, \theta_k, v_k)$  with  $O_{\underline{K},2k}, G_{\underline{K},k}, vK, \theta_k, v_k$  as above and call it the mixed k-structure assigned to the valued field  $\underline{K}$ . In particular, for  $p=1$ ,  $\underline{K}_k = \underline{K}_0$  is the triple  $(\bar{K}, G_{\underline{K}}, vK)$  together with the exact sequence

$$1 \rightarrow \bar{K}^{\times} \rightarrow G_{\underline{K}} \rightarrow vK \rightarrow 0.$$

The mixed  $k$ -structures introduced above play a key role in the model theory of Henselian fields of characteristic zero as shows the following elementary equivalence theorem:

Theorem A. Given the valued field extensions  $\underline{L}/\underline{K}$  and  $\underline{F}/\underline{K}$ , where  $\underline{K}$  is of characteristic zero and residue characteristic exponent  $p$ , and  $\underline{L}, \underline{F}$  are Henselian, the necessary and sufficient condition for the valued fields  $\underline{L}, \underline{F}$  to be elementarily equivalent over  $\underline{K}$  is that for all  $k \in \mathbb{N}$ , the corresponding mixed  $k$ -structures  $\underline{L}_k, \underline{F}_k$  are elementarily equivalent over  $\underline{K}_k$ .

Given a sentence  $\varphi$  in the language  $\underline{L}_k$  of mixed  $k$ -structures,  $k \in \mathbb{N}$ , one may assign effectively to  $\varphi$  a formula  $\text{tr}_k(\varphi)(z)$ , with one variable  $z$ , in the language  $\underline{L}$  of valued fields in such a way that for every valued field  $\underline{K}$  of characteristic zero and residue characteristic exponent  $p$ ,  $\underline{K}$  satisfies  $\text{tr}_k(\varphi)(p)$ , written  $\underline{K} \models \text{tr}_k(\varphi)(p)$ , iff  $\underline{K}_k$  satisfies  $\varphi$ , written  $\underline{K}_k \models \varphi$ . The correspondence above  $\varphi \mapsto \text{tr}_k(\varphi)$  extends naturally to a translation map  $\text{tr}_k$  from the arbitrary formulas in  $\underline{L}_k$  to formulas in  $\underline{L}$ .

As a consequence of Theorem A we get the second main result of the paper concerning the relative elimination of quantifiers for Henselian fields of characteristic zero.

Theorem B. Let  $p$  be either 1 or a prime number, and denote by  $T_p$  the theory of Henselian valued fields of characteristic zero and residue characteristic exponent  $p$ . For every formula  $\varphi(\underline{x})$ ,  $\underline{x} = (x_1, \dots, x_n)$  in  $\underline{L}$  there exist  $k \in \mathbb{N}$ , formulas  $\psi_1(\underline{y}), \dots, \psi_\ell(\underline{y})$  in  $\underline{L}_k$ ,  $\underline{y} = (y_1, \dots, y_m)$ , quantifierless formulas  $\lambda_1(\underline{x}), \dots, \lambda_\ell(\underline{x})$  in  $\underline{L}$  and polynomials  $f_i, g_i \in \mathbb{Z}[\underline{x}]$ ,  $1 \leq i \leq m$ , such that  $\varphi(\underline{x})$  is equivalent in  $T_p$  with the following formula

$$\bigvee_{j=1}^{\ell} [\lambda_j(\underline{x}) \wedge \text{tr}_k(\psi_j)(f_1(\underline{x})g_1(\underline{x})^{-1} : 1 \leq i \leq m; p)]$$



One may derive Weisbfenning's main Theorem 4.3 [32] (with recursive instead of primitive-recursive) from Theorem B above; details will be contained in a forthcoming paper. In the last section of the present work we shall show only that Prestel-Roquette theorem [24] on quantifier elimination for  $p$ -adically closed fields is a consequence of Theorem B.

# 1. THE RADICAL STRUCTURE THEOREM

Consider a valued field  $\underline{K} = (K, v)$  of characteristic zero and residue characteristic exponent  $p$ . We define the canonical decomposition of the valuation  $v$  as follows. Denote by  $\Delta = \Delta_{\underline{K}}$  the smallest convex subgroup of  $vK$  containing  $vp$ .  $\Delta = 0$  iff  $p=1$ , i.e.  $\bar{K}$  is of characteristic zero. Let  $\dot{v}K$  be the factor group  $vK/\Delta$ , and  $\dot{v}: K^x \rightarrow \dot{v}K: a \mapsto \dot{v}a$  be the group epimorphism induced by  $v: K^x \rightarrow vK$ . Since  $\Delta$  is convex in  $vK$ ,  $\dot{v}K$  inherits from  $vK$  the structure of a totally ordered group and hence the map  $\dot{v}$  is a valuation of the field  $K$ , called the course valuation assigned to  $v$ . Denote by  $\dot{\underline{K}}$  the valued field  $(K, \dot{v})$ . The valuation ring  $O_{\dot{\underline{K}}}^*$  of  $\dot{\underline{K}}$  is characterized as the smallest overring of  $O_{\underline{K}}$  in which  $p$  becomes a unit, i.e.  $O_{\dot{\underline{K}}}^*$  is the ring of fractions of  $O_{\underline{K}}$  with respect to the multiplicatively closed set  $\{p^k: k \in \mathbb{N}\}$ . Note that  $\dot{v}=v$  iff  $p=1$ , and  $\dot{v}$  is trivial iff  $vK = \Delta$ .

Let  $\underline{m}_{\dot{\underline{K}}}^*$  be the maximal ideal of  $O_{\dot{\underline{K}}}^*$ ; then  $\underline{m}_{\dot{\underline{K}}}^* \subset \underline{m}_{\underline{K}} \subset O_{\underline{K}} \subset O_{\dot{\underline{K}}}^*$ . Denote by  $K^o$  the residue field  $O_{\dot{\underline{K}}}^* / \underline{m}_{\dot{\underline{K}}}^*$  of the valued field  $\dot{\underline{K}}$ . For  $a \in O_{\dot{\underline{K}}}^*$  let  $a^o$  be its residue in  $K^o$ . The field  $K^o$ , called the core field of the valuation  $v$  of  $K$ , carries naturally a valuation whose valuation ring is the image  $O_{\dot{\underline{K}}}^* / \underline{m}_{\dot{\underline{K}}}^*$  of  $O_{\dot{\underline{K}}}^*$ . Denote also by  $v$  this valuation and by  $\underline{K}^o$  the core valued field  $(K^o, v)$ . The value group  $vK^o$  is identified with the convex subgroup  $\Delta$  of  $vK$  and the residue field  $\bar{K}^o$  is identified with the residue field  $\bar{K}$  of  $\underline{K}$ . Thus the core valued field  $\underline{K}^o$  is of characteristic zero and residue charac

teristic exponent  $p$ ;  $K^0 = \bar{K}$  iff  $p=1$ .

Consider a valued field extension  $\underline{L}=(L,v)$  of  $\underline{K}$ . Then the course valuation of  $\underline{L}$  is a prolongation of the course valuation of  $\underline{K}$ ; hence both may be denoted by the same symbol  $v$ . The core valued field  $\underline{L}^0$  is an extension of the core valued field  $\underline{K}^0$ .

We say that the extension  $\underline{L}/\underline{K}$  is core-dense if for every  $\alpha \in \Delta = \Delta_{\underline{K}}$  and for every  $b \in L$  there exists  $a \in K$  such that  $v(b-a) > \alpha$ ; in particular  $\Delta_{\underline{L}} = \Delta_{\underline{K}}$ . This is equivalent with the fact that for every  $k \in \mathbb{N}$ , the ring embedding  $O_{\underline{K},k} = O_{\underline{K}^0,k} \rightarrow O_{\underline{L}^0,k} = O_{\underline{L},k}$  is an isomorphism. In other words,  $\underline{L}/\underline{K}$  is core-dense iff the core extension  $\underline{L}^0/\underline{K}^0$  is dense. If  $p=1$  then  $\underline{L}/\underline{K}$  is core-dense iff  $\bar{L}=\bar{K}$ .

A main ingredient in the proof of Theorem A is the following natural generalization of Prestel-Roquette radical structure theorem [24], Theorem 3.8.

PROPOSITION 1.1. Let  $\underline{K}=(K,v)$  be a Henselian valued field of characteristic zero and  $\underline{L}$  be an algebraic core-dense extension of  $\underline{K}$ . Then  $\underline{L}/\underline{K}$  is generated by radicals, i.e.  $L=K(T)$  where  $T=T_{\underline{L}/\underline{K}} = \{t \in L^X : \bigvee_{n \geq 1} t^n \in K\}$  is the multiplicative group of radical elements of  $L/K$ . The radical value group  $vT$  equals the full value group  $vL$  of  $\underline{L}$  and the valuation map  $v:T \rightarrow vL$  induces a group isomorphism  $T/K^X \xrightarrow{\sim} vL/v_K$ . If  $\underline{L}/\underline{K}$  is a finite extension then  $[L:K] = (T:K^X)$ .

Proof. As  $\underline{K}$  is Henselian and  $\underline{L}/\underline{K}$  is algebraic,  $\underline{L}$  is Henselian too and  $vL/v_K$  is a torsion group. First of all let us show that  $L^0=K^0$  and  $vL=vT$ . Let  $a \in O_{\underline{L}^0}$ . We have to show that  $a \in O_{\underline{K}^0}$ . As  $L^0/K^0$  is algebraic and  $K^0$  is of characteristic zero, there exists  $f \in O_{\underline{K}^0}[X]$  such that  $f(a)=0$  and  $f'(a) \neq 0$ . Let  $c \in vK^0 = vL^0$  be such that  $v(a-c) > 2vf'(a)$ . Since  $\underline{L}^0/\underline{K}^0$  is dense there exists  $c \in O_{\underline{K}^0}$  such that  $v(a-c) > \gamma$ . Thus  $vf(c) > \gamma + 2vf'(a) = 2vf'(c)$  and hence by Newton's lemma [24] p.20, there is one and only one  $b \in O_{\underline{K}^0}$  such that  $f(b)=0$  and  $v(b-c) > vf'(c) = vf'(a)$ . (Note that  $\underline{K}^0$  is Henselian



since  $\underline{K}$  is Henselian). As  $f(a)=0$  and  $v(a-c) > 2vf'(a) > vf'(a)$ , we get  $a=bc \in \underline{O}_{\underline{K}}^\times$ , as contended. Now let  $a \in L^\times$ . We have to show that  $va=vt$  for some  $t \in T$ . As  $vL/vK$  is a torsion group,  $a^n=bu$  with  $b \in K$ ,  $n \geq 1$ ,  $u \in \underline{O}_{\underline{L}}^\times$ . As  $L^\circ = K^\circ$ , there exists  $u' \in \underline{O}_{\underline{K}}^\times$  such that  $\dot{v}(u-u') > 0$ . Consider the polynomial  $f(X) = X^n - uu'^{-1} \in \underline{O}_{\underline{L}}^\times[X]$ . Since  $\dot{v}f(1) > 0 = \dot{v}n = \dot{v}f'$  and  $\underline{L}$  is Henselian (as  $\underline{L}$  is Henselian) there exists  $t' \in L^\times$  such that  $f(t')=0$ , i.e.  $t'^n = a^n(bu')^{-1}$ . Let  $t = at'^{-1}$ ; then  $t^n = bu' \in K^\times$ , i.e.  $t \in T$ , and  $vt=va$  as contended.

Consider the intermediate field  $L' = K(T)$  between  $K$  and  $L$ . We have to show that  $L' = L$ . The value group  $vL'$  of  $\underline{L}' = (L', v)$  contains  $vT = vL$  and hence  $vL' = vL$ ; in particular,  $\dot{v}L' = \dot{v}L$ . On the other hand,  $L'^\circ = L^\circ$  since  $K^\circ = L^\circ$ . Thus the valued field extension  $\underline{L}'/\underline{L}$  is immediate. As an algebraic extension of the Henselian valued field  $\underline{K}$ ,  $\underline{L}'$  is Henselian and hence algebraically complete being of residue characteristic zero [1], Proposition 15. Since  $\underline{L}'/\underline{L}$  is algebraic immediate, we conclude that  $L' = L$ .

As  $vT = vL$ , the valuation  $v$  induces a group epimorphism  $v: T/K^\times \rightarrow vL/vK$ . We have to show that this is an isomorphism. Let  $t \in T$  be such that  $vt \in vK$ . Assume that the order of  $t$  modulo  $K^\times$  is  $n$  and  $t^n = a \in K$ . As  $vt \in vK$  by assumption,  $t = bu$  with  $b \in K$ ,  $u \in \underline{O}_{\underline{L}}^\times$ . Since  $K^\circ = L^\circ$ , there exists  $u' \in \underline{O}_{\underline{K}}^\times$  such that  $\dot{v}(u-u') > 0$ ; therefore  $t = (bu')(uu'^{-1}) \in K^\times(1 + \underline{m}_{\underline{L}})$ . To show that  $t \in K$  we may replace  $t$  by any other element in its coset modulo  $K^\times$ , so we may assume without loss of generality that  $\dot{v}(1-t) > 0$  and hence  $\dot{v}(1-t^n) = \dot{v}(1-a) > 0$ . Consider the polynomial  $f(X) = X^n - a \in \underline{O}_{\underline{K}}^\times[X]$ . As  $\underline{K}$  is Henselian and  $\dot{v}f(1) = \dot{v}(1-a) > 0 = \dot{v}(n) = \dot{v}f'(1)$ , there exists one and only one  $c \in K$  such that  $f(c) = c^n - a = 0$  and  $\dot{v}(1-c) > 0$ . Since this uniqueness statement holds not only in  $\underline{K}$  but also in the Henselian field  $\underline{L}$  and since  $f(t) = 0$ ,  $\dot{v}(1-t) > 0$ , we conclude that  $t = c \in K$ .

We have shown that  $T/K^\times \cong vL/vK$ . If one of these groups is finite then the other is finite too and  $(T:K^\times) = (vL:vK)$ . On the other hand, if  $[L:K]$  is finite then  $[L:K] = (\dot{v}L:\dot{v}K) = (vL:vK)$  since

$K$  is algebraically complete and  $[L^0 : K^0] = 1$ . We conclude that  $[L : K] = (T : K^X)$ .

Q.E.D.

The following explicit description of the field structure of  $L/K$  in terms of radicals is an immediate consequence of Proposition 1.1; see the proof of [24] Corollary 3.9.

Corollary 1.2. In the same situation as in Proposition 1.1, assume that  $L/K$  is finite. Then  $L/K$  can be generated by finitely many radicals such that the product of their radical exponents equals the field degree:

$$L = K(t_1, \dots, t_r)$$

$$t_i^{n_i} = a_i \in K^X \quad (1 \leq i \leq r)$$

$$[L : K] = n_1 n_2 \dots n_r.$$

The substitution  $x_i \mapsto t_i$  ( $1 \leq i \leq r$ ) extends to a  $K$ -isomorphism of the factor algebra  $K[x_1, \dots, x_r]/I$ , where  $I$  is the ideal generated by the polynomials  $x_i^{n_i} - a_i$  ( $1 \leq i \leq r$ ), onto the field  $L$ .

## 2. AN EMBEDDING THEOREM FOR HENSELIAN FIELDS

It is well known the key role played by the embedding theorems in the investigation of the model theoretic properties of the Henselian valued fields (see for instance [21], [24] Theorem 4.1, [7] Theorem 1.2, [9] Proposition 2.2). In this section we prove a general embedding theorem for Henselian valued fields of characteristic zero that will be the main tool for the proof of Theorem A. A basic ingredient in the proof of this embedding theorem is Proposition 1.1.

Given a valued field  $\underline{K} = (K, v)$  of characteristic zero, let us



denote by  $\underline{\underline{K}}_0$  the mixed structure assigned to the coarse <sup>valued</sup> field  $\underline{\underline{K}} = (K, v)$ , namely the system  $(K^\circ, G_{\underline{\underline{K}}}^\circ = K^\times / 1 + m_{\underline{\underline{K}}}^\circ, vK)$  together with the exact sequence

$$1 \rightarrow K^{\circ \times} \rightarrow G_{\underline{\underline{K}}}^\circ \xrightarrow{v} vK \rightarrow 0.$$

In fact we shall consider the core field  $K^\circ$  not only as an abstract field but also as a valued field  $\underline{\underline{K}}^\circ = (K^\circ, v)$  with the valuation naturally induced by the valuation  $v$  of  $\underline{\underline{K}}$ . Thus it seems natural to consider systems  $\underline{\underline{S}} = (\underline{\underline{M}}, H, \overset{\circ}{\Gamma}, i, \overset{\circ}{\mathcal{N}})$  where  $\underline{\underline{M}} = (M, v)$  is a valued field of characteristic zero whose value group  $vM$  equals the smallest convex subgroup containing  $vp$ ,  $p$  = the characteristic exponent of the residue field  $\bar{M}$ ,  $H$  is a multiplicative Abelian group,  $\overset{\circ}{\Gamma}$  is an additive totally ordered group,  $i: M^\times \rightarrow H$  is a group monomorphism and  $\overset{\circ}{\mathcal{N}}: H \rightarrow \overset{\circ}{\Gamma}$  is a group epimorphism such that the sequence  $1 \rightarrow M^\times \xrightarrow{i} H \xrightarrow{\overset{\circ}{\mathcal{N}}} \overset{\circ}{\Gamma} \rightarrow 0$  is exact. Call such a system a mixed  $\omega$ -structure. Thus the system  $\underline{\underline{K}}_0$  above is the mixed  $\omega$ -structure naturally associated to  $\underline{\underline{K}}$ . If the residue characteristic of  $\underline{\underline{K}}$  is zero then  $\underline{\underline{K}}_0 = \underline{\underline{K}}_0$  is the mixed 0-structure  $(\bar{K}, G_{\underline{\underline{K}}}^\circ = K^\times / 1 + m_{\underline{\underline{K}}}^\circ, vK; 1 \rightarrow \bar{K}^\times \rightarrow G_{\underline{\underline{K}}}^\circ \rightarrow vK \rightarrow 0)$  assigned to  $\underline{\underline{K}}$ .

Given a mixed  $\omega$ -structure  $\underline{\underline{S}} = (\underline{\underline{M}}, H, \overset{\circ}{\Gamma}, i, \overset{\circ}{\mathcal{N}})$ , we obtain the canonic exact sequence

$$0 \rightarrow vM \cong M^\times / O_{\underline{\underline{M}}}^\times \rightarrow \overset{\circ}{\Gamma} \rightarrow \overset{\circ}{\Gamma} \rightarrow 0$$

where  $\overset{\circ}{\Gamma} = H / i(O_{\underline{\underline{M}}}^\times)$ . Thus  $\overset{\circ}{\Gamma}$  inherits a naturally structure of totally ordered group with the order given by  $\alpha \leq \beta$  iff either  $\alpha < \beta$  or  $0 \leq \beta - \alpha \in vM$ ; with respect to this order,  $vM$  is identified with a convex subgroup of  $\overset{\circ}{\Gamma}$  and  $\overset{\circ}{\Gamma} \cong \overset{\circ}{\Gamma} / vM$  as ordered group. Thus the mixed  $\omega$ -structure  $\underline{\underline{S}}$  may be seen also as a system  $\underline{\underline{S}}' = (\underline{\underline{M}}, H, \overset{\circ}{\Gamma}, i, \overset{\circ}{\mathcal{N}})$  where  $\underline{\underline{M}}, H, i$  are as above,  $\overset{\circ}{\Gamma}$  is an additive totally ordered group and

$\pi: H \rightarrow \Gamma$  is a group epimorphism such that the kernel of  $\pi \circ i: M^X \rightarrow \Gamma$  is the group  $O_M^X$  of units of  $O_M$  and the value group  $vM \in M^X / O_M^X$  is identified through  $\pi \circ i$  with a convex subgroup of  $\Gamma$ . In the present section we prefer to use mixed  $\omega$ -structures in the first acceptance reserving the second equivalent definition for the next sections of the work.

Given two mixed  $\omega$ -structures  $S_i$ ,  $i=1,2$ , an embedding  $\mu: S_1 \rightarrow S_2$  will be a system  $(\mu', \mu, \mu'')$  where  $\mu': M_1 \rightarrow M_2$  is a valued field embedding,  $\mu: H_1 \rightarrow H_2$  is a group monomorphism and  $\mu'': \dot{\Gamma}_1 \rightarrow \dot{\Gamma}_2$  is a monomorphism of ordered groups such that the next diagram

$$\begin{array}{ccccc} M_1^X & \xrightarrow{i_1} & H_1 & \xrightarrow{\pi_1} & \dot{\Gamma}_1 \\ \mu' \downarrow & & \mu \downarrow & & \downarrow \mu'' \\ M_2^X & \xrightarrow{i_2} & H_2 & \xrightarrow{\pi_2} & \dot{\Gamma}_2 \end{array}$$

is commutative.

With this preparation we are now able to state the general embedding theorem for Henselian valued fields of characteristic zero.

Theorem 2.1. Let  $\underline{K}=(K,v)$  be a valued field of characteristic zero and  $\underline{L}=(L,v)$ ,  $\underline{F}=(F,v)$  be Henselian valued fields extending  $\underline{K}$ . Assume that  $\dot{\underline{F}}$  is  $|L|$  - pseudo-complete, where  $|L|$  denotes the cardinality of  $L$ . Given a  $\underline{K}_0$  - embedding  $\mu: \dot{\underline{L}}_0 \rightarrow \dot{\underline{F}}_0$  of mixed  $\omega$  - structures there exists a  $\underline{K}$ -embedding  $\eta: \underline{L} \rightarrow \underline{F}$  of valued fields inducing the given  $\underline{K}_0$  - embedding  $\mu$ .

Proof. Consider the family  $\Lambda$  of pairs  $(L', \eta)$  where  $L'$  is an intermediate field between  $K$  and  $L$ , and  $\eta: \underline{L}' \rightarrow \underline{F}$  is a  $\underline{K}$ -embedding of valued fields inducing the restriction embedding  $\mu|_{\dot{\underline{L}}'_0}: \dot{\underline{L}}'_0 \rightarrow \dot{\underline{F}}_0$  of mixed  $\omega$ -structures.  $\Lambda$  is non-empty since the pair



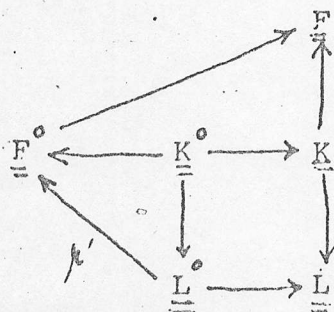
$(K, K \hookrightarrow F)$  belongs to  $\Lambda$ . Consider the partial order on  $\Lambda$ :

$(L', \eta') \leq (L'', \eta'')$  iff  $L' \subset L''$  and  $\eta' = \eta''|_{L'}$ . As the non-empty partial ordered set  $(\Lambda, \leq)$  is inductive there exists by Zorn's lemma a maximal pair  $(L', \eta') \in \Lambda$ . We have to show that  $L' = L$ .

Without loss of generality we may assume that  $L' = K$ , i.e.  $\Lambda$  is the singleton  $\{(K, K \hookrightarrow F)\}$ ; so we have to show that  $K = L$ . We proceed step by step as follows.

1) First let us show that  $\underline{\underline{K}} = (K, \underline{\underline{v}})$  is Henselian. Let  $\underline{\underline{K'}} = (K', \underline{\underline{v}})$  be the Henselization of  $\underline{\underline{K}}$ . As  $\underline{\underline{L}}$  is Henselian,  $\underline{\underline{L}} = (L, \underline{\underline{v}})$  is Henselian too and hence  $\underline{\underline{K'}}$  is identified with a subextension of  $\underline{\underline{L}}/\underline{\underline{K}}$ . Let  $\underline{\underline{K'}} = (K', \underline{\underline{v}})$  be the field  $K'$  with the valuation induced by the valuation  $\underline{\underline{v}}$  of  $\underline{\underline{L}}$ . As the residue field  $K'^{\circ}$  of  $\underline{\underline{K'}}$  equals  $K^{\circ}$ , the valuation of  $\underline{\underline{K'}}$  is exactly the valuation on the field  $K'$  induced by the valuation  $\underline{\underline{v}}$  of  $\underline{\underline{K'}}$  and the valuation  $v$  of  $\underline{\underline{K}}^{\circ}$ . Since  $\underline{\underline{F}}$  is Henselian,  $\underline{\underline{F}}$  is Henselian too and hence we get a canonic  $\underline{\underline{K}}$ -embedding  $\eta: \underline{\underline{K'}} \rightarrow \underline{\underline{F}}$ . In fact  $\eta$  is a  $\underline{\underline{K}}$ -embedding of  $\underline{\underline{K'}}$  into  $\underline{\underline{F}}$ . It remains to check that  $\underline{\underline{K'}}^{\circ} = \underline{\underline{K}}^{\circ}$  in order to conclude that  $(K', \eta) \in \Lambda$  and hence  $K' = K$  by maximality of  $K$ . The equalities  $\underline{\underline{K'}}^{\circ} = \underline{\underline{K}}^{\circ}$ ,  $\underline{\underline{v}}K' = \underline{\underline{v}}K$  are trivial, so it remains to verify that  $G_{\underline{\underline{K'}}}^{\circ} = G_{\underline{\underline{K}}}^{\circ}$ . Let  $x \in K'^{\times}$ . As  $\underline{\underline{v}}K' = \underline{\underline{v}}K$ ,  $x = ay$  with  $a \in K^{\times}$ ,  $y \in O_{\underline{\underline{K}}}^{\times}$ . Since  $K'^{\circ} = K^{\circ}$ , there is  $b \in O_{\underline{\underline{K}}}^{\times}$  such that  $\underline{\underline{v}}(y-b) > 0$ . It follows  $x = (ab)(yb^{-1}) \in K^{\times}(1 + \mathfrak{m}_{\underline{\underline{K}}}^{\circ})$ . Therefore the canonic morphism  $G_{\underline{\underline{K'}}}^{\circ} \rightarrow G_{\underline{\underline{K}}}^{\circ}$  is an isomorphism as contended.

Now since  $\underline{\underline{K}}$  is Henselian of residue characteristic zero, we may assume by [1] Proposition 16 that we have the following commutative diagram of valued fields:



Note also that  $K, L^\circ$  and respectively  $K, F^\circ$  are linearly disjoint over  $K^\circ$ .

2) Let us show that  $K^\circ$  is algebraically closed in  $L^\circ$ .

Let  $x \in L^\circ$  be algebraic over  $K^\circ$  and  $\underline{K}'$  be the field  $K(x)$  with the valuation  $v$  induced from  $\underline{L}$ . By linear disjointness we get  $[K':K] = [K^\circ(x):K^\circ]$  and hence  $\dot{v}K' = \dot{v}K$  and  $K'^\circ = K^\circ(x)$ . The correspondence  $x \mapsto \mu'(x)$  defines a field  $K$ -embedding  $\eta: K' \rightarrow F$ . Moreover,  $\eta$  is a  $\underline{K}$ -embedding of valued fields. Indeed, let  $w$  be the valuation of  $K'$  induced through  $\eta$  by the valuation of  $\underline{F}$ . As the residue field of  $K'$  with respect to the coarse valuation  $\dot{w}$  equals  $K'^\circ$ , the valuations  $w$  and  $v$  of  $K'$  induce the same valuation on  $K'^\circ$  ( $\mu'$  is a  $\underline{K}^\circ$ -embedding) and  $\dot{w}$  equals the valuation  $\dot{v}$  of  $K'$ , the unique valuation of  $K'$  extending the valuation  $\dot{v}$  of  $K$ , it follows that  $w$  equals the valuation  $v$  of  $\underline{K}'$ , as contended. Thus it remains to show that the canonic  $\underline{K}_\circ$  - embedding  $\hat{\eta}: \underline{K}'_\circ \rightarrow \underline{F}_\circ$  of mixed  $w$  - structures induced by  $\eta$  coincides with the restriction of  $\mu$  to  $\underline{K}'_\circ$ . As  $\hat{\eta}$  coincides with  $\mu'$  on  $\underline{K}'^\circ$  by definition of  $\eta$  and  $\dot{v}K' = \dot{v}K$  it remains to show that the canonic monomorphism  $\hat{\eta}: G_{\underline{K}'}^\circ \rightarrow G_{\underline{F}}^\circ$  induced by  $\eta$  equals the restriction of  $\mu: G_{\underline{L}}^\circ \rightarrow G_{\underline{F}}^\circ$  to  $G_{\underline{K}'}^\circ$ . Let us show that  $G_{\underline{K}'}^\circ = G_{\underline{K}}^\circ(K'^\circ)^X$ . Indeed, let  $z \in K'^X$ . As  $\dot{v}K' = \dot{v}K$ ,  $z = au$  with  $a \in K^X$ ,  $u \in O_{\underline{K}}^X$ , and hence  $z(1 + \frac{m_\circ}{\underline{K}}) = [a(1 + \frac{m_\circ}{\underline{K}})][u(1 + \frac{m_\circ}{\underline{K}})]$ , therefore  $z(1 + \frac{m_\circ}{\underline{K}}) \in G_{\underline{K}}^\circ(K'^\circ)^X$  as contended. Now the statement above is immediate since  $\mu$  and  $\hat{\eta}$  coincide on  $G_{\underline{K}}^\circ$  and  $(K'^\circ)^X$  by definition of  $\eta$ . Consequently,  $(K', \eta) \in \Lambda$  and hence  $K' = K$  since  $\Lambda = \{K, K \hookrightarrow F\}$  by assumption. As  $x$  is arbitrary, we conclude that  $K^\circ$  is algebraically closed in the Henselian field  $\underline{L}^\circ$ . In particular,  $\underline{K}^\circ$  is Henselian.

3) Moreover, we claim that  $K$  is algebraically closed in  $L$ .

First let us observe that  $\underline{K}$  is Henselian since  $\underline{K}^\circ$  is Henselian ~~from~~ ~~by 1)~~ and  $\underline{K}^\circ$  is Henselian by 2). Let  $\underline{K}' \subset \underline{L}$  be a finite extension of  $\underline{K}$ . Note that  $K'^\circ = K^\circ$  by 2). Here is the point where we use the radical field structure of  $K'/K$  given by Corollary



ry 12:  $K' = K(t_1, \dots, t_r) \simeq K[X_1, \dots, X_r]/I$ , where  $I$  is the ideal generated by  $r$  polynomials  $c_i X_i^{n_i} - 1$ ,  $1 \leq i \leq r$ , with  $c_i \in K^X$  such that  $c_i t_i^{n_i} = 1$ ,  $1 \leq i \leq r$ . Let  $T = K^X t_1^{\frac{Z}{1}} t_2^{\frac{Z}{2}} \dots t_r^{\frac{Z}{r}}$  be the multiplicative group of radicals of  $K'/K$ . Consider the chain of isomorphisms  $T/K^X \xrightarrow{\sim} vK'/vK \xrightarrow{\sim} vK'/vK \simeq \prod_{1 \leq i \leq r} \underline{Z}/n_i \underline{Z}$ . Let us choose  $y_1, \dots, y_r \in F$  such that  $y_i(1 + \underline{m}_F^{\circ}) = \mu(t_i(1 + \underline{m}_F^{\circ}))$ ,  $1 \leq i \leq r$ . We get  $c_i y_i^{n_i} \in 1 + \underline{m}_F^{\circ}$  since  $c_i t_i^{n_i} = 1$ ,  $1 \leq i \leq r$ . Consider the polynomials  $f_i(X) = X^{n_i} - c_i y_i^{n_i} \in O_F^{\circ}[X]$ ,  $1 \leq i \leq r$ . As  $v f_i(1) > 0$ ,  $v f_i'(1) = v(n_i) = 0$ , we get by Hensel lemma applied to  $\underline{F}$  some elements  $x_1, \dots, x_r \in F$  uniquely determined by the conditions  $x_i^{n_i} = c_i y_i^{n_i}$ ,  $v(x_i - 1) > 0$ ,  $1 \leq i \leq r$ . Let us put  $z_i = y_i \cdot x_i^{-1}$ ,  $1 \leq i \leq r$ . The substitution  $t_i \mapsto z_i$  ( $1 \leq i \leq r$ ) defines a field  $K$ -embedding  $\eta: K' \rightarrow F$ . Moreover,  $\eta: \underline{K}' \rightarrow \underline{F}$  is a valued field  $\underline{K}$ -embedding since  $\underline{K}$  is Henselian and  $K'/K$  is algebraic. Let us show that the  $\underline{K}_0$ -embedding  $\hat{\eta}: \underline{K}' \rightarrow \underline{F}_0$  of mixed  $\omega$ -structures induced by  $\eta$  equals the restriction of  $\mu: \underline{L} \rightarrow \underline{F}_0$  to  $\underline{K}'$ . First note that  $K'^{\circ} = K^{\circ}$  and  $vK'/vK$  is finite since the extension  $K'/K$  is finite by assumption. Consequently, the embedding of totally ordered groups  $vK' \rightarrow vF$  induced by  $\eta$  equals the restriction of  $\mu: vL \rightarrow vF$  to  $vK'$ , so it remains to show that the group monomorphism  $G_{\underline{K}'}^{\circ} \rightarrow G_{\underline{F}}^{\circ}$  induced by  $\eta$  equals the restriction of  $\mu: G_{\underline{L}}^{\circ} \rightarrow G_{\underline{F}}^{\circ}$  to  $G_{\underline{K}'}^{\circ}$ . By construction of  $\eta$  it suffices to verify the equality  $G_{\underline{K}'}^{\circ} = G_{\underline{K}}^{\circ} \cdot \hat{t}_1^{\frac{Z}{1}} \hat{t}_2^{\frac{Z}{2}} \dots \hat{t}_r^{\frac{Z}{r}}$ , where  $\hat{t}_i$  is the coset of  $t_i$  modulo  $1 + \underline{m}_{\underline{K}'}^{\circ}$ ,  $1 \leq i \leq r$ . In order to do this we have to show that the canonical group morphism  $T \rightarrow G_{\underline{K}'}^{\circ}$  is onto. Let  $z$  be an element of  $K'^X$ . As  $T/K^X \simeq vK'/vK$ ,  $z = tu$  with  $t \in T$ ,  $u \in O_{\underline{K}'}^X$ . Since  $K'^{\circ} = K^{\circ}$ ,  $u = u'a$  with  $u' \in O_{\underline{K}}^X$ ,  $a \in 1 + \underline{m}_{\underline{K}'}^{\circ}$ . Thus  $z = (tu')a \in T(1 + \underline{m}_{\underline{K}'}^{\circ})$  as contended.

We have shown that  $(K', \eta) \in \Lambda$  and hence  $K' = K$  since  $\Lambda = \{(K, K \hookrightarrow F)\}$ .  $K'$  being arbitrary we conclude that  $K$  is algebraically closed in  $L$ .

4) Now it is the time to show that  $K^{\circ} = L^{\circ}$ . Assuming the contrary, let  $x \in L^{\circ} \setminus K^{\circ}$ . As  $K^{\circ}$  is algebraically closed in  $L^{\circ}$  by 2),  $x$  is transcendental over  $K^{\circ}$ . Let  $\underline{K}'$  be the rational function field  $K(x)$  with the valuation induced by the valuation  $v$  of  $\underline{L}$ . According to [10] Ch. 6, §10, Proposition 2, the restriction of  $\hat{v}$  to  $K'$  is the

unique valuation of  $K'$  extending the valuation  $\dot{v}$  of  $K$  subject to  $\dot{v}x=0$  and  $x^0=x$  transcendental over  $K^0$ . Note that  $\dot{v}K'=\dot{v}K$  and  $K'^0=K^0(x)$ ; therefore the correspondence  $x \mapsto \mu'(x)$  defines a valued field  $\underline{K}$ -embedding  $\eta: \underline{K}' \rightarrow \underline{F}$ . It remains to verify that the  $\underline{K}_0$ -embedding  $\hat{\eta}: \underline{K}'_0 \rightarrow \underline{F}_0$  induced by  $\eta$  equals the restriction of  $\mu$  to  $\underline{K}'_0$  in order to get  $(K', \eta) \in \Lambda$  contrary to the fact that  $\Lambda = \{(K, K \hookrightarrow F)\}$ . As  $\hat{\eta}$  and  $\mu$  coincide on  $\underline{K}'^0$  by definition of  $\eta$ , and  $\dot{v}K'=\dot{v}K$ , it suffices to observe that  $G_{\underline{K}}^{\dot{v}} = G_{\underline{K}}^{\dot{v}}(K'^0)^X$ . Indeed, any element  $z \in K'^X$  can be written in the form  $z=au$  with  $a \in K^X$ ,  $u \in O_{\underline{K}}^X$ , since  $\dot{v}K'=\dot{v}K$ .

5) Let us show that  $\dot{v}L=\dot{v}K$ . As  $K^0=L^0$  by 4), it suffices to verify that  $\dot{v}L=\dot{v}K$ . Assuming the contrary, let  $x \in L \setminus K$  be such that  $\dot{v}x \notin \dot{v}K$ . As  $K$  is algebraically closed in  $L$  by 3) and  $K^0=L^0$ , the factor group  $\dot{v}L/\dot{v}K$  is torsion free. Indeed, let  $y \in L^X$  be such that  $n\dot{v}y=\dot{v}a$  with  $a \in K^X$ ,  $n \geq 1$  i.e.  $\dot{v}(y^n a^{-1})=0$ . Since  $K^0=L^0$ ,  $y^n a^{-1}=bu$  with  $b \in O_{\underline{K}}^X$ ,  $u \in 1+\underline{m}_{\underline{L}}$ . As  $\underline{L}$  is Henselian of residue characteristic zero,  $u=z^n$  for some  $z \in O_{\underline{L}}^X$ ; therefore  $(yz^{-1})^n=ab \in K$ . Since  $K$  is algebraically closed in  $L$ ,  $yz^{-1} \in K$  and  $\dot{v}y=\dot{v}(yz^{-1}) \in \dot{v}K$  as contended.

Let  $\underline{K}'$  be the rational function field  $K(x)$  with the valuation induced by the valuation  $v$  of  $\underline{L}$ . Since  $\dot{v}K'/\dot{v}K \neq 0$  is torsion free as a subgroup of  $\dot{v}L/\dot{v}K$ , it follows by [10], Ch.6, §10, Proposition 1, that  $\dot{v}K'=\dot{v}K \oplus \underline{\mathbb{Z}} \dot{v}x$  and  $\dot{v}(\sum_{i=1}^1 a_i x^i) = \min_i (\dot{v}a_i + i\dot{v}x)$  for arbitrary  $a_i \in K$ ,  $1 \leq i \leq 1$ . Let us choose an element  $y \in F^X$  whose coset modulo  $1+\underline{m}_{\underline{F}}$  is  $\mu(x(1+\underline{m}_{\underline{L}}))$ ; in particular  $\dot{v}y=\mu''(\dot{v}x)$  is of infinite order modulo  $\dot{v}K$  and hence  $y$  is transcendental over  $K$ . Thus the substitution  $x \mapsto y$  defines a field  $K$ -embedding  $\eta: K' \rightarrow F$ . We claim that  $\eta: \underline{K}' \rightarrow \underline{F}$  is a valuation field  $\underline{K}$ -embedding such that the induced  $\dot{v}K$ -embedding  $\dot{v}K' \rightarrow \dot{v}F$  equals restriction of  $\mu''$  to  $\dot{v}K'$ . Indeed, let  $f(x) = \sum a_i x^i \in K'^X$ . Then  $\dot{v}f(x) = \min_i (\dot{v}a_i + i\dot{v}x)$ . On the other hand, [10], Ch.6, §10, Proposition 1 may also be applied to  $K(v)$  instead of  $K'=K(x)$ ; we get  $\dot{v}f(y) = \min_i (\dot{v}a_i + i\dot{v}y)$ , i.e.  $\dot{v}(\eta(f(x))) = \mu''(\dot{v}f(x))$  as contended. Moreover, as  $K^0=K'^0=L^0$ ,  $\eta$  is in fact a  $\underline{K}$ -embedding



of the valued field  $\underline{K}'$  into  $\underline{F}$ . Thus it remains to show that the  $\underline{K}'_0$ -embedding  $\hat{\eta} : \underline{K}'_0 \rightarrow \underline{F}_0$  of mixed  $\omega$ -structures induced by  $\eta$  equals the restriction of  $\mu$  to  $\underline{K}'_0$  in order to obtain  $(K', \eta) \in \Lambda$  contrary to the assumption that  $\Lambda = \{(K, K \hookrightarrow F)\}$ . Since  $K'^0 = K^0$  and  $\hat{v}K' = \hat{v}K \oplus \oplus \hat{v}x$  we obtain  $G_{\underline{K}'}^e = G_{\underline{K}}^e \cdot \hat{x}^{\mathbb{Z}}$ , where  $\hat{x}$  is the coset of  $x$  modulo  $1 + m_{\underline{K}}^e$ ; therefore  $\hat{\eta}$  and  $\mu$  coincide on  $G_{\underline{K}'}^e$ , by definition of  $\eta$ .

6) Finally we are ready to prove that  $K=L$ . Assuming the contrary, let  $x \in L \setminus K$  and  $K' = K(x)$ . The element  $x$  is transcendental over  $K$  by 3),  $K^e = K'^e$  by 4),  $vK = vK'$  by 5); consequently, we get also  $G_{\underline{K}}^e = G_{\underline{K}'}^e$ , and hence  $\underline{K}_0 = \underline{K}'_0$ . Thus it remains to show that there exists a  $\underline{K}$ -embedding  $\eta : \underline{K}' \rightarrow \underline{F}$  in order to get  $(K', \eta) \in \Lambda$  contrary to the assumption that  $\Lambda = \{(K, K \hookrightarrow F)\}$ . As  $K^0 = K'^0$ , it suffices to show that there exists a  $\underline{K}$ -embedding  $\eta : \underline{K}' \rightarrow \underline{F}$ .

As the valued field extension  $\underline{K}'/\underline{K}$  is immediate, there exists a pseudo-convergent sequence  $\underline{a} = (a_\gamma)_{\gamma < \lambda}$  in  $\underline{K}$ ,  $\lambda \leq |K|$ , without pseudo-limits in  $K$  such that  $x$  is a pseudo-limit of  $\underline{a}$ . Indeed, let  $C = \{\hat{v}(x-c) : c \in K\}$ ; we show that  $C$  has no greatest element. Let  $\alpha = \hat{v}(x-c) \in C$ ; then there exists  $b \in K^X$  such that  $\alpha = \hat{v}b$ , since  $\hat{v}K = \hat{v}K'$ . Also there is  $d \in O_{\underline{K}}^X$  such that  $\hat{v}(d - (x-c)b^{-1}) > 0$  since  $K^0 = K'^0$ . Consequently,  $\hat{v}(x - (c+bd)) > \alpha$ , giving a greater element of  $C$ . Thus there exist a limit ordinal  $\lambda \leq |K|$  and an increasing sequence  $(\alpha_\gamma)_{\gamma < \lambda}$  of elements of  $C$  cofinal with  $C$ . Let us choose  $a_\gamma \in K$  such that  $\hat{v}(x - a_\gamma) = \alpha_\gamma$ ,  $\gamma < \lambda$ . Then  $\underline{a} = (a_\gamma)_{\gamma < \lambda}$  is a pseudo-convergent sequence and  $x$  is a pseudo-limit of  $\underline{a}$ , written  $\underline{a} \rightarrow x$ . If  $\underline{a} \rightarrow b$  for some  $b \in K$ , then  $\hat{v}(x-b) > \alpha_\gamma$  for all  $\gamma < \lambda$ , contrary to the fact that  $(\alpha_\gamma)_{\gamma < \lambda}$  is cofinal with  $C$ . Thus,  $\underline{a}$  has no pseudo-limit in  $K$ .

Let us show that  $\underline{a}$  is a transcendental sequence, i.e.  $f(\underline{a}) = (f(a_\gamma))_{\gamma < \lambda} \neq 0$  for all  $f \in K[X] \setminus \{0\}$ . Assuming the contrary, let  $f \in K[X] \setminus \{0\}$  be a counterexample of smallest degree:  $f(\underline{a}) \rightarrow 0$ ,  $n = \deg f$  minimal. Obviously  $n \neq 0$ . Moreover  $n > 1$ . Indeed, if  $f(X) = c(X-b)$  with  $b \in K$ ,  $c \in K^X$  then  $\underline{a} \rightarrow b$  contradicting the fact that  $\underline{a}$

has no pseudo-limit in  $K$ . If  $f(X) = g(X)h(X)$  would be reducible then its factors  $g, h$  have smaller degree and hence  $f(\underline{a}) \not\rightarrow 0$  since  $g(\underline{a}) \not\rightarrow 0$  and  $h(\underline{a}) \not\rightarrow 0$ , contrary to the choice of  $f$ . Thus  $f(X)$  is irreducible and the factor  $K$ -algebra  $M = K[X]/f(X)$  is a field extension of  $K$  of finite degree  $[M:K] = n > 1$ . Now we define a prolongation  $w$  of the valuation  $\dot{v}$  of  $K$  to  $M$ , in such a way that the extension  $(M, w)/\dot{K}$  is immediate. For each  $g \in K[X] \setminus \{0\}$  of degree  $< n$  there is  $\tau(g) < \lambda$  such that  $\dot{v}g(\underline{a}_\xi) = \dot{v}g(\underline{a}_{\tau(g)})$  for all  $\xi < \lambda$  subject to  $\tau(g) < \xi$ , since  $g(\underline{a}) \not\rightarrow 0$  by assumption; as  $\underline{a} \rightarrow x$  we obtain  $\dot{v}g(x) = \dot{v}g(\underline{a}_{\tau(g)})$ . For every such  $g$  let us put  $w(g(X) \bmod f(X)) = \dot{v}g(x)$ . Thus we get a map  $w: M \rightarrow \dot{v}K$  satisfying all properties of valuations except possibly the multiplication rule  $w(gh \bmod f) = w(g \bmod f) + w(h \bmod f)$  for  $g, h \in K[X] \setminus \{0\}$ . We claim that this rule is also satisfied. Indeed let  $g, h \in K[X] \setminus \{0\}$  of degree  $< n$ . By Euclid's algorithm we obtain  $gh = qf + r$  with  $q, r \in K[X]$  of degree  $< n$ . We have to show that  $\dot{v}r(x) = \dot{v}g(x) + \dot{v}h(x)$ . Obviously, it suffices to show that  $\dot{v}r(x) < \dot{v}g(x) + \dot{v}h(x)$ . Assume the contrary and let us choose an ordinal  $\tau < \lambda$  such that  $\tau > \max(\tau(g), \tau(h), \tau(r), \tau(q))$ ,  $\dot{v}f(\underline{a}_\eta) > \dot{v}f(\underline{a}_\xi)$  for  $\tau \leq \xi < \eta < \lambda$  and  $\dot{v}f(x) > \dot{v}f(\underline{a}_\xi)$  for  $\tau \leq \xi < \lambda$ . Such a  $\tau$  exists since  $f(\underline{a}) \rightarrow 0$  and  $\underline{a} \rightarrow x$ ; therefore  $f(\underline{a}) \rightarrow f(x)$  by continuity for pseudo-convergence. For  $\tau \leq \xi < \lambda$  we get  $\dot{v}r(\underline{a}_\xi) = \dot{v}r(x) \geq \dot{v}g(x) + \dot{v}h(x) > \dot{v}g(\underline{a}_\xi) + \dot{v}h(\underline{a}_\xi)$ . Consequently, for  $\tau < \xi < \lambda$  we obtain  $\dot{v}g(\underline{a}_\xi) + \dot{v}h(\underline{a}_\xi) = \dot{v}g(\underline{a}_\tau) + \dot{v}h(\underline{a}_\tau) = \dot{v}q(\underline{a}_\tau) + \dot{v}f(\underline{a}_\tau) < \dot{v}q(\underline{a}_\xi) + \dot{v}f(\underline{a}_\xi) = \dot{v}g(\underline{a}_\xi) + \dot{v}h(\underline{a}_\xi)$ , a contradiction. We conclude that  $w$  is a valuation of  $M$  which extends the valuation  $\dot{v}$  of  $K$  and  $wM = \dot{v}K$ . It remains to verify that the residue field of  $w$  equals  $K^0$ ; i.e. the extension  $(M, w)/\dot{K}$  is immediate: let  $g \in K[X] \setminus \{0\}$  of degree  $< n$  be such that  $w(g(X) \bmod f(X)) = \dot{v}g(x) = 0$ ; as  $K^0 = K'^0$  there is  $b \in \dot{K}$  such that  $w((g(X) - b) \bmod f(X)) = \dot{v}(g(x) - b) > 0$ . Since  $\dot{K}$  is algebraically complete (as a Henselian field of residue characteristic zero) and  $(M, w)$  is an algebraic immediate extension of  $\dot{K}$  we obtain  $K = M$ , contradicting the fact  $[M:K] = n > 1$ . We conclude that the sequence  $\underline{a}$  is transcendental.



Here it is the point where we use the assumption that  $\underline{\dot{F}}$  is  $|\underline{L}|$  - pseudo-complete; therefore  $\underline{a}$  has a pseudo-limit  $\underline{y}$  in  $\underline{F}$ . As  $\underline{a}$  is transcendental, its pseudo-limit  $\underline{y}$  is transcendental over  $\underline{K}$ ; thus the substitution  $x \mapsto y$  defines a field  $\underline{K}$ -embedding  $\eta: \underline{K}' \rightarrow \underline{F}$ . We claim that  $\eta$  is a  $\underline{K}$ -embedding of the valued field  $\underline{K}'$  into  $\underline{\dot{F}}$ . Indeed, let  $f \in \underline{K}[X] \setminus \{0\}$ . As  $\underline{a}$  is transcendental,  $f(\underline{a}) \neq 0$  and hence there is  $\tau < \lambda$  such that  $\dot{v}f(x) = \dot{v}f(\underline{a}_\tau) = \dot{v}f(\underline{y})$  for all  $\tau \leq \lambda$ .

O.E.D.

The next embedding criterion for Henselian fields of characteristic zero is an immediate consequence of Theorem 2.1.

Corollary 2.2. Let  $\underline{K}=(\underline{K}, \underline{v})$  be a valued field of characteristic zero and  $\underline{L}=(\underline{L}, \underline{v})$ ,  $\underline{F}=(\underline{F}, \underline{v})$  be Henselian valued fields extending  $\underline{K}$ . Suppose that  $\underline{\dot{F}}$  is  $|\underline{L}|$  - pseudo-complete. The necessary and sufficient condition for  $\underline{L}$  to be  $\underline{K}$ -isomorphically embeddable into  $\underline{F}$  is that the mixed  $\omega$ -structure  $\underline{\dot{L}}_0$  is  $\underline{K}_0$ -embeddable in  $\underline{\dot{F}}_0$ .

### 3. Mixed structures

We have introduced in the previous section the so called mixed  $\omega$  - structures and we have seen the key role played by these ones in embedding problems for Henselian fields of characteristic zero. The mixed  $\omega$ -structure  $\underline{\dot{K}}_0$  assigned to a valued field  $\underline{K}=(\underline{K}, \underline{v})$  of characteristic zero is constructed with the help of the coarse valuation  $\dot{v}$  induced by  $\underline{v}$  and hence it is not an elementary object assigned to  $\underline{K}$ , except the special case when the residue characteristic of  $\underline{K}$  is zero. As the model theoretic investigation of Henselian fields requires more elementary invariants, it seems natural to approximate the global object  $\underline{\dot{K}}_0$  above by a family of suitable objects that are definable in elementary terms. Possible candidates for these elementary objects are the mixed  $k$ -structures  $\underline{K}_k$ ,  $k \in \mathbb{N}$ ,

defined in Introduction.

In abstract terms, a mixed k-structure ( $k \in \mathbb{N}$ ) is a system  $\underline{\Lambda} = (\Lambda, H, \Gamma, \Theta, v)$  where  $\Lambda$  is a commutative ring with 1,  $H$  is a multiplicative Abelian group,  $\Gamma$  is an additive totally ordered group,  $\Theta$  is a partial map from  $\Lambda$  into  $H$  and  $v: (\Lambda \setminus \{0\}) \cup H \rightarrow \Gamma$  is a map subject to the next conditions:

- 1.1) For  $a, b \in \Lambda \setminus \{0\}$ ,  $a$  divides  $b$  (written  $a|b$ ) iff  $va \leq vb$ ;
- 1.2) For  $a, b \in \Lambda \setminus \{0\}$  such that  $ab \neq 0$ ,  $v(ab) = va + vb$ ; in particular,  $va \geq 0$  for all  $a \in \Lambda \setminus \{0\}$  and  $vu = 0$  for all units  $u \in \Lambda^\times$  by 1.1); set by convention  $v0 = \infty$  with the usual rules for the symbol  $\infty$ ;
- 1.3) For  $a, b \in \Lambda$ ,  $v(a+b) \geq \min(va, vb)$ ; thus  $\Lambda$  is a local ring with maximal ideal  $\underline{m}_\Lambda = \{a \in \Lambda : va > 0\}$ ; let  $p$  be the characteristic exponent of the residue field  $\bar{\Lambda} = \Lambda / \underline{m}_\Lambda$ ;
- 1.4) The image  $v\Lambda := v(\Lambda \setminus \{0\})$  equals the convex subset  $\Gamma_{2k} = \{\alpha \in \Gamma : 0 \leq \alpha \leq vp^{2k}\}$  of  $\Gamma$ ; consequently,  $\Lambda \setminus \{0\} = \{a \in \Lambda : a|p^{2k}\}$ , the characteristic exponent of  $\Lambda$  is  $p^{2k+1}$  and  $\Lambda = \bar{\Lambda}$  iff  $p=1$ ;
- 2) The restriction  $v|_H: H \rightarrow \Gamma$  is a group epimorphism;
- 3.1) The domain of the partial map  $\Theta$  is the complement in  $\Lambda$  of the ideal  $\underline{m}_{\Lambda, k} := \{a \in \Lambda : va > vp^k\}$ ;
- 3.2) The following diagram

$$\begin{array}{ccccc}
 \Lambda \setminus \{0\} & \xrightarrow{v|_\Lambda} & \Gamma_{2k} & \hookrightarrow & \Gamma \\
 \uparrow & & & & \uparrow \\
 \Lambda \setminus \underline{m}_{\Lambda, k} & \xrightarrow{\Theta} & H & & 
 \end{array}
 \quad \begin{array}{c} \\ \\ v|_H \end{array}$$

commutes;

- 3.3) For  $a, b \in \Lambda \setminus \underline{m}_{\Lambda, k}$  such that  $ab \notin \underline{m}_{\Lambda, k}$ ,  $\Theta(ab) = \Theta(a)\Theta(b)$ ; in particular  $\Theta|_{\Lambda^\times}: \Lambda^\times \rightarrow H$  is a group morphism;



3.4)  $\text{Ker}(v|_H) \subset \Theta(A^X)$ ; in fact we have equality by 3.2);

3.5) For  $a, b \in A \setminus \underline{m}_{A,k}$ ,  $\Theta(a) = \Theta(b)$  iff  $v(a-b) > v p^k + v b$ ; thus  $A^X / 1 + \underline{m}_{A,k} \simeq \text{Ker}(v|_H)$  by 3.4.

Note also that the image of the map  $\Theta$  equals  $\{h \in H: 0 \leq v h \leq v p^k\}$ . For suppose  $a \in A \setminus \underline{m}_{A,k}$ , i.e.  $v a \leq v p^k$ ; we get  $v \Theta(a) = v a \leq v p^k$  by 3.2). Conversely, let  $h \in H$  be such that  $0 \leq v h \leq v p^k$ . By 1.4) there is  $a \in A$  such that  $va = vh$ . As  $a \in \underline{m}_{A,k}$ , we get  $v \Theta(a) = va = vh$  by 3.2). Thus  $h \Theta(a)^{-1} \in \text{Ker}(v|_H)$  and hence  $h \Theta(a)^{-1} = \Theta(b)$  for some  $b \in A^X$  by 3.4). It follows  $h = \Theta(a) \Theta(b) = \Theta(ab)$  by 3.3).

Given  $k, k' \in \mathbb{N}$ ,  $k \geq k'$ , an Abelian totally ordered group  $\Gamma$ , a mixed  $k$ -structure  $\underline{A} = (A, H, \Gamma, \Theta, v)$  and a mixed  $k'$ -structure  $\underline{A}' = (A', H', \Gamma, \Theta', v')$  such that  $A$  and  $A'$  have the same residue characteristic exponent, say  $p$ , a restriction map  $\pi: \underline{A} \rightarrow \underline{A}'$  is a map  $\pi: A \cup H \rightarrow A' \cup H'$  subject to the next conditions:

i)  $\pi|_A$  is an unitary ring morphism from  $A$  onto  $A'$  which induces an isomorphism  $A / \underline{m}_{A,2k} \simeq A'$ ; in particular,  $\pi|_A$  induces an isomorphism of residue fields  $\bar{A} \simeq \bar{A}'$ ;

ii) if  $a \in A \setminus \underline{m}_{A,2k'}$  then  $va = v' \pi(a)$ ; in particular, for  $k' > 0$ , we get  $vp = v'p$ ;

iii)  $\pi|_H$  is a group epimorphism from  $H$  onto  $H'$  such that  $vh = v' \pi(h)$  for all  $h \in H$ ;

iv)  $\pi(\Theta(a)) = \Theta'(\pi(a))$  for all  $a \in A \setminus \underline{m}_{A,k}$ .

Note that the restriction map  $\pi$  induces an isomorphism  $H / \Theta(1 + \underline{m}_{A,k}) \simeq H'$ . For suppose  $a \in 1 + \underline{m}_{A,k}$ , i.e.  $v(a-1) > v p^k$ . By ii) we get  $v' \pi(a-1) > v' p^k$  i.e.  $\pi(a) \in 1 + \underline{m}_{A',k'}$  and hence  $\Theta'(\pi(a)) = 1$  since  $A'^X / 1 + \underline{m}_{A',k'} \simeq \text{Ker}(v'|_{H'})$  by 3.5). On the other hand,  $\pi(\Theta(a)) = \Theta'(\pi(a)) = 1$  by iv). Conversely, let  $h \in H$  be such that  $\pi(h) = 1$ . In particular  $vh = 0$  by iii) and hence there is  $a \in A^X$  such that  $h = \Theta(a)$  since  $A^X / 1 + \underline{m}_{A,k} \simeq \text{Ker}(v|_H)$ . By iv), we obtain

$\theta'(\pi(a)) = \pi(\theta(a)) = \pi(h) = 1$ ; therefore  $\pi(a) \in 1 + \underline{m}_{\Lambda', k'}$ . According to

ii) it follows  $a \in 1 + \underline{m}_{\Lambda, k}$ , as contended.

By a projective system of mixed structures we mean a system  $\underline{\Lambda} = (\underline{\Lambda}_k; \pi_{k, \ell})_{k \in \mathbb{N}, k \leq \ell}$ , where  $\underline{\Lambda}_k = (\underline{\Lambda}_k, H_k, \Gamma_k, \theta_k, v_k)$  is a mixed  $k$ -structure with  $\Gamma_k = \Gamma$  for all  $k \in \mathbb{N}$ , and  $\pi_{k, \ell}: \underline{\Lambda}_\ell \rightarrow \underline{\Lambda}_k$  for  $k \leq \ell$  is a restriction map as defined above such that  $\pi_{k, \ell} = \pi_{k, s} \circ \pi_{s, \ell}$  for  $k \leq s \leq \ell$  and  $\pi_{k, k} = 1_{\underline{\Lambda}_k}$ .

To a mixed  $\omega$ -structure  $\underline{S} = (\underline{M}, H, \Gamma, \theta, v)$  one assigns naturally a projective system of mixed structures  $\underline{P}(\underline{S}) = (\underline{S}_k; \pi_{k, \ell})_{k \in \mathbb{N}, k \leq \ell}$ , where  $\underline{S}_k = (O_{\underline{M}, 2k}, H_k, \Gamma, \theta_k, v_k)$  with  $O_{\underline{M}, 2k} = O_{\underline{M}} / \underline{m}_{\underline{M}, 2k}$ ,  $H_k = H / \theta(1 + \underline{m}_{\underline{M}, k})$  and  $\theta_k: O_{\underline{M}} \setminus \underline{m}_{\underline{M}, k} / \underline{m}_{\underline{M}, 2k} \rightarrow H_k$ , by  $\theta: M^x \hookrightarrow H$ ,  $v: H \rightarrow \Gamma$ , and  $\pi_{k, \ell}: \underline{S}_\ell \rightarrow \underline{S}_k$  for  $k \leq \ell$  are the canonic restriction maps. Note that  $\underline{P}(\underline{S})$  is identified with  $\underline{S}$  if the residue characteristic of  $\underline{M}$  is zero.

As we are interested in organizing the projective systems of mixed structures into a category we have to define morphisms between such objects.

First we define the embeddings of mixed  $k$ -structures for given  $k \in \mathbb{N}$ . Let  $\underline{A} = (A, H, \Gamma, \theta, v)$  and  $\underline{A}' = (A', H', \Gamma', \theta', v')$  be mixed  $k$ -structures. An embedding  $\varphi: \underline{A} \rightarrow \underline{A}'$  is a map  $\varphi: A \cup H \cup \Gamma \rightarrow A' \cup H' \cup \Gamma'$  subject to the next conditions:

- i)  $\varphi|_A$  is a ring embedding of  $A$  into  $A'$ ;
- ii)  $\varphi|_H$  is a group embedding of  $H$  into  $H'$ ;
- iii)  $\varphi|_\Gamma$  is an ordered group embedding of  $\Gamma$  into  $\Gamma'$ ;
- iv)  $v' \varphi(a) = \varphi(va)$  for all  $a \in A$ ;
- v)  $v' \varphi(h) = \varphi(vh)$  for all  $h \in H$ ;
- vi)  $\theta'(\varphi(a)) = \varphi(\theta(a))$  for all  $a \in A \setminus \underline{m}_{\Lambda, k}$ .

Now, given the projective systems of mixed structures



Let  $\underline{A} = (\underline{A}_k; \pi_{k,l})_{k \in \mathbb{N}, k \leq l}$  and  $\underline{A}' = (\underline{A}'_k; \pi'_{k,l})_{k \in \mathbb{N}, k \leq l}$ , an embedding of  $\underline{A}$  into  $\underline{A}'$  is a family  $\varphi_k: \underline{A}_k \rightarrow \underline{A}'_k$  of embeddings of mixed  $k$ -structures,  $k \in \mathbb{N}$ , subject to the compatibility condition  $\varphi_k \pi_{k,l} = \pi'_{k,l} \varphi_l$  for  $k \leq l$ .

If  $\underline{A}, \underline{B}$  are respectively mixed  $k$ -structures,  $\omega$ -structures, projective systems of mixed structures, denote by  $\text{Hom}(\underline{A}, \underline{B})$  the set of embeddings of  $\underline{A}$  into  $\underline{B}$ .

Call a mixed  $\omega$ -structure  $\underline{S} = (\underline{M}, H, \Gamma, \theta, v)$  complete if  $\underline{M}$  is Cauchy complete, i.e.  $\underline{O}_{\underline{M}} \simeq \varprojlim \underline{O}_{\underline{M},k}$ , and  $H$  is Cauchy complete with respect to the topology given by the system of open neighborhoods of 1

$$\{\theta(1 + \underline{m}_{\underline{M},k})\}_{k \in \mathbb{N}}, \text{ i.e. } H \simeq \varprojlim H / \theta(1 + \underline{m}_{\underline{M},k}).$$

The next lemmas are immediate.

Lemma 3.1. Given a mixed  $\omega$ -structure  $S$  there exist a complete mixed  $\omega$ -structure  $\hat{S}$  and an embedding  $i: S \rightarrow \hat{S}$  such that for every embedding  $\varphi: S \rightarrow S'$  with  $S'$  complete, there is a unique embedding  $\psi: \hat{S} \rightarrow S'$  subject to  $\varphi = \psi i$ .

Call the mixed  $\omega$ -structure  $\hat{S}$  above the completion of  $S$ ;  $\hat{S}$  is unique up to an isomorphism over  $S$ .

Lemma 3.2. Let  $S$  and  $S'$  be mixed  $\omega$ -structures and assume  $S'$  is complete. The canonic map  $\text{Hom}(S, S') \rightarrow \text{Hom}(P(S), P(S'))$  is bijective.

Given  $k \in \mathbb{N}$ , the category of mixed  $k$ -structures, with embeddings as morphisms, can be seen as the category of models of a theory  $T_k$  in a suitable first order language  $L_k$ . We may consider the mixed  $k$ -structures as one-sorted structures, as well as (finitely) many sorted structures with sorts for the ring  $A$ , the group  $H$  and the ordered group  $\Gamma$ ; we choose in the following the many-sorted approach. The languages  $L_k$ ,  $k \in \mathbb{N}$ , are related each to other thanks to the next translation procedure.

Lemma 3.3. Let  $k, \ell \in \mathbb{N}$  be such that  $k \leq \ell$ . Given an  $L_k$ -formula  $\Psi(\underline{x})$ ,  $\underline{x} = (x_1, \dots, x_n)$ , one may assign effectively to it an  $L_\ell$ -formula  $\text{tr}_{k, \ell}(\Psi)(\underline{x}; y)$ , where  $y$  is a new variable of sort  $A$ , in such a way that for every restriction map  $\pi$  defined on a mixed  $\ell$ -structure  $A$  of residue characteristic exponent  $p$  with values into a mixed  $k$ -structure  $A'$  and for every  $\underline{a} = (a_1, \dots, a_n)$  in  $A$  of suitable sorts,  $A \models \Psi(\pi(\underline{a}))$  iff  $A' \models \text{tr}_{k, \ell}(\Psi)(\underline{a}; p)$ .

Proof. We define the translation map  $\text{tr}_{k, \ell}: L_k \rightarrow L_\ell$  by induction on the complexity of the formula  $\Psi$ .

1) Assuming  $\Psi$  atomic, we distinguish the next cases:

i)  $\Psi := (f(\underline{x}) = 0)$ , where  $f \in \mathbb{Z}[\underline{x}]$ ,  $\underline{x} = (x_1, \dots, x_n)$  variables of sort  $A$ ; let us put  $\text{tr}_{k, \ell}(\Psi) := v f(\underline{x}) > v^{2k}$ ;

ii)  $\Psi := (t = 1)$ , where  $t$  is a term of sort  $\Pi$ , i.e.  $t$  is a word



$w_1^{n_1} \dots w_m^{n_m}$  with  $n_1, \dots, n_m \in \mathbb{Z}$  and  $w_1$  has either the form  $\theta(f(\underline{x}))$  with  $f \in \mathbb{Z}[\underline{x}]$ ,  $\underline{x} = (x_1, \dots, x_n)$  variables of sort  $A$ , or it is a variable of sort  $H$ ; define  $\text{tr}_{k,l}(\psi) := [(\exists z) (v(z-1) > v\psi^k) \wedge (\theta(z)=t)]$  where  $z$  is a new variable of sort  $A$ ;

iii)  $\psi := (\xi = 0)$ , where  $\xi$  is a term of sort  $\Gamma$ , i.e.

$\xi = \sum_{i=1}^m n_i \alpha_i$  with  $n_1, \dots, n_m \in \mathbb{Z}$  and  $\alpha_i$  has either the form  $v f(\underline{x})$  with  $f \in \mathbb{Z}[\underline{x}]$ ,  $\underline{x} = (x_1, \dots, x_n)$  variables of sort  $A$ , or the form  $vt$ , where  $t$  is a term of sort  $H$ ; let us put  $\text{tr}_{k,l}(\psi) := (\xi = 0)$ ;

iv)  $\psi := (\xi > 0)$ , where  $\xi$  is a term of sort  $\Gamma$ ; set  $\text{tr}_{k,l}(\psi) := (\xi > 0)$ .

$$2) \text{tr}_{k,l}(\psi_1 \wedge \psi_2) := \text{tr}_{k,l}(\psi_1) \wedge \text{tr}_{k,l}(\psi_2);$$

$$3) \text{tr}_{k,l}(\neg \psi) := \neg \text{tr}_{k,l}(\psi);$$

$$4) \text{tr}_{k,l}((\exists z) \psi(z)) := (\exists z) \text{tr}_{k,l}(\psi(z)).$$

O.E.D.

Similarly we may interpret the category of projective systems of mixed structures, with embeddings as morphisms, as the category of models of a theory  $T_\omega$  in a suitable first order  $\omega$ -sorted language  $L_\omega$  with sorts for the ring  $A_k$  ( $k \in \mathbb{N}$ ), the group  $H_k$  ( $k \in \mathbb{N}$ ) and the ordered group  $\Gamma$ . The next lemma is an immediate consequence of Lemma 3.3.

Lemma 3.4. Let  $\psi(x_1, \dots, x_n)$  be an  $L_\omega$ -formula. For

$$1 \leq i \leq n, \text{ let } m_i = \begin{cases} k & \text{if } x_i \text{ is of sort } \Lambda_k \text{ or } H_k \\ 0 & \text{if } x_i \text{ is of sort } \Gamma. \end{cases}$$

One may assign effectively to  $\psi$  a natural number  $m \geq \max_i(m_i)$  and an  $L_m$ -formula  $\varphi(y_1, \dots, y_n; z)$  in such a way that for every projective system of mixed structures  $\underline{A} = (\underline{A}_k; \pi_{k,l})_{k \in \mathbb{N}, k \leq l}$  of residue characteristic exponent  $p$  and for arbitrary  $a_1, \dots, a_n$  in  $\underline{A}_m$  of suitable sorts,  $\underline{A} \models \psi(\pi_{m,1,m}(a_1), \dots, \pi_{m,n,m}(a_n))$  iff  $\underline{A}_m \models \varphi(a_1, \dots, a_n; p)$ .

Given an uncountable cardinal  $K$ , a projective system of mixed structures  $\underline{A} = (\underline{A}_k; \pi_{k,l})_{k \in \mathbb{N}, k \leq l}$  is  $K$ -saturated if for every set  $\phi$  of  $L_\omega$ -formulas with parameters from  $\underline{A}$  (i.e. involving constants for elements of  $\underline{A}$ ) the following hold: if  $\phi$  has cardinality less than  $K$  and every finite set of formulas from  $\phi$  is realized in  $\underline{A}$  then all formulas from  $\phi$  are simultaneously realized in  $\underline{A}$ .

Lemma 3.5. Let  $\underline{S} = (\underline{M}, H, \Gamma, \Theta, v) \hookrightarrow \underline{S}^{(1)} = (\underline{M}^{(1)}, H^{(1)}, \Gamma^{(1)}, \Theta^{(1)}, v^{(1)})$ ,  $i=1,2$ , be embeddings of mixed  $\omega$ -structures. Suppose that  $\underline{S}^{(2)}$  is complete and  $\underline{P}(\underline{S}^{(2)})$  is  $K$ -saturated for some cardinal  $K > |\underline{S}^{(1)}| = |H^{(1)}|$ . The next statements are equivalent:

- i)  $\text{Hom}_{\underline{S}}(\underline{S}^{(1)}, \underline{S}^{(2)})$  is non-empty;
- ii)  $\text{Hom}_{\underline{S}_k}(\underline{S}_k^{(1)}, \underline{S}_k^{(2)})$  is non-empty for all  $k \in \mathbb{N}$ .

Proof. The implication i)  $\rightarrow$  ii) is obvious.

ii)  $\rightarrow$  i) As  $\underline{S}^{(2)}$  is complete, it suffices to show that  $\text{Hom}_{\underline{P}(\underline{S})}(\underline{P}(\underline{S}^{(1)}), \underline{P}(\underline{S}^{(2)}))$  is non-empty, according to Lemma 3.2. Let  $X = \{x_a : a \in \underline{P}(\underline{S}^{(1)}) \setminus \underline{P}(\underline{S})\}$  be a set of  $L_\omega$ -variables such that  $x_a$  has the same sort as  $a$ , and consider the set  $\phi$  of all basic  $L_\omega$ -formulas (atomic and negated atomic formulas)  $\varphi(x_{a_1}, \dots, x_{a_n})$  with parameters in  $\underline{P}(\underline{S})$  subject to  $\underline{P}(\underline{S}^{(1)}) \models \varphi(a_1, \dots, a_n)$ . Obviously,  $\text{Hom}_{\underline{P}(\underline{S})}(\underline{P}(\underline{S}^{(1)}), \underline{P}(\underline{S}^{(2)}))$  is non-empty iff  $\phi$  is realized in



$\mathcal{P}(S^{(2)})$ . By  $\underline{K}$ -saturation property of  $\mathcal{P}(S^{(2)})$  it suffices to show that each finite subset of  $\phi$  is realized on  $\mathcal{P}(S^{(2)})$ . Let  $\varphi_i(x_{a_1}, \dots, x_{a_n})$ ,  $1 \leq i \leq m$ , be formulas from  $\phi$ . According to Lemma 3.3, we may assume without loss of generality that there is  $k \in \mathbb{N}$  such that all parameters occurring in  $\varphi_i$ ,  $1 \leq i \leq m$ , denote constants from  $S_k$ ; we may also assume that  $a_1, \dots, a_n$  belong to  $S_k^{(1)}$ . Since  $\text{Hom}_{S_k}(S_k^{(1)}, S_k^{(2)})$  is non-empty by assumption, we conclude that the formula  $\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_m$  is realized on  $S_k^{(2)}$  and hence it is also realized on  $\mathcal{P}(S^{(2)})$ .

O.E.D.

#### 4. Proof of the main results

Denote by  $\underline{L}$  the first order language of valued fields, whose vocabulary contains, besides the logical symbols, constants  $0, 1$ , function symbols for the field operations  $(+, \cdot, -, {}^{-1})$  with convention  $0^{-1} = 0$  and one unary predicate  $\underline{O}$  which in a valued field is interpreted as the valuation ring. The axioms of valued fields, as well as the axioms of Henselian valued fields can be formulated in this language.

The language  $\underline{L}$  and the languages  $\underline{L}_k$  of mixed  $k$ -structures,  $k \in \mathbb{N}$ , introduced in Section 3, are related via the following translation procedure.

Given a valued field  $\underline{K} = (K, v)$  of characteristic zero and a natural number  $k$ , let us consider the canonic maps  $\pi_{A,k}: \underline{O}_K \rightarrow \underline{O}_{K,k} = \underline{O}_K / \underline{m}_{K,k}$ ,  $\pi_{H,k}: K^x \rightarrow K^x / 1 + \underline{m}_{K,k}$ .

Lemma 4.1. Let  $k \in \mathbb{N}$  and  $\Psi(x_1, \dots, x_n; y_1, \dots, y_m; \zeta_1, \dots, \zeta_l)$  be an  $\underline{L}_k$ -formula, where the  $x_i$ 's are variables of sort  $A$ , the  $y_i$ 's are variables of sort  $H$  and the  $\zeta_i$ 's are variables of sort  $\Gamma$ . One assigns effectively to  $\Psi$  an  $\underline{L}$ -formula  $\text{tr}_k(\Psi)(z_1, z_2, \dots, z_{n+m+l+1})$  in such a way that for every valued field  $\underline{K}$  of characteristic zero

and residue characteristic exponent  $p$ , and for arbitrary  $a_1 \in O_K (1 \leq i \leq n+m+1)$ ,  $a_i \in K^X (n+1 \leq i \leq n+m+1)$ ,  $\underline{K}_k \models \Psi(\mathcal{A}_{\Lambda, 2k}(a_i) : 1 \leq i \leq n; \mathcal{A}_{\Pi, k}(a_i) : n+1 \leq i \leq n+m; va_1 : n+m+1 \leq i \leq n+m+1)$  iff  $\underline{K} \models \text{tr}_k(\Psi)(a_1, \dots, a_{n+m+1}, p)$ .

The proof is immediate.

The next embedding theorem for Henselian fields of characteristic zero is a consequence of Corollary 2.2 and Lemma 3.5.

Theorem 4.2. Let  $\underline{K} = (K, v)$  be a valued field of characteristic zero and  $\underline{L}, \underline{F}$  be Henselian valued fields extending  $\underline{K}$ . Suppose that the valued field  $\underline{F}$  is  $\underline{K}$ -saturated for some cardinal  $\underline{K} > |L|$ . The necessary and sufficient condition for the valued field  $\underline{L}$  to be  $\underline{K}$ -embeddable into  $\underline{F}$  is that the mixed  $k$ -structure  $\underline{L}_k$  is  $\underline{K}_k$ -embeddable into  $\underline{F}_k$  for all  $k \in \mathbb{N}$ .

Proof. An implication is trivial. Conversely, assume that  $\text{Hom}_{\underline{K}_k}(\underline{L}_k, \underline{F}_k)$  is non-empty for all  $k \in \mathbb{N}$ . By  $\underline{K}$ -saturation property of  $\underline{F}$ , it follows that the mixed  $\omega$ -structure  $\underline{F}_\omega$  is complete, the projective system  $\underline{P}(\underline{F}_\omega)$  is  $\underline{K}$ -saturated and the valued field  $\underline{F}$  is  $|L|$ -pseudo-complete. According to Lemma 3.5,  $\text{Hom}_{\underline{K}_\omega}(\underline{L}_\omega, \underline{F}_\omega)$  is non-empty and hence  $\underline{L}$  is  $\underline{K}$ -embeddable into  $\underline{F}$  by Corollary 2.2.

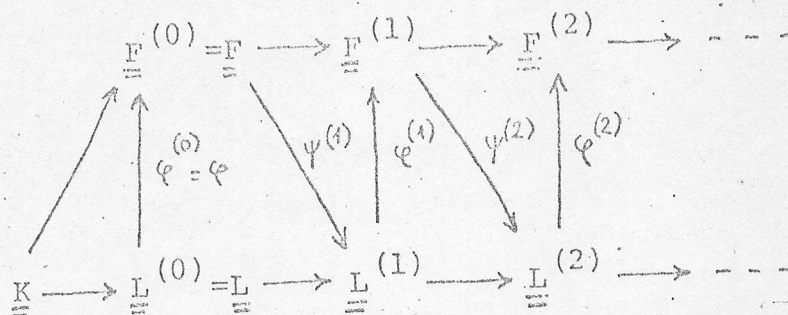
Q.E.D.

#### (4.3) Proof of Theorem A from Introduction

An implication is obvious. Conversely, let us assume that the mixed  $k$ -structures  $\underline{L}_k, \underline{F}_k$  are elementarily equivalent over  $\underline{K}_k$  (written  $\underline{L}_k \equiv_{\underline{K}_k} \underline{F}_k$ ) for all  $k \in \mathbb{N}$ . We may assume without loss of generality that  $\underline{F}$  is  $\underline{K}$ -saturated for some cardinal  $\underline{K} > |L|$ . For otherwise we may consider a  $\underline{K}$ -saturated elementary extension  $\underline{F}'$  of  $\underline{F}$  according to [29] Theorem 16.4. In order to get  $\underline{L} \equiv_{\underline{K}} \underline{F}$  it suffices to show that there exists a  $\underline{K}$ -embedding  $\varphi: \underline{L} \rightarrow \underline{F}$  such that the in-



duced  $\underline{K}$ -embedding of mixed  $\underline{k}$ -structures  $\varphi_k: \underline{L}_k \rightarrow \underline{F}_k$  is an elementary one for all  $k \in \mathbb{N}$ . For in this case we may construct by iteration an infinite commutative diagram



where  $\underline{L}^{(i+1)}/\underline{L}^{(i)}$ ,  $\underline{F}^{(i+1)}/\underline{F}^{(i)}$  are elementary extensions,  $\underline{L}^{(i+1)}$  is  $|\underline{F}_i|^+$ -saturated,  $\underline{F}^{(i)}$  is  $|\underline{L}_i|^+$ -saturated and  $\varphi_k^{(i)}: \underline{L}_k^{(i)} \rightarrow \underline{F}_k^{(i)}$ ,  $\psi_k^{(i)}: \underline{F}_k^{(i)} \rightarrow \underline{L}_k^{(i+1)}$  are elementary embeddings for all  $k \in \mathbb{N}$ . Let  $\underline{F}^{(\omega)} = \varinjlim \underline{F}^{(i)} = \varinjlim \underline{L}^{(i)}$ . By Tarski-Vaught Theorem [29] Theorem 10.1,  $\underline{F}^{(\omega)}/\underline{L}$  and  $\underline{F}^{(\omega)}/\underline{F}$  are elementary extensions and hence  $\underline{L} \equiv_{\underline{K}} \underline{F}$ .

Thus it remains to show that there exists a  $\underline{K}$ -embedding  $\varphi: \underline{L} \rightarrow \underline{F}$  with the required properties. As  $\underline{L} \equiv_{\underline{K}} \underline{F}$  and  $\underline{F}_k$  is  $\underline{K}$ -saturated,  $\underline{K} > |\underline{L}| > |\underline{L}_k|$ , there exist elementary  $\underline{K}_k$ -embeddings  $\underline{L}_k \rightarrow \underline{F}_k$  for all  $k \in \mathbb{N}$ ; therefore, by a slight adjustment of Lemma 3.5, we obtain a  $\underline{K}_0$ -embedding  $\varphi_\omega: \underline{L}_0 \rightarrow \underline{F}_0$  of mixed  $\omega$ -structures inducing elementary  $\underline{K}_k$ -embeddings  $\varphi_k: \underline{L}_k \rightarrow \underline{F}_k$  for all  $k \in \mathbb{N}$ . It remains to extend  $\varphi_\omega$  to a  $\underline{K}$ -embedding  $\varphi: \underline{L} \rightarrow \underline{F}$  by Theorem 2.1.

Q.E.D.

#### (4.4) Proof of Theorem B from Introduction

Let  $p$  be either 1 or a fixed prime number. Denote by  $\underline{T}_p$  the  $\underline{L}$ -theory of Henselian valued fields of characteristic zero and residue characteristic exponent  $p$ . Let  $\varphi(\underline{x})$ ,  $\underline{x} = (x_1, \dots, x_n)$ , be an  $\underline{L}$ -formula. Denote by  $W$  the set of sentences in the language  $\underline{L}$  augmented

with the constants  $\underline{c} = (c_1, \dots, c_n)$ , consisting of:

- i) the axioms of  $T_p$ ;
- ii) the sentence  $\varphi(\underline{c})$ ;
- iii) the sentences of the form

$$\neg \left[ \lambda(\underline{c}) \wedge \text{tr}_k(\psi)(f_i(\underline{c})g_i(\underline{c})^{-1} : 1 \leq i \leq m; p) \right],$$

where  $\lambda(\underline{x})$  is a quantifier-less  $L$ -formula,  $k$  is a natural number,  $\psi(\underline{y})$  with  $\underline{y} = (y_1, \dots, y_m)$  is an  $L_k$ -formula and  $f_i, g_i \in \mathbb{Z}[\underline{x}]$ ,  $1 \leq i \leq m$ , such that  $T_p \vdash \left[ \lambda(\underline{x}) \wedge \text{tr}_k(\psi)(f_i(\underline{x})g_i(\underline{x})^{-1} : 1 \leq i \leq m; p) \right] \rightarrow \varphi(\underline{x})$ .

We claim that  $W$  is inconsistent. Assuming the contrary, let us choose, by Gödel's completeness theorem [29] Theorem 7.1, a model  $(\underline{L}; \underline{c})$  of  $W$ . In particular,  $\underline{L} = (L, v)$  is a Henselian valued field of characteristic zero and residue characteristic exponent  $p$ , and  $\underline{c} = (c_1, \dots, c_n) \in L^n$ . Let  $\underline{K} = (\underline{O}(\underline{c}), v)$  be the smallest valued subfield of  $\underline{L}$  containing the  $c_i$ 's. Denote by  $D(\underline{K})$  the diagram of  $\underline{K}$ , i.e. the set of basic sentences (atomic and negated atomic) in the language of  $\underline{K}$  that are true on  $\underline{K}$ , and by  $D(\underline{L}/\underline{K})$  the set of all sentences having the form  $\text{tr}_k(\psi)(\underline{a}; p)$ , where  $k \in \mathbb{N}$ ,  $\psi(\underline{y})$  with  $\underline{y} = (y_1, \dots, y_m)$  is an  $L_k$ -formula and  $\underline{a} = (a_1, \dots, a_m) \in K^m$  such that  $\pi_k(\underline{a})$  makes sense (i.e.  $a_i \in O_K$  if  $y_i$  is a variable of sort  $A$ ,  $a_i \in K^X$  if  $y_i$  is a variable of sort  $H$  or  $\Gamma$ ; then  $\pi_k(a_i)$  is respectively  $\pi_{A, k}(a_i)$ ,  $\pi_{H, k}(a_i)$ ,  $v a_i$ ) and  $\underline{L}_k \models \psi(\pi_k(\underline{a}))$ . According to Theorem A, the theory  $S = T_p \cup D(\underline{K}) \cup D(\underline{L}/\underline{K})$  is complete; therefore  $S \vdash \varphi(\underline{c})$ , since  $\underline{L} \models S$  and  $\underline{L} \models \varphi(\underline{c})$ . Consequently there are finitely many sentences  $\lambda_1, \dots, \lambda_s \in D(\underline{K})$ ,

$\eta_1, \dots, \eta_t \in D(\underline{L}/\underline{K})$  such that  $T_p \vdash \bigwedge_{i=1}^s \lambda_i \wedge \bigwedge_{j=1}^t \eta_j \rightarrow \varphi(\underline{c})$ . Let  $\lambda(\underline{x})$  be a quantifier-less  $L$ -formula such that  $\lambda(\underline{c}) \leftrightarrow \bigwedge_{i=1}^s \lambda_i$ . On the other hand, by Lemma 3.3, there exist  $k \in \mathbb{N}$ , an  $L_k$ -formula  $\psi(\underline{y})$ ,  $\underline{y} = (y_1, \dots, y_m)$ , and some polynomials  $f_i, g_i \in \mathbb{Z}[\underline{x}]$  such that  $\text{tr}_k(\psi)(f_i(\underline{c})g_i(\underline{c})^{-1} : 1 \leq i \leq m; p) \leftrightarrow \bigwedge_{j=1}^t \eta_j$ . Thus  $T_p \vdash \left[ \lambda(\underline{x}) \wedge \text{tr}_k(\psi)(f_i(\underline{x})g_i(\underline{x})^{-1} : 1 \leq i \leq m; p) \right] \rightarrow \varphi(\underline{x})$  since  $\underline{c}$  does not occur in  $T_p$ ; therefore  $\underline{L} \models \neg \left[ \lambda(\underline{c}) \wedge \text{tr}_k(\psi)(f_i(\underline{c})g_i(\underline{c})^{-1} : 1 \leq i \leq m; p) \right]$ .



$1 \leq i \leq m; p)$ ] since  $(\underline{L}, \underline{c})$  is a model of  $W$ , a contradiction.

We conclude that  $W$  is inconsistent and hence there exist quantifierless  $L$ -formulas  $\lambda_1(\underline{x}), \dots, \lambda_f(\underline{x})$ , natural numbers  $k_1, \dots, \dots, k_f$ ,  $L_{k_j}$ -formulas  $\psi_j(\underline{y})$ ,  $1 \leq j \leq f$ ,  $\underline{y} = (y_1, \dots, y_m)$ , and polynomials  $f_i, g_i \in \mathbb{Z}[\underline{x}]$ ,  $1 \leq i \leq m$ , such that  $\mathbb{Z} \vdash \mathbb{Z}_p \vdash \varphi(\underline{x}) \leftrightarrow \bigvee_{1 \leq i \leq f} [\lambda_i(\underline{x}) \wedge \text{tr}_{k_j}(\psi_j)]$   $(f_i(\underline{x})g_i(\underline{x})^{-1} : 1 \leq i \leq m; p)$ . Using again Lemma 3.3, we may assume that  $k_1 = k_2 = \dots = k_f = k$ .

Q.E.D.

### 5. Application to $p$ -adically closed fields

In the last section of the work we shall show that Macintyre theorem [23] on quantifier elimination for  $p$ -adic fields and its generalization to  $p$ -adically closed fields due to Prestel-Roquette [2, Theorem 5.6, can be obtained as consequences of the general Theorem B.

Let us fix in the following a prime number  $p$ . A valued field  $\underline{K} = (K, v)$  is called  $p$ -valued if  $K$  is of characteristic zero, the residue field  $\bar{K}$  is of characteristic  $p$  and the  $\mathbb{F}_p$ -space  $\underline{O}_K / \underline{pO}_K$  is finite. Call the dimension of the space  $\underline{O}_K / \underline{pO}_K$  the  $p$ -rank of the  $p$ -valued field  $\underline{K}$ .

The above condition implies that the residue field  $\bar{K}$  is finite, say  $\bar{K} \simeq \mathbb{F}_q$  with  $q = p^f$ ,  $f = [\bar{K} : \mathbb{F}_p]$  and the absolute ramification index  $e$  of  $\underline{K}$ , i.e. the number of positive elements in  $vK$  which are  $\leq vp$ , is finite. The  $p$ -rank  $d$  of the  $p$ -valued field  $\underline{K}$  satisfies the relation  $d = ef$ .

A  $p$ -valued field  $\underline{K}$  of  $p$ -rank  $d$  is called  $p$ -adically closed if  $\underline{K}$  does not admit any proper  $p$ -valued algebraic extension of the same  $p$ -rank. It turns out by [24] Theorem 3.1, that a  $p$ -valued field  $\underline{K}$  is  $p$ -adically closed iff  $\underline{K}$  is Henselian and its value group  $vK$  is a  $\mathbb{Z}$ -group, i.e. the coarse value group  $\hat{v}K$  is divisible.

Given  $p$  and  $d \geq 1$ , let us consider the augmentation  $L^{(d)}$  of the language  $L$  of valued fields with  $d-1$  new constants  $u_2, \dots, u_d$ . The class of  $p$ -valued fields of  $p$ -rank  $d$  is axiomatizable by universal axioms in the language  $L^{(d)}$  in such a way that for any  $p$ -valued field  $K$  of  $p$ -rank  $d$  the constants  $u_1=1, u_2, \dots, u_d$  denote an  $\mathbb{F}_p$ -basis of  $O_K/pO_K$ . With respect to the modified language  $L^{(d)}$ , every substructure of a  $p$ -valued field of  $p$ -rank  $d$  is again a  $p$ -valued field of  $p$ -rank  $d$ . Note also, by [24] Theorem 3.1, that the class of  $p$ -adically closed fields of  $p$ -rank  $d$  is axiomatizable in  $L$  and hence in  $L^{(d)}$ .

Now extend again the language  $L^{(d)}$  by unary predicates  $P_n (n \in \mathbb{N})$  and add to the  $L^{(d)}$ -axioms for the theory  $T_{p,d}$  of  $p$ -adically closed fields of  $p$ -rank  $d$  the new defining axioms

$$P_n(x) \leftrightarrow (\exists y) xy^n = 1.$$

The basic result of Prestel-Roquette [24] Theorem 5.6, including Macintyre's result [23] as a special case, can be stated as follows:

Theorem 5.1. In the language  $L^{(d)}$  extended by unary predicates  $P_n (n \in \mathbb{N})$ , the theory  $T_{p,d}$  together with the defining axioms above admits elimination of quantifiers.

The main goal of this section is to show that Theorem 5.1 is a consequence of Theorem B. First of all we state an equivalent version of Theorem 5.1.

Extend the language  $L^{(d)}$  by unary predicates  $W_{n,k} (n, k \in \mathbb{N})$  and add to the  $L^{(d)}$ -theory  $T_{p,d}$  of  $p$ -adically closed fields of  $p$ -rank  $d$  the defining axioms

$$W_{n,k}(x) \leftrightarrow (\exists y) \left[ (1-xy^n)^{-k} p^{-k} \in O \wedge (1-xy^n)^{-1} p^k \notin O \right].$$

We show that Theorem 5.1 is equivalent with the following statement



Theorem 5.1.a. In the language  $L^{(d)}$  extended by unary predicates  $W_{n,k}$  ( $n, k \in \mathbb{N}$ ), the theory  $T_{p,d}$  together with the defining axioms above admits elimination of quantifiers.

Proof of the equivalence (5.1)  $\leftrightarrow$  (5.1.a):

(5.1)  $\rightarrow$  (5.1.a): Given  $n \in \mathbb{N}$ , let  $\ell \in \mathbb{N}$  be such that  $\ell > 2v_p(n)$ , where  $v_p(n)$  is the p-adic value of  $n$ , i.e.  $n = p^{v_p(n)} m$  with  $(m, p) = 1$ . By Newton's lemma,  $P_n(x) \leftrightarrow W_{n,\ell}(x)$  on any Henselian valued field of characteristic zero and residue characteristic  $p$ ; in particular we obtain  $T_{p,d} \vdash P_n(x) \leftrightarrow W_{n,\ell}(x)$ .

(5.1.a)  $\rightarrow$  (5.1): Given  $n, k \in \mathbb{N}$ , let  $\ell \in \mathbb{N}$  be such that  $\ell > \max(k, 2v_p(n))$ . It suffices to show that  $T_{p,d} \vdash W_{n,k}(x) \leftrightarrow$

$(\alpha_1, \dots, \alpha_d) \in S \left[ P_n\left(\sum_{i=1}^d \alpha_i u_i x\right) \wedge v\left(1 - \sum_{i=1}^d \alpha_i u_i\right) > kvp \right]$ , where  $S = \{(\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d \setminus (0, \dots, 0) : 0 \leq \alpha_i \leq p^{\ell-1} (1 \leq i \leq d)\}$ . Let  $\underline{K} = (K, v)$  be a p-adically closed field of p-rank  $d$  and  $u_1 = 1, u_2, \dots, u_d$  be the given constants in  $O_{\underline{K}}$  defining a basis of  $O_{\underline{K}}/pO_{\underline{K}}$ . Clearly, the  $u_i$ 's define also a  $\mathbb{Z}/p\mathbb{Z}$ -basis of  $O_{\underline{K}}/pO_{\underline{K}}$ . Given  $a \in K$ , let us assume  $\underline{K} \models W_{n,k}(a)$  i.e.  $v(1 - ab^n) > kvp$  for some  $b \in K$ . As  $ab^n \in O_{\underline{K}}^\times$ , there exists  $(\alpha_1, \dots, \alpha_d) \in S$  such that  $v(1 - \sum_{i=1}^d \alpha_i u_i ab^n) > \ell vp$ . As  $\ell > 2v_p(n)$  it follows by Newton's lemma that  $(\sum_{i=1}^d \alpha_i u_i a)(bc)^n = 1$  for some  $c \in K$ , i.e.

$\underline{K} \models P_n(\sum_{i=1}^d \alpha_i u_i a)$ . On the other hand,  $v(1 - \sum_{i=1}^d \alpha_i u_i) = v\left[\left(1 - \sum_{i=1}^d \alpha_i u_i ab^n\right) - \left(\sum_{i=1}^d \alpha_i u_i\right)(1 - ab^n)\right] > kvp$  since  $v(\sum_{i=1}^d \alpha_i u_i) = 0$  and  $\ell > k$ .

Conversely, let us assume that there exists  $(\alpha_1, \dots, \alpha_d) \in S$  such that  $v(1 - \sum_{i=1}^d \alpha_i u_i) > kvp$  and  $\sum_{i=1}^d \alpha_i u_i ab^n = 1$  for some  $b \in K$ . Then  $ab^n \in O_{\underline{K}}^\times$  and  $v(1 - ab^n) = v(ab^n(\sum_{i=1}^d \alpha_i u_i - 1)) > kvp$ , i.e.  $\underline{K} \models W_{n,k}(a)$ .

O.E.D.

As the residue rings  $O_{\underline{K},k} = O_{\underline{K}}/p^k O_{\underline{K}}$ ,  $k \in \mathbb{N}$ , are finite with

$|O_{K,k}| \leq p^{(k+1)d}$ , any sentence about such rings is equivalent to a quantifierless sentence involving the constants  $u_1=1, u_2, \dots, u_d$ . Consequently, Theorem 5.1.a. is an immediate consequence of Theorem B and of the elimination of quantifiers for the structures defined as follows.

Given a finite Abelian group  $A$ , let us consider the systems  $(H, \Gamma, v, t)$  where  $(H, \cdot, 1)$  is an Abelian group,  $(\Gamma, +, \leq, 0)$  is a totally ordered Abelian group,  $v: H \rightarrow \Gamma$  is a group epimorphism and  $t$  is an element of  $H$  such that  $\text{Ker } v \cong A$  and  $vt$  is the smallest positive element of  $\Gamma$ . The class of these systems can be axiomatized by universal axioms in a first order language  $L_A$  whose vocabulary contains symbols for group operations on  $H$  and  $\Gamma$ , a predicate for the order on  $\Gamma$ , a function symbol for the map  $v$  and constants for the elements of  $A$ , the neutral element  $0$  of  $\Gamma$  and the distinguished element  $t$  of  $H$ .

Denote by  $T_A$  the  $L_A$ -theory whose models are the systems  $(H, \Gamma, v, t)$  above satisfying the supplementary condition that  $\Gamma$  is a  $\mathbb{Z}$ -group, i.e.  $\Gamma / \mathbb{Z}vt$  is divisible.

Now extend the language  $L_A$  by unary predicates  $R_n$  ( $n \in \mathbb{N}$ ) and add to the axioms for the theory  $T_A$  the defining axioms

$$R_n(x) \leftrightarrow (\exists y \in H) \quad x = y^n.$$

Proposition 5.2. In the language  $L_A$  extended with unary predicates  $R_n$  ( $n \in \mathbb{N}$ ), the theory  $T_A$  together with the defining axioms above admits elimination of quantifiers.

Proof. Let  $\underline{H}' = (H', \Gamma', v, t)$ ,  $\underline{H}'' = (H'', \Gamma'', v, t)$  be models of  $T_A$  and  $\underline{H} = (H, \Gamma, v, t)$  be a common  $L_A$ -substructure of  $\underline{H}'$ ,  $\underline{H}''$  such that  $H'^n \cap H = H''^n \cap H$  for all  $n \in \mathbb{N}$ . According to [29] Theorem 13.2 we have to show that  $\underline{H}' \equiv \underline{H}''$ .



Denote by  $T(H'/H) = \{x \in H' : \bigvee_{n \geq 1} x^n \in H\}$  the group of the torsion elements of  $H'$  over  $H$ . As an  $L_A$ -structure,  $T(H'/H)$  is a model of  $T_A$  since  $H'/T(H'/H)$  is torsion free and  $\Gamma'/n\Gamma' \cong \mathbb{Z}/n\mathbb{Z}$  for all  $n \in \mathbb{N}$ . First of all we show that  $T(H'/H) \cong_{\underline{H}} T(H''/H)$ . It suffices to show that  $T(H'/H)$  is  $\underline{H}$ -embeddable into  $\underline{H''}$ . For suppose that  $\varphi: T(H'/H) \rightarrow \underline{H''}$  is a  $\underline{H}$ -embedding, and let  $y \in T(H''/H)$ ; we have to show that  $y \in \text{Im}(\varphi)$ . Let  $a \in H$ ,  $n \geq 1$  be such that  $y^n = a$ . As  $H'^n \cap H = H''^n \cap H$  there is  $x \in T(H'/H)$  such that  $x^n = a$ . It follows  $(\varphi(x)y^{-1})^n = 1$  and hence  $\varphi(x)y^{-1} \in A = \text{Ker } v$ . As  $A \subset H$ , we get  $y \in \text{Im}(\varphi)$  as contended.

Thus it remains to show that  $T(H'/H)$  is  $\underline{H}$ -embeddable into  $\underline{H''}$ . Since the equation  $x^n = a$  with  $a \in H$  has finitely many solutions in  $H'$  (assume  $x_0 \in H'$  is a solution of the equation above; then any other solution has the form  $x_0 u$  with  $u \in A$ ), it suffices, by a standard compactness argument, to show that any intermediate group  $G$  between  $H$  and  $T(H'/H)$  which is finitely generated over  $H$  can be embedded over  $\underline{H}$  (as an  $L_A$ -structure) into  $\underline{H''}$ . Let  $G$  be such a group. As  $G/H$  is finite, let us consider a direct decomposition into finite cyclic groups:

$$G/H = C_1 \times C_2 \times \dots \times C_\ell.$$

Let  $n_i$  be the order of  $C_i$ ; then  $n_1 \dots n_\ell = n := (G:H)$ . Let  $x_i \in G$  be such that  $x_i$  generates  $C_i$  modulo  $H$ ; then  $x_i$  is of order  $n_i$  modulo  $H$  and hence  $x_i^{n_i} = a_i \in H$ ,  $1 \leq i \leq \ell$ . By assumption there exist  $y_i \in H''$  such that  $y_i^{n_i} = a_i$  for  $1 \leq i \leq \ell$ . The substitution  $x_i \mapsto y_i$  ( $1 \leq i \leq \ell$ ) defines a group morphism over  $H$ ,  $\varphi: G \rightarrow H''$ . First note that  $\varphi$  is injective. Indeed, let  $z \in G$  be such that  $\varphi(z) = 1$ . In particular,  $\varphi(z^n) = 1$  and hence  $z^n = \varphi(z^n) = 1$  since  $z^n \in H$ ; it follows  $z \in A \subset H$  and  $z = \varphi(z) = 1$ , as contended. Next let us show that  $\varphi$  is an  $L_A$ -embedding, i.e.  $v\varphi(z) \geq 0$  for all  $z \in G$  subject to  $vz \geq 0$ . Consider such an element  $z$ ; as  $z^n = b \in H$  and  $vz \geq 0$ , we get  $vb \geq 0$ ,  $\varphi(z)^n = b$  and hence  $v\varphi(z) \geq 0$ .

Thus we have shown that  $\underline{T}(\underline{H}'/\underline{H})$  and  $\underline{T}(\underline{H}''/\underline{H})$  are  $\underline{H}$ -isomorphic as  $L_A$ -structures. Identifying  $\underline{T}(\underline{H}'/\underline{H}) = \underline{T}(\underline{H}''/\underline{H})$ , it suffices to show that  $\underline{H}'$  and  $\underline{H}''$  are elementarily equivalent over  $\underline{T}(\underline{H}'/\underline{H})$  (rather than over  $\underline{H}$ ). Hence after replacing  $\underline{H}$  by  $\underline{T}(\underline{H}'/\underline{H})$  we may assume that  $\underline{H}$  is a model of  $T_A$ . The required fact  $\underline{H}' \equiv_{\underline{H}} \underline{H}''$  is an immediate consequence of the following model completeness result:

Proposition 5.3. The  $L_A$ -theory  $T_A$  is model complete, i.e.  
 $\underline{H} \hookrightarrow \underline{G}$  is an  $L_A$ -embedding of models of  $T_A$ , then  $\underline{G}$  is an elementary extension of  $\underline{H}$ .

Proof. By Robinson's test [12] Proposition 3.17. it suffices to show for each such embedding  $\underline{H} \hookrightarrow \underline{G}$  that any primitive existential sentence with parameters from  $\underline{H}$  which holds in  $\underline{G}$  also holds in  $\underline{H}$ .

First let us show that each primitive existential sentence with parameters from  $\underline{H}$  has the form

$$\exists \underline{x} \varphi(\underline{x}), \text{ where } \underline{x} = (x_1, \dots, x_n) \text{ and } \varphi(\underline{x}) := \left( \bigwedge_{1 \leq i \leq l} w_i(\underline{x}) = a_i \right) \wedge \bigwedge_{1 \leq i \leq m} v w_i(\underline{x}) \geq v a_i \text{ with } w_i(\underline{x}) = \prod_{1 \leq j \leq n} x_j^{s_{ij}}, s_{ij} \in \mathbb{Z}, a_i \in \underline{H}.$$

It suffices to observe that

$$\begin{aligned} T_A \vdash x \neq y &\leftrightarrow (vx \neq vy) \vee \left( \bigvee_{1 \neq u \in A} y = xu \right) \leftrightarrow (v(xy^{-1}) \geq vt^{-1}) \vee \\ &\vee (v(yx^{-1}) \geq vt^{-1}) \vee \left( \bigvee_{1 \neq u \in A} yx^{-1} = u \right) \end{aligned}$$

and

$$T_A \vdash vx < 0 \leftrightarrow vx^{-1} \geq vt.$$

So let  $\varphi(\underline{x})$  be as above and assume  $\underline{G} \models (\exists \underline{x}) \varphi(\underline{x})$ ; let  $\underline{b} = (b_1, \dots, b_n) \in G^n$  be such that  $\underline{G} \models \varphi(\underline{b})$ . Consider the subgroup  $\underline{G}' = \underline{H}.b_1^{\mathbb{Z}}.b_2^{\mathbb{Z}} \dots b_n^{\mathbb{Z}}$  of  $G$ . As  $G/\underline{H}$  is torsion free, the finitely generated subgroup  $G'/\underline{H}$  is free by [22] Ch.1, §10, Theorem 7; in particu-



lar,  $G'$  has a direct decomposition  $G' \simeq H \oplus G'/H$ . Let  $\mathcal{C} = (c_1, \dots, c_k) \in G'^k$  be such that  $G' \simeq H \oplus \bigoplus_{j=1}^k \mathbb{Z} c_j$ . Thus each  $b_i$  admits a unique

representation

$$b_i = a_i + \sum_{j=1}^k \alpha_{ij} c_j \quad (1 \leq i \leq n)$$

with  $a_i \in H$ ,  $\alpha_{ij} \in \mathbb{Z}$

Putting  $x_i = a_i + \sum_{j=1}^k \alpha_{ij} y_j$ ,  $1 \leq i \leq n$ , the

equalities which occur in  $\mathcal{Q}(\underline{x})$  become identities trivially satisfied thanks to the above direct decomposition of  $G'$ . The inequalities which occur in  $\mathcal{Q}(\underline{x})$  become inequalities in the new variables  $y_1, \dots, y_k$  of the form

$$(*) \quad \sum_{j=1}^k \beta_{ij} \cdot v y_j \geq v a_i \quad (1 + 1 \leq i \leq m)$$

with  $\beta_{ij} \in \mathbb{Z}$ ,  $a_i \in H$ . The system of inequalities  $(*)$  admits the solution  $\underline{c} \in G'^n$ . By model completeness of the theory of  $\mathbb{Z}$ -groups

[27], we obtain  $\gamma_1, \dots, \gamma_k \in v H$  such that

$$\sum_{j=1}^k \beta_{ij} \gamma_j \geq v a_i \quad (1 + 1 \leq i \leq m).$$

Let  $d_1 \dots d_k \in H$  be such that  $vd_j = \gamma_j$  for  $1 \leq j \leq k$ ,  
 and  $e_i = a_i \prod d_j^{\alpha_{ij}}$  for  $1 \leq i \leq n$ . Then  $\underline{H} \models \varphi(\underline{e}_1, \dots, \underline{e}_n)$   
 and hence  $\underline{H} \models (\exists \underline{x}) \varphi(\underline{x})$  as contended. Q.E.D.

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