# AN ISOMORPHISM THEOREM FOR ALGEBRAIC EXTENSIONS OF VALUED FIELDS

by

Serban A. BASARAB*)

February 1986

*) The National Institute for Scientific and Tehnical Creation
Bd.Pacii 220, 79622 Bucharest, Romania

# AN ISOMORPHISM THEOREM FOR ALGEBRAIC EXTENSIONS OF VALUED FIELDS

by

Serban A. BASARAB

PREPRINT SERIES IN MATHEMATICS
No.8/1986

# AN ISOMORPHISM THEOREM FOR ALGEBRAIC EXTENSIONS OF VALUED FIELDS

Şerban A. BASARAB

## Introduction

Consider the following problem: given a valued field $\underline{\underline{K}}=(K,v)$, find elementary invariants to classify, up to isomorphism over $\underline{\underline{K}}$, the algebraic Henselian valued field extensions of $\underline{\underline{K}}$.

In an important special case, an answer to the question above is given by the following result due to Prestel-Roquette [3] Corollary 3.11:

Theorem I. Let $\underline{\underline{K}}=(K,v)$ be a p-valued field and $\underline{\underline{L}}=(L,v)$, $\underline{\underline{F}}=(F,v)$ be two Henselian p-valued algebraic extensions of the same p-rank as $\underline{\underline{K}}$. The necessary and sufficient condition for $\underline{\underline{L}}$, $\underline{\underline{F}}$ to be $\underline{\underline{K}}$-isomorphic (as valued fields) is that $K\cap L^n=K\cap F^n$ for each $n\in\underline{N}$.

The main goal of this paper is to extend suitably Theorem I to the more general case of valued fields of characteristic zero.

Given a valued field $\underline{\underline{K}}=(K,v)$ let us denote by $O_{\underline{\underline{K}}}$ the valuation ring, by $\overline{K}$ the residue field and by $vK$ the value group. Assume that the characteristic of K is zero and let p be the characteristic exponent of $\overline{K}$. For $k\in\underline{N}$, let $\underline{m}_{\underline{\underline{K}},k}$ be the ideal $\{a\in O_{\underline{\underline{K}}}: :va>kvp\}$ of $O_{\underline{\underline{K}}}$. In particular, $\underline{m}_{\underline{\underline{K}},0}=\underline{m}_{\underline{\underline{K}}}$ is the maximal ideal of the valuation ring $O_{\underline{\underline{K}}}$. Denote by $O_{\underline{\underline{K}},k}$ the factor ring $O_{\underline{\underline{K}}}/\underline{m}_{\underline{\underline{K}},k}$; $O_{\underline{\underline{K}},k}$ is a local ring with maximal ideal $\underline{m}_{\underline{\underline{K}}}/\underline{m}_{\underline{\underline{K}},k}$. In particular, for $p=1$, $O_{\underline{\underline{K}},k}=\overline{K}$ for each $k\in\underline{N}$.

On the other hand, consider the multiplicative groups

$G_{\underline{\underline{K}},k}=K^{\times}/_{1+\underline{m}_{\underline{\underline{K}},k}}$ for $k\in\underline{N}$. If $p=1$ then $G_{\underline{\underline{K}},k}=G_{\underline{\underline{K}},0}=G_{\underline{\underline{K}}}=K^{\times}/_{1+\underline{m}_{\underline{\underline{K}}}}$ for each $k\in\underline{N}$. Given $k\in\underline{N}$, the local ring $O_{\underline{\underline{K}},2k}$ and the group $G_{\underline{\underline{K}},k}$ are related through a natural map $\Theta_k$ defined on the subset $\{a\in O_{\underline{\underline{K}},2k}: a|p^k\}=O_{\underline{\underline{K}},2k}\setminus \underline{m}_{\underline{\underline{K}},k}/_{\underline{m}_{\underline{\underline{K}},2k}}$ of $O_{\underline{\underline{K}},2k}$ with values in $G_{\underline{\underline{K}},k}$: $\Theta_k(a+\underline{m}_{\underline{\underline{K}},2k})=a(1+\underline{m}_{\underline{\underline{K}},k})$ for $a\in O_{\underline{\underline{K}}}$ subject to $va\leq kvp$.

For $k\in\underline{N}$, the valuation v induces a map $v_k:(O_{\underline{\underline{K}},2k}\setminus\{0\})\,\dot\cup\, G_{\underline{\underline{K}},k}\to vK$; the image of $v_k$ restricted to $O_{\underline{\underline{K}},2k}\setminus\{0\}$ is the convex subset $\{\alpha\in vK: 0\leq\alpha\leq 2kvp\}$ of vK, and $v_k$ restricted to $G_{\underline{\underline{K}},k}$ is a group epimorphism onto vK. Among other properties, the map $\Theta_k$ satisfies the following compatibility condition: $v_k\,\Theta_k(a)=v_k\,a$ for all $a\in O_{\underline{\underline{K}},2k}\setminus \underline{m}_{\underline{\underline{K}},k}/_{\underline{m}_{\underline{\underline{K}},2k}}$.

For $k\in\underline{N}$, consider the system $\underline{\underline{K}}_k=(O_{\underline{\underline{K}},2k}, G_{\underline{\underline{K}},k}, vK, \Theta_k, v_k)$ and call it the <u>mixed</u> <u>k-structure</u> assigned to $\underline{\underline{K}}$. In particular, for $p=1$, $\underline{\underline{K}}_k=\underline{\underline{K}}_0$ is the triple $(\bar{K}, G_{\underline{\underline{K}}}, vK)$ together with the exact sequence $1\to\bar{K}^{\times}\to G_{\underline{\underline{K}}}\to vK\to 0$.

Given a sentence $\varphi$ in the language $\underline{L}_k$ of mixed k-structures, $k\in\underline{N}$, one may assign effectively to it a formula $tr_k(\varphi)(z)$ of one variable z, in the language $\underline{L}$ of valued fields in such a way that for every valued field $\underline{\underline{K}}$ of characteristic zero and residue characteristic exponent p, $tr_k(\varphi)(p)$ is true on $\underline{\underline{K}}$, written $\underline{\underline{K}}\models tr_k(\varphi)(p)$ iff $\varphi$ is true on $\underline{\underline{K}}_k$, written $\underline{\underline{K}}_k\models\varphi$. The correspondence above $\varphi\mapsto tr_k(\varphi)$ extends naturally to a translation map $tr_k$ from the arbitrary $\underline{L}_k$ - formulas to $\underline{L}$-formulas.

Given a valued field $\underline{\underline{K}}=(K,v)$ of characteristic zero and residue characteristic exponent p, and a valued field extension $\underline{\underline{L}}=(L,v)$ of $\underline{\underline{K}}$, let us denote by $\underline{T}(\underline{\underline{L}}/\underline{\underline{K}})$ the existential $\underline{L}$-theory with parameters from $\underline{\underline{K}}$ (i.e. involving constants for elements of K) consisting of all sentences $tr_k(\varphi)(p)$, where $k\in\underline{N}$, the $\underline{L}_k$-sentence $\varphi$ has the form

$$(\exists x \in O_{2k})(\exists y \in G_k^r)\left[(f(x)=0)\wedge(\bigwedge_{1\leq i\leq m} v_k g_i(x)=\alpha_i)\wedge(\bigwedge_{1\leq i\leq r} \theta_k(h_i(x))= =b_i y_i^{m_i})\right]$$, with $f, g_i, h_i \in O_{K,2k}[X]$, $\alpha_i \in vK$, $0 \leq \alpha_i \leq 2kvp$, $b_i \in G_{K,k}$, and $L_k \models \varphi$.

The existential theory $T(L/K)$ defined above plays a key role in the study of algebraic extensions of valued fields as shows the following embedding theorem, which extends [3] Theorem 3.10.

*Theorem II. Let $K=(K,v)$ be a valued field of characteristic zero and $L=(L,v)$ be a Henselian algebraic extension of $K$. The next statements are equivalent for a Henselian valued field extension $F=(F,v)$ of $K$:*

i) *$L$ is $K$-embeddable in $F$.*

ii) *$L_k$ is $K_k$ - embeddable in $F_k$ for each $k \in N$.*

iii) *$T(L/K) \subset T(F/K)$.*

The next isomorphism theorem which extends Theorem I is an immediate consequence of Theorem II.

*Theorem III. Let $K=(K,v)$ be a valued field of characteristic zero and $L=(L,v)$, $F=(F,v)$ be two Henselian algebraic extensions of $K$. The next statements are equivalent:*

i) *$L$ and $E$ are $K$-isomorphic.*

ii) *$L_k$ and $F_k$ are $K_k$ - isomorphic for each $k \in N$.*

iii) *$T(L/K)=T(F/K)$.*

## 1. Preliminary results

Let $K=(K,v)$ be a valued field and $L=(L,v)$ be a finite simple extension of $K$, i.e. $L=K(c)$ with $c \in L$ algebraic over $K$. We may assume without loss of generality that $c \in O_L$ and its monic minimal polynomial belongs to $O_K[X]$. Denote by $\underline{S}(L/K)$ the set of all pairs $(g,\alpha)$ where $g\in O_K[X]$, $\alpha\in vK$, $\alpha\geqslant 0$, and $vg(c)=\alpha$ (of course, we may consider only polynomials $g$ of degree less than the degree of $f$). If $S\subset \underline{S}(L/K)$ and $\underline{F}=(F,v)$ is a valued field extension of $\underline{K}$, let us denote by $X_{S,\underline{F}}$ the set $\{z\in F: f(z)=0 \wedge \bigwedge_{(g,\alpha)\in S} vg(z)=\alpha\}$. Denote by $\mathrm{Hom}_{\underline{K}}(\underline{L},\underline{F})$ the set of $\underline{K}$-embeddings of $\underline{L}$ into $\underline{F}$. The canonic map $\lambda_{S,\underline{F}}:\mathrm{Hom}_{\underline{K}}(\underline{L},\underline{F})\to X_{S,\underline{F}}: \varphi\mapsto\varphi(c)$ is obviously injective, and the cardinality $|X_{S,\underline{F}}|$ of $X_{S,\underline{F}}$ is $\leqslant$ the number of roots of $f$ in $F$. We claim that the map $\lambda_{\underline{S}(L/K),\underline{F}}$ is bijective. Indeed, let $z\in X_{\underline{S}(L/K),\underline{F}}$ and $\varphi: L\to F$ be the field K-embedding given by the substitution $c\mapsto z$. Let $w$ be the valuation of L induced by the valuation $v$ of F through $\varphi$. We have to show that $w$ equals the valuation $v$ of L, i.e. $\varphi$ is a $\underline{K}$-embedding of valued fields. Let $y\in L$. As $L/K$ is finite there is $n\geqslant 1$ such that $nvy$, $nwy\in vK$; therefore it suffices to show that $vy^n=wy^n$. Thus we may assume $vy$, $wy\in vK$. Write $y$ in the form $y=g(c)b^{-1}$ with $g\in O_K[X]$, $0\neq b\in O_K$; then $vg(c)=\alpha\in vK$ and $(g,\alpha)\in \underline{S}(L/K)$. By assumption, $wg(c)=v_{\underline{F}}\,g(z)=\alpha$ and hence $wy=\alpha-vb=$ $=vy$, as contended.

Moreover, we obtain

<u>Lemma 1.1. There exists a finite subset S of $\underline{S}(L/K)$ such that the map $\lambda_{S',\underline{F}}$ is bijective for every valued field extension $\underline{F}$ of K and for every subset S' of $\underline{S}(L/K)$ containing S.</u>

<u>Proof</u>. Assuming the contrary, let us choose, for each finite subset S of $\underline{S}(L/K)$, a valued field extension $\underline{F}_S=(F_S,v_S)$ of $\underline{K}$ such that $\lambda_{S,\underline{F}_S}$ is not onto. For each finite set S, denote by $Y_S$ the set of all valuations of L extending $v_{\underline{K}}$ such that $w\neq v_{\underline{L}}$ and $wg(c)=\alpha$ for each $(g,\alpha)\in S$. As L/K is finite, the sets

$Y_S$ are finite. Moreover, these sets are non-empty. For, let $z \in X_{S,\underline{F}_S} \setminus Im(\lambda_{S,\underline{F}_S})$; as $f(z)=0$ and $f$ is irreducible over K, the substitution $c \mapsto z$ defines some $\varphi \in Hom_K(L,F_S) \setminus Hom_{\underline{K}}(\underline{L},\underline{F}_S)$. Let w be the valuation of L induced by $v_S$ through the field embedding $\varphi$; clearly $w \in Y_S$.

Consider the directed projective system of non-empty finite sets $Y_S$ with canonic maps $\psi_{S,S'}: Y_{S'} \hookrightarrow Y_S$ for $S \subset S'$. Then $\varprojlim Y_S = \bigcap Y_S$ is non-empty, i.e. there exists a valuation w of L extending $v_{\underline{K}}$ such that $w \neq v_{\underline{L}}$ and $wg(c) = \alpha$ for each $(g,\alpha) \in \underline{S}(\underline{L}/\underline{K})$. Consequently, $w = v_{\underline{L}}$, a contradiction.

Q.E.D.

Corollary 1.2. Let $S \subset \underline{S}(\underline{L}/\underline{K})$ be as in Lemma 1.1. Given a valued field extension $\underline{F}$ of $\underline{K}$, the next statements are equivalent:

i) $\underline{L}$ is $\underline{K}$-embeddable in $\underline{F}$.

ii) $\underline{F} \models (\exists x)\varphi(x)$, where $\varphi$ is the following L-sentence with parameters from $\underline{K}$:

$$\varphi(x) :\equiv f(x)=0 \wedge \bigwedge_{(g,\alpha)\in S} vg(x) = \alpha.$$

For $\gamma \in vK$, $\gamma \geq 0$, denote by $m_{\underline{K},\gamma}$ the ideal $\{a \in O_{\underline{K}} : va > \gamma\}$ of $O_{\underline{K}}$ and by $O_{\underline{K},\gamma}$ the factor ring $O_{\underline{K}}/m_{\underline{K},\gamma}$; $O_{\underline{K},\gamma}$ is a local ring with maximal ideal $m_{\underline{K}}/m_{\underline{K},\gamma}$. Define as follows a map $v_\gamma : O_{\underline{K},\gamma} \to vK \cup \{\infty\}$: $v_\gamma(a + m_{\underline{K},\gamma}) = \begin{cases} va & \text{if } va \leq \gamma \\ \infty & \text{if } va > \gamma \end{cases}$. If $g \in O_{\underline{K}}[X]$, let $g_\gamma$ be the image of g in $O_{\underline{K},\gamma}[X]$.

For $\underline{L}/\underline{K}$ as above, let $S \subset \underline{S}(\underline{L}/\underline{K})$, $\underline{F}=(F,v)$ be a valued field extension of $\underline{K}$ and $\gamma \in vK$, $\gamma \geq 0$. Denote by $X_{S,\underline{F},\gamma}$ the set

$\{z\in O_{\underline{F},\gamma} : f_\gamma(z)=0 \wedge \bigwedge_{(g,\alpha)\in S,\alpha\leq\gamma} v_\gamma g_\gamma(z)=\alpha\}$. Consider the canonic maps

$$\pi_{\gamma,\delta} : X_{S,\underline{F},\delta} \to X_{S,\underline{F},\gamma} \text{ for } \gamma\leq\delta,\ \pi_\gamma : X_{S,\underline{F}} \longrightarrow X_{S,\underline{F},\gamma}$$

and $\lambda_{S,\underline{F},\gamma} : \mathrm{Hom}_{\underline{K}}(\underline{L},\underline{F}) \to X_{S,\underline{F},\gamma} : \varphi\mapsto\varphi(c) + \underline{m}_{\underline{F},\gamma}$.

Lemma 1.3. In the same situation as in Lemma 1.1, suppose that L/K is separable. There exist a finite subset S of $\underline{S}(\underline{L}/\underline{K})$ and $\delta\in vK$, $\delta\geq 0$, such that the canonic map $\mathrm{Hom}_{\underline{K}}(\underline{L},\underline{F}) \to \pi_{\gamma,\omega}(X_{S,\underline{F},\omega})$ is bijective for every Henselian valued field extension $\underline{F}=(F,v)$ of $\underline{K}$ and for all $\gamma,\omega\in vK$ subject to $\gamma\geq\delta$, $\omega\geq 2\gamma$.

Proof. By Lemma 1.1 there is a finite $S\subset\underline{S}(L/K)$ such that $\lambda_{S,\underline{F}} : \mathrm{Hom}_{\underline{K}}(\underline{L},\underline{F}) \longrightarrow X_{S,\underline{F}}$ is bijective for each valued field extension $\underline{F}$ of $\underline{K}$. As the finite extension L/K is separable, $f'(c)\neq 0$ and $vf'(c)^n\in vK$ for some $n\geq 1$, so we may assume $(f'(X)^n, nvf'(c))\in S$. Let L' be the splitting field of the polynomial f. As L'/K is finite, there is $\delta'\in vK$, $\delta'\geq 0$, such that $0\leq w(c_i-c_j)\leq\delta'$ for all valuations w of L' extending $v_{\underline{K}}$ and all roots $c_i\neq c_j$ of f. Let $\delta=\max(\delta',\alpha:(g,\alpha)\in S)$, $\gamma,\omega\in vK$, $\gamma\geq\delta$, $\omega\geq 2\gamma$ and $\underline{F}$ be a Henselian valued field extension of $\underline{K}$. We have to show that

1) $\pi_\gamma : X_{S,\underline{F}} \to X_{S,\underline{F},\gamma}$ is injective, and

2) $\pi_{\gamma,\omega}(X_{S,\underline{F},\omega}) \subset \pi_\gamma(X_{S,\underline{F}})$ [note that the opposite inclusion is trivial].

1): Let $z,z'\in X_{S,\underline{F}}$ be such that $\pi_\gamma(z)=\pi_\gamma(z')$, i.e. $v(z-z')>\gamma$. As $f(z)=f(z')=0$ and $\gamma\geq\delta\geq\delta'$, we get $z=z'$.

2): Let $z \in O_{\underline{F}}$ be such that $z + \underline{m}_{\underline{F},\omega} \in X_{S,\underline{F},\omega}$, i.e. $vf(z) > \omega$ and $vg(z) = \alpha$ for $(g,\alpha) \in S$. As $(f'(X)^n, nvf'(c)) \in S$, it follows $nvf'(z) = nvf'(c) \leq \delta \leq \gamma$; therefore $vf'(z) \leq \gamma$. Since $vf(z) > \omega \geq 2\gamma$, we conclude by Newton's lemma [3] p.20 that there is one and only one $z' \in O_{\underline{F}}$ such that $f(z') = 0$ and $v(z-z') > \gamma$. On the other hand, $vg(z') = vg(z) = \alpha$ for $(g,\alpha) \in S$ since $\alpha \leq \delta \leq \gamma$ and $v(z-z') > \gamma$. Consequently, $z' \in X_{S,\underline{F}}$ and $\pi_\gamma(z') = \pi_{\gamma,\omega}(z + \underline{m}_{\underline{F},\omega})$.

Q.E.D.

Corollary 1.4. Let $S \subset \underline{S}(\underline{L}/\underline{K})$, $\delta \in vK$, $\delta \geq 0$ be as in Lemma 1.3. Given a Henselian valued field extension $\underline{F} = (F,v)$ of $\underline{K}$, the next statements are equivalent:

i) $\underline{L}$ is $\underline{K}$-embeddable in $\underline{F}$.

ii) $O_{\underline{F},2\delta} \models (\exists x)\varphi(x)$, where

$$\varphi(x) := f_{2\delta}(x) = 0 \wedge \bigwedge_{(g,\alpha) \in S} v_{2\delta} g_{2\delta}(x) = \alpha$$

By a standard compactness argument we obtain the following embedding criterion.

Proposition 1.5. Let $\underline{K} = (K,v)$ be a valued field and $\underline{L} = (L,v)$ be an algebraic separable valued field extension of $\underline{K}$. The next statements are equivalent for a Henselian valued field extension $\underline{F} = (F,v)$ of $\underline{K}$:

i) $\underline{L}$ is $\underline{K}$ - embeddable in $\underline{F}$.

ii) For arbitrary $\gamma \in vK$, $\gamma \geq 0$, $f, g_1, \ldots, g_r \in O_{\underline{K}}[X]$, $\alpha_1, \ldots, \alpha_r \in vK$, $0 \leq \alpha_i \leq \gamma$ $(1 \leq i \leq r)$, if $O_{\underline{L},2\gamma} \models (\exists x)\varphi(x)$, where $\varphi(x) := f_{2\gamma}(x) = 0 \wedge \bigwedge_{1 \leq i \leq r} v_{2\gamma}\, g_{2\gamma}(x) = \alpha_i$, then $O_{\underline{F},2\gamma} \models (\exists x)\varphi(x)$.

Proposition 1.5 is in general unsatisfactory because of the too large amount of values $\gamma \in vK$ which occur in the condition ii) above.

## 2. Nice extensions of valued fields

Consider a valued field $\underline{\underline{K}}=(K,v)$ of characteristic zero and residue characteristic exponent p. The canonical decomposition of the valuation v is defined as follows. Denote by $\Delta=\Delta_{\underline{\underline{K}}}$ the smallest convex subgroup of vK containing vp. Let $\dot{v}K$ be the factor group $vK/\Delta$ , and $\dot{v}:K^{x} \to \dot{v}K: a \mapsto \dot{v}a$ be the group epimorphism induced by v. Since $\Delta$ is convex in vK, $\dot{v}K$ inherits from vK the structure of a totally ordered group and hence the map $\dot{v}$ is a valuation of the field K, called the coarse valuation assigned to v. Denote by $\dot{\underline{\underline{K}}}$ the valued field $(K,\dot{v})$. The valuation ring $O_{\dot{\underline{\underline{K}}}}$ of $\dot{\underline{\underline{K}}}$ is characterized as the smallest overring of $O_{\underline{\underline{K}}}$ in which p becomes a unit, i.e. $O_{\dot{\underline{\underline{K}}}}$ is the ring of fractions of $O_{\underline{\underline{K}}}$ with respect to the multiplicatively closed set $\{p^k : k \in \underset{\sim}{N}\}$. Note that $\dot{v}=v$ iff p=1, and $\dot{v}$ is trivial iff $vK=\Delta$.

Let $m_{\dot{\underline{\underline{K}}}}$ be the maximal ideal of $O_{\dot{\underline{\underline{K}}}}$; then $m_{\dot{\underline{\underline{K}}}} \subset m_{\underline{\underline{K}}} \subset O_{\underline{\underline{K}}} \subset O_{\dot{\underline{\underline{K}}}}$. Denote by $K^{\circ}$ the residue field $O_{\dot{\underline{\underline{K}}}}/m_{\dot{\underline{\underline{K}}}}$ of the valued field $\dot{\underline{\underline{K}}}$. For $a \in O_{\dot{\underline{\underline{K}}}}$ let $a^{\circ}$ be its residue in $K^{\circ}$. The field $K^{\circ}$, called the core field of the valuation v of K, carries naturally a valuation whose valuation ring is the image $O_{\underline{\underline{K}}}/m_{\dot{\underline{\underline{K}}}}$ of $O_{\underline{\underline{K}}}$. Denote also by v this valuation and by $\underline{\underline{K}}^{\circ}$ the core valued field $(K^{\circ},v)$. The value group $vK^{\circ}$ is identified with the convex subgroup $\Delta$ of vK and the residue field $\overline{K^{\circ}}$ is identified with the residue field $\overline{K}$ of $\underline{\underline{K}}$. Thus the core valued field $\underline{\underline{K}}^{\circ}$ is of characteristic zero and residue characteristic exponent p; $K^{\circ}=\overline{K}$ iff p=1. For a valued field extension $\underline{\underline{L}}=(L,v)$ of $\underline{\underline{K}}$, the coarse valuation of $\underline{\underline{L}}$ is a prolongation of the coarse valuation of $\underline{\underline{K}}$; hence both may be denoted by the same symbol $\dot{v}$. The core valued field $\underline{\underline{L}}^{\circ}$ is an extension of the core valued field $\underline{\underline{K}}^{\circ}$.

Given a valued field $\underline{\underline{K}}=(K,v)$ of characteristic zero, a finite valued field extension $\underline{\underline{L}}=(L,v)$ is called <u>nice</u> if the next conditions are satisfied:

i) There exists a ring K-morphism from the polynomial algebra $K[X,Y_1,\ldots,Y_r]$ onto $L: X \mapsto c$, $Y_i \mapsto t_i$ $(1\leq i\leq r)$, whose kernel I is generated by the polynomials $f(X)$, $b_i Y_i^{m_i} - h_i(X)$ $(1\leq i\leq r)$;

ii) $c \in O_{\underline{\underline{L}}}$ and the residue $f^{\circ} \in O_{\underline{\underline{K}}^{\circ}}[X]$ of the minimal monic polynomial $f \in O_{\underline{\underline{K}}}[X]$ of $c$ is irreducible over $K^{\circ}$;

iii) $h_i \in O_{\underline{\underline{K}}}[X]$ and $h_i^{\circ}(c^{\circ}) \neq 0$ for $1\leq i\leq r$;

iv) $[L:K(c)] = (\dot{v}L:\dot{v}K)$.

The next lemma is immediate.

<u>Lemma 2.1. Suppose that the finite extension $\underline{\underline{L}}/\underline{\underline{K}}$ is nice of the form above. Then</u>

a) $L^{\circ} = K^{\circ}(c^{\circ})$, $[K(c):K] = [L^{\circ}:K^{\circ}]$ <u>and</u> $\dot{v}K = \dot{v}K(c)$;

b) $[L:K(c)] = m_1 m_2 \ldots m_r$;

c) <u>Let</u> T <u>be the subgroup of</u> $L^{\times}$ <u>generated by</u> $K(c)^{\times} \cup \{t_1,\ldots,t_r\}$; <u>the valuation</u> v <u>induces the isomorphisms</u> $T/K(c)^{\times} \xrightarrow{\sim} vL/vK(c) \xrightarrow{\sim} \dot{v}L/\dot{v}K$.

In the situation above, let $\underline{S}(\underline{\underline{L}}/\underline{\underline{K}}) = \{(g,\alpha) \in O_{\underline{\underline{K}}}[X] \times vK^{\circ} : vg(c) = \alpha\}$. For $S \subset \underline{S}(\underline{\underline{L}}/\underline{\underline{K}})$ and a valued field extension $\underline{\underline{F}}=(F,v)$ of $\underline{\underline{K}}$, denote by $Y_{S,\underline{\underline{F}}}$ the set of those $(r+1)$-tuples $(x,y_1,\ldots,y_r) \in F^{r+1}$ for which $f(x)=0$, $vg(x)=\alpha$ for $(g,\alpha)\in S$ and $b_i y_i^{m_i} - h_i(x)=0$ for $1\leq i\leq r$; $Y_{S,\underline{\underline{F}}}$ is a finite set and $|Y_{S,\underline{\underline{F}}}| \leq \deg f . m_1 \ldots m_r = [L:K]$. Consider the injective map $\mu_{S,\underline{\underline{F}}}: \mathrm{Hom}_{\underline{\underline{K}}}(\underline{\underline{L}},\underline{\underline{F}}) \to Y_{S,\underline{\underline{F}}}: \varphi \mapsto (\varphi(c), \varphi(t_1),\ldots,\varphi(t_r))$.

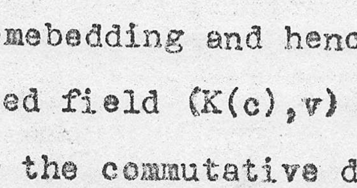
<u>Lemma 2.2.</u> <u>With the data above, there exists a finite subset S of $\underline{S}(\underline{\underline{L}}/\underline{\underline{K}})$ such that $\mu_{S,\underline{\underline{F}}}$ is bijective for each</u>

$S' \subset \underline{\underline{S}}(L/K)$ _containing_ $S$ _and for each valued field extension_ $\underline{\underline{F}}$ _of_ $\underline{\underline{K}}$.

_Proof_. By Lemma 1.1 applied to $\underline{\underline{L}}^{\circ}/\underline{\underline{K}}^{\circ}$ there exists a finite subset $S$ of $\underline{\underline{S}}(L/K)$ such that the canonic map $\lambda_{S^{\circ},\underline{\underline{F}}'}$ : $: \mathrm{Hom}_{\underline{\underline{K}}^{\circ}}(\underline{\underline{L}}^{\circ},\underline{\underline{F}}') \to X_{S^{\circ},\underline{\underline{F}}'} : \varphi \mapsto \varphi(c^{\circ})$ is bijective for each valued field extension $\underline{\underline{F}}'$ of $\underline{\underline{K}}^{\circ}$ [Here $S^{\circ} = \{(g^{\circ},\alpha) : (g,\alpha) \in S\}$ and $X_{S^{\circ},\underline{\underline{F}}'} = \{z \in F' : f^{\circ}(z) = 0 \wedge \bigwedge_{(g,\alpha)\in S} vg^{\circ}(z) = \alpha\}$ ]. It remains to show that $\mu_{S,\underline{\underline{F}}}$ is onto for each valued field extension $\underline{\underline{F}}$ of $\underline{\underline{K}}$. Let $(x,\underline{\underline{y}}) \in Y_{S,\underline{\underline{F}}}$, and consider the morphism of K-algebras $\varphi : K[X,\underline{\underline{Y}}] \to F :$ $: (X,\underline{\underline{Y}}) \mapsto (x,\underline{\underline{y}})$. As $K(x,\underline{\underline{y}})$ is algebraic over K, the image of $\varphi$ is a field, i.e. $\ker \varphi$ is a maximal ideal. As the maximal ideal I defining $L \simeq K[X,\underline{\underline{Y}}]/_{I}$ is contained in $\ker\varphi$, we conclude that $\varphi$ induces a field K-embedding denoted also by $\varphi : L \to F : (c,\underline{\underline{t}}) \mapsto (x,\underline{\underline{y}})$. On the other hand, since $(x,\underline{\underline{y}}) \in Y_{S,\underline{\underline{F}}}$, it follows $x^{\circ} \in X_{S^{\circ},F^{\circ}}$; therefore $\varphi^{\circ} : \underline{\underline{L}}^{\circ} \to \underline{\underline{F}}^{\circ} : c^{\circ} \mapsto x^{\circ}$ is a $\underline{\underline{K}}^{\circ}$- embedding according to Lemma 1.1.

We claim that the field embedding $\varphi|_{K(c)} : K(c) \to F : c \mapsto x$ is a $\underline{\underline{K}}$-embedding of valued fields. For, let w be the valuation of K(c) induced by $\dot{v}_F$ through $\varphi$. As $\dot{v}_F$ extends $\dot{v}_K$, w is a prolongation of $\dot{v}_K$ too. Since $[K(c):K] = [L^{\circ}:K^{\circ}]$, $\dot{v}_{K(c)}$ is the unique prolongation of $\dot{v}_K$ to K(c). Consequently, $w = \dot{v}_{K(c)}$ and $\varphi|_{K(c)}$ is a $\dot{\underline{\underline{K}}}$-embedding of the valued field $(K(c),\dot{v})$ into $\dot{\underline{\underline{F}}}$. On the other hand, $\varphi^{\circ} : \underline{\underline{L}}^{\circ} \to \underline{\underline{F}}^{\circ}$ is a $\underline{\underline{K}}^{\circ}$-emebedding and hence $\varphi|_{K(c)}$ is in fact a $\underline{\underline{K}}$-embedding of the valued field $(K(c),v)$ into $\underline{\underline{F}}$.

Now consider the commutative diagram of field embeddings



Note that i and $\varphi|_{K(c)}$ are valued field embeddings. Let w be the valuation of L induced by $\dot{v}_F$ through $\varphi$ ; w is a prolongation of $\dot{v}_{K(c)}$. As $[L:K(c)]=(\dot{v}L:\dot{v}K(c))=(\dot{v}L:\dot{v}K)$, $\dot{v}_L$ is the unique prolongation of $\dot{v}_{K(c)}$ to L. Consequently, $w=\dot{v}_L$ and $\varphi$ is a $(K(c),\dot{v})$-embedding of $\dot{\underline{\underline{L}}}$ into $\dot{\underline{\underline{F}}}$. Since $\varphi^\circ:\underline{\underline{L}}^\circ\to\underline{\underline{F}}^\circ$ is a valued field embedding, we conclude that $\varphi$ is a $(K(c),v)$ - embedding of $\underline{\underline{L}}$ into $\underline{\underline{F}}$.

Q.E.D.

In the same situation as above, consider the mixed k-structures $\underline{\underline{K}}_k=(O_{K,2k}, G_{K,k}, vK, \Theta_k, v_k)$, $k\in\underline{N}$, defined in Introduction. For $S\subset\underline{\underline{S}}(L/\underline{\underline{K}})$, $k\in\underline{N}$ and a valued field extension $\underline{\underline{F}}$ of $\underline{\underline{K}}$, let us denote by $Y_{S,\underline{\underline{F}},k}$ the set of these (r+1)-tuples $(x,\underline{y})\in O_{F,2k}\times G^r_{\underline{\underline{F}},k}$ for which $f_k(x)=0$, $v_k\, g_k(x)=\alpha$ for $(g,\alpha)\in S$ and $\alpha\leqslant 2kvp$, and $\Theta_k((h_i)_k(x))=(b_i)_k\, y_i^{m_i}$ if $v_k(h_i)_k(x)\leqslant kvp$ $(1\leqslant i\leqslant r)$. [For $g\in O_{\underline{\underline{K}}}[X]$, $g_k$ denotes its image in $O_{K,2k}[X]$; for $1\leqslant i\leqslant r$, $(b_i)_k$ is the image of $b_i\in K^\times$ in $G_{\underline{\underline{K}},k}=K^\times/1+\underline{m}_{\underline{\underline{K}},k}$]. Consider the canonic maps $\pi_{k,l}:Y_{S,\underline{\underline{F}},l}\to Y_{S,\underline{\underline{F}},k}$ for $k\leqslant l$, $\pi_k:Y_{S,\underline{\underline{F}}}\to Y_{S,\underline{\underline{F}},k}$, and $\mu_{S,\underline{\underline{F}},k}: Hom_{\underline{\underline{K}}}(\underline{\underline{L}},\underline{\underline{F}})\to Y_{S,\underline{\underline{F}},k}:\varphi\mapsto(\varphi(c)+\underline{m}_{\underline{\underline{F}},2k}, \varphi(t_i)(1+\underline{m}_{\underline{\underline{F}},k}):1\leqslant i\leqslant r)$.

Lemma 2.3. In the situation above, there exist $k\in\underline{N}$ and a finite subset S of $\underline{\underline{S}}(\underline{\underline{L}}/\underline{\underline{K}})$ such that the canonic map $Hom_{\underline{\underline{K}}}(\underline{\underline{L}},\underline{\underline{F}})\to\pi_{k,3k}(Y_{S,\underline{\underline{F}},3k})$ is bijective for each Henselian valued field extension $\underline{\underline{F}}$ of $\underline{\underline{K}}$.

Proof. By Lemma 2.2, there is a finite subset S of $\underline{\underline{S}}(\underline{\underline{L}}/\underline{\underline{K}})$ such that $\mu_{S,\underline{\underline{F}}}: Hom_{\underline{\underline{K}}}(\underline{\underline{L}},\underline{\underline{F}})\to Y_{S,\underline{\underline{F}}}$ is bijective for each valued field extension $\underline{\underline{F}}$ of $\underline{\underline{K}}$. As $K^\circ$ is of characteristic zero and $f^\circ\in O_{\underline{\underline{K}}^\circ}[X]$ is irreducible, it follows $f'^\circ(c^\circ)\neq 0$. As L/K is finite, we get $0\leqslant vf'(c)^n\in vK^\circ$ for some $n\geqslant 1$; therefore we may assume

$(f'(X)^n, nvf'(c)) \in S$. Similarly, since $h_i^{\circ}(c^{\circ}) \neq 0$, we may assume $(h_i(X)^n, nvh_i(c)) \in S$, $1 \leq i \leq r$. Now let $L'$ be the splitting field of $f$ over $K$. As $f^{\circ} \in O_{\underline{\underline{K}}}[X]$ is irreducible and separable, there exists $k' \in \underline{N}$ such that $0 \leq w(c_i - c_j) \leq k'vp$ for all valuations $w$ of $L'$ extending $v_K$ and for all roots $c_i \neq c_j$ of $f$. Let $k \in \underline{N}$ be such that $k \geq k'$, $kvp \geq v([L:K(c)])$ and $kvp \geq \alpha$ for $(g,\alpha) \in S$. For any Henselian valued field extension $\underline{\underline{F}}$ of $\underline{\underline{K}}$, we have to show that

1) $\pi_k : Y_{S,\underline{\underline{F}}} \longrightarrow Y_{S,\underline{\underline{F}},k}$ is injective, and

2) $\pi_{k,3k}(Y_{S,\underline{\underline{F}},3k}) \subset \pi_k(Y_{S,\underline{\underline{F}}})$.

1): Let $(x,\underline{\underline{y}})$, $(x',\underline{\underline{y}}') \in Y_{S,\underline{\underline{F}}}$ be such that $\pi_k(x,\underline{\underline{y}}) = \pi_k(x',\underline{\underline{y}}')$, i.e. $v(x-x') > 2kvp$ and $v(1-y_i y_i'^{-1}) > kvp$ $(1 \leq i \leq r)$. As $f(x) = f(x') = 0$ and $v(x-x') > 2kvp \geq k'vp$, it follows $x = x'$ and $b_i y_i^{m_i} = h_i(x) = h_i(x') = b_i y_i'^{m_i}$ for $1 \leq i \leq r$; thus $(y_i y_i'^{-1})^{m_i} = 1$ since $b_i \neq 0$ $(1 \leq i \leq r)$. Consider the polynomials $P_i(X) = X^{m_i} - 1 \in O_{\underline{\underline{F}}}[X]$, $1 \leq i \leq r$. By Newton's lemma, 1 is the unique root $\zeta$ of $P_i$ in $F$ subject to $v(\zeta - 1) > vm_i$. As $P_i(y_i y_i'^{-1}) = 0$ and $v(y_i y_i'^{-1} - 1) > kvp \geq v([L:K(c)]) \geq vm_i$, we obtain $y_i = y_i'$ for $1 \leq i \leq r$.

2): Let $(x,\underline{\underline{y}}) \in O_{\underline{\underline{F}}} \times F^{\times r}$ be such that $(x + m_{\underline{\underline{F}},6k}, y_i(1 + m_{\underline{\underline{F}},3k}) : 1 \leq i \leq r) \in Y_{S,\underline{\underline{F}},3k}$. Thus $vf(x) > 6kvp$; $vg(x) = \alpha$ for $(g,\alpha) \in S$ since $6kvp \geq kvp \geq \alpha$ for $(g,\alpha) \in S$; $v(1 - h_i(x)(b_i y_i^{m_i})^{-1}) > 3kvp$ since $(h_i(X)^n, nvh_i(c)) \in S$ and $3kvp \geq kvp \geq nvh_i(c) = nvh_i(x) \geq vh_i(x)$ for $1 \leq i \leq r$. We have to show that there exists $(x', \underline{\underline{y}}') \in Y_{S,\underline{\underline{F}}}$ such that $v(x-x') > 2kvp$ and $v(1 - y_i y_i'^{-1}) > kvp$ for $1 \leq i \leq r$. As $vf(x) > 6kvp$, $(f'(X)^n, nvf'(c)) \in S$ and $3kvp \geq kvp \geq nvf'(c) = nvf'(x) \geq vf'(x)$, it follows by Newton's lemma that there exists $x' \in O_{\underline{\underline{F}}}$ such that $f(x') = 0$ and $v(x-x') > 3kvp$. For $(g,\alpha) \in S$, $vg(x) = \alpha \leq kvp$ and hence $vg(x') = \alpha$. As $v(b_i y_i^{m_i}) = vh_i(x) \leq kvp$ and $v(1 - h_i)(b_i y_i^{m_i})^{-1}) > 3kvp$, it follows $v(1 - h_i(x')(b_i y_i^{m_i})^{-1}) > 2kvp$ for $1 \leq i \leq r$. Consider the polynomials $P_i(X) = X^{m_i} - h_i(x')(b_i y_i^{m_i})^{-1} \in O_{\underline{\underline{F}}}[X]$, $1 \leq i \leq r$. We get

$vP_i(1) > 2kvp$ and $vP_i'(1) = vm_i \leq v([L:K(c)]) \leq kvp$; therefore, by Newton's lemma, there exists $\eta_i \in O_{\mathbf{F}}$ such that $P_i(\eta_i)=0$ and $v(1-\eta_i) > kvp$. Set $y_i' = y_i \eta_i$ $(1 \leq i \leq r)$. Then $(x', \mathbf{y}') \in Y_{S,\mathbf{F}}$ and $v(x-x') > 2kvp$, $v(1-y_i y_i'^{-1}) > kvp$ $(1 \leq i \leq r)$, as contended.

Q.E.D.

Corollary 2.4. In the situation above, there exist $l \in N$ and a finite subset $S$ of $S(\mathbf{L}/\mathbf{K})$ such that the next statements are equivalent for a Henselian valued field extension $\mathbf{F}$ of $\mathbf{K}$:

i) $\mathbf{L}$ is $\mathbf{K}$-embeddable in $\mathbf{F}$.

ii) $Y_{S,\mathbf{F},l}$ is non-empty, i.e. $\mathbf{F}_l$ satisfies the following existential sentence with parameters from $\mathbf{K}_l$:

$$(\exists x \in O_{\mathbf{F},2l})(\exists y \in G^r_{\mathbf{F},l})\Big[f_l(x)=0 \wedge \bigwedge_{(g,\alpha)\in S} v_l g_l(x) = \alpha \wedge \bigwedge_{1\leq i\leq r} \Theta_l((h_i)_l(x)) = (b_i)_l y_i^{m_i}$$

## 3. A structure theorem for algebraic extensions of valued fields

The present section is devoted to a structure theorem for a class of algebraic extensions of valued fields of characteristic zero (see Theorem 3.2 below), which will be the main ingredient in the proof of Theorem II stated in Introduction. Its proof is based on the following natural generalization of Prestel-Roquette radical structure theorem [3], Theorem 3.8.

Proposition 3.1. Let $\mathbf{K}=(K,v)$ be a Henselian valued field of characteristic zero and $\mathbf{L}$ be an algebraic extension of $\mathbf{K}$ such that $K^o = L^o$. Then $L/K$ is generated by radicals, i.e. $L=K(T)$ where $T=\{t \in L^{\times} : \bigvee_{n\geq 1} t^n \in K\}$ is the multiplicative group of radical elements of $L/K$. The radical value group $vT$ equals the full value group $vL$ of $\underline{L}$ and the valuation map $v:T \to vL$ induces a group isomorphism $T/_{K^{x}} \xrightarrow{\sim} vL/_{vK}$. If $L/K$ is a finite extension then $[L:K]=(T:K^{x})$.

For a proof of Proposition 3.1 see [2] Proposition 1.1.

Theorem 3.2. Let $\underline{K}=(K,v)$ be a valued field of characteristic zero such that $\underline{\dot{K}}=(K,\dot{v})$ is Henselian, and $\underline{L}=(L,v)$ be a Henselian algebraic extension of $\underline{K}$. $\underline{L}$ is a directed union of finite nice extensions of $\underline{K}$.

Proof. Since $\underline{\dot{K}}$ is Henselian of residue characteristic zero, we may assume by [1] Proposition 16 that we have the following commutative diagram of valued fields:

$$\begin{array}{ccccc} \underline{K} & \longrightarrow & \underline{L'} & \longrightarrow & \underline{L} \\ \uparrow & & \uparrow & & \\ \underline{K}^{o} & \longrightarrow & \underline{L}^{o} & & \end{array}$$

where $L'$ is the composite $L'=K.L^{o}$ over $K^{o}$ with the valuation induced from $\underline{L}$. Note also that $K, L^{o}$ are linearly disjoint over $K^{o}$.

As $\underline{\dot{K}}$ is Henselian and $L'/K$ is algebraic, $\underline{\dot{L}'}$ is Henselian too. On the other hand, $\underline{L'}^{o}=\underline{L}^{o}$ is Henselian since $\underline{L}$ is Henselian; therefore $\underline{L'}$ is Henselian. By Proposition 3.1, it follows $L=L'(T)$ where $T=\{t\in L^{x}: \bigvee_{n\geq 1} t^{n}\in L'\}$ is the multiplicative group of radical elements of $L/L'$. As $K$, $L^{o}$ are linearly disjoint over $K^{o}$, we also obtain $\dot{v}L'=\dot{v}K$, i.e. $vL'=vK+vL^{o}$. By Proposition 3.1 we get the canonic isomorphisms

$T/_{L'^{x}} \xrightarrow{\sim} vL/_{vL'} \xrightarrow{\sim} \dot{v}L/_{\dot{v}K}$ .

Now we have to show that $\underline{\underline{L}}$ is a directed union of finite nice extensions of $\underline{\underline{K}}$; it suffices to show that each finite subextension of $\underline{\underline{L}}/\underline{\underline{K}}$ is contained in some nice finite subextension of $\underline{\underline{L}}/\underline{\underline{K}}$. As K is of characteristic zero, each finite subextension of L/K has the form K(a) for some $a \in L$. It follows $K(a) \subset L'(a) \subset \subset L'(\underline{\underline{t}})$ for some $\underline{\underline{t}}=(t_1,\ldots,t_s) \in T^s$. Let $T_{\underline{\underline{t}}}$ be the subgroup of T generated by $L'^{x} \cup \{t_1,\ldots,t_s\}$. Consider a direct decomposition into finite cyclic groups of the finite factor group $T_{\underline{\underline{t}}}/_{L'^{x}}$:

$$T_{\underline{\underline{t}}}/_{L'^{x}} = C_1 \times C_2 \times \ldots \times C_r.$$

Let $m_i$ be the order of $C_i$; then $m_1 m_2 \ldots m_r = (T_t : L'^{x})$. Let $t_i' \in T_{\underline{\underline{t}}}$ be such that $t_i'$ generates $C_i$ modulo $L'^{x}$; then $t_i'$ is of order $m_i$ modulo $L'^{x}$ and hence $t_i'^{m_i} = c_i \in L'$ $(1 \leq i \leq r)$. Obviously, $T_{\underline{\underline{t}}}$ is generated over $L'^{x}$ by $t_1',\ldots,t_r'$ , and $L'(\underline{\underline{t}}) = L'(t_1',\ldots,t_r')$, so we may assume without loss of generality that $s=r$ and $t_i=t_i'$ $(1 \leq i \leq r)$.

On the other hand, $a=P(\underline{\underline{t}})$, where $P \in L'[\underline{\underline{Y}}]$, $\underline{\underline{Y}}=(Y_1,\ldots,Y_r)$. Let $N \subset L^{o}$ be a finite extension of $K^{o}$ such that the coefficients of P and the elements $c_i$ $(1 \leq i \leq r)$ above belong to K(N), and M be the field $K(N)(\underline{\underline{t}})$ containing K(a). Denote by $T_{\underline{\underline{t}}}'$ the subgroup of $M^{x}$ generated by $K(N)^{x} \cup \{t_1,\ldots,t_r\}$; $T_{\underline{\underline{t}}}'$ is contained in the group of radical elements of M/K(N) and the canonic group morphism $T_{\underline{\underline{t}}}'/_{K(N)^{x}} \longrightarrow T/_{L'^{x}}$ is an embedding.

It remains to show that the valued field $\underline{\underline{M}}=(M,v)$ with the valuation induced from $\underline{\underline{L}}$ is a nice extension of $\underline{\underline{K}}$. As $K^{o}$ is of characteristic zero, $N=K^{o}(c)$ for some $c \in L^{o}$ . We may assume without loss of generality that $c \in O_{L^{o}}$ and its minimal monic polynomial $f \in O_{\underline{\underline{K}}^{o}}[X]$. Consequently, K(N)=K(c) and f is also the minimal polynomial fo c over K. Consider the commutative diagram

$$\begin{array}{ccccc} T'_{\underline{t}}/K(c)^{\times} & \xrightarrow{\mu} & vM/_{vK(c)} & \xrightarrow{\nu} & \dot{v}M/_{\dot{v}K} \\ \downarrow & & \downarrow & & \downarrow \\ T/_{L'^{\times}} & \xrightarrow{\sim} & vL/_{vL'} & \xrightarrow{\sim} & \dot{v}L/_{\dot{v}K} \end{array}$$

It follows that the canonic morphisms $\mu, \nu\mu$ above are embeddings. In particular, $(T'_{\underline{t}}:K(c)^{\times}) \leqslant (vM:vK(N)) \leqslant [M:K(c)]$ . Consider the canonic epimorphism of K(c) - algebras $K(c)[\underline{Y}] \to M$: $:\underline{Y} \mapsto \underline{t}$, and let J be its kernel; thus $M \simeq K(c)[\underline{Y}]/_J$. Let $J' \subset J$ be the ideal generated by the polynomials $Y_i^{m_i} - c_i$ $(1 \leqslant i \leqslant r)$. We claim that $J'=J$. Indeed, let $M'$ be the factor ring $K(c)[\underline{Y}]/_{J'}$ . Considering $M, M'$ as K(c) - spaces, we get $\dim M = [M:K(c)] \leqslant$ $\leqslant \dim M'$. By induction on r, we obtain $\dim M' = m_1 m_2 \ldots m_r =$ $= (T'_{\underline{t}}:K(c)^{\times}) \leqslant [M:K(c)]$, concluding that $\dim M = \dim M'$, i.e. $J=J'$. It follows also that $[M:K(c)] = (\dot{v}M:\dot{v}K)$.

In order to conclude that $\underline{M}$ is a nice extension of $\underline{K}$, it remains to show that $M \simeq K[X,\underline{Y}]/_I$ , where the ideal I is generated by f(X) and some polynomials $b_i Y_i^{m_i} - h_i(X)$ $(1 \leqslant i \leqslant r)$ subject to: $h_i \in O_{\underline{K}}[X]$ and $h_i^{\circ}(c) \neq 0$ $(1 \leqslant i \leqslant r)$.

As $\dot{v}K = \dot{v}K(c)$ and $c_1, \ldots, c_r \in K(c)^{\times}$, we obtain $c_i = d_i u_i$ with $d_i \in K^{\times}$, $u_i \in K(c)$, $\dot{v}(u_i) = 0$ $(1 \leqslant i \leqslant r)$. Thus $u_i = u_i^{\circ}(u_i u_i^{\circ -1})$ and $\dot{v}(1 - u_i u_i^{\circ -1}) > 0$. Consider the polynomials $P_i(X) = X^{m_i} - u_i u_i^{\circ -1} \in$ $O_{\underline{K}(c)}[X]$, $1 \leqslant i \leqslant r$. As $\dot{v}P'_i(1) = \dot{v}m_i = 0 < \dot{v}P_i(1)$, there is $z_i \in O^{\times}_{\underline{K}(c)}$ such that $z_i^{m_i} = u_i u_i^{\circ -1}$ according to Hensel's lemma applied to the Henselian valued field $(K(c), \dot{v})$. Thus $t_i^{m_i} = c_i = d_i u_i^{\circ} z_i^{m_i}$ $(1 \leqslant i \leqslant r)$. As $u_i^{\circ} \in K^{\circ}(c) = N$, we obtain $u_i^{\circ} = h_i(c) w_i^{-1}$ with $h_i \in O_{\underline{K}^{\circ}}[X]$, $w_i \in O_{\underline{K}^{\circ}}$. Let $t'_i = t_i z_i^{-1}$ , $b_i = w_i d_i^{-1} \in K^{\times}$ for $1 \leqslant i \leqslant r$. Therefore $M = K(c)(\underline{t}) =$ $= K(c)(\underline{t}') \simeq K[X,\underline{Y}]/_I$ thanks to the substitution $X \mapsto c$, $\underline{Y} \mapsto \underline{t}'$, the ideal I is generated by the polynomials $f(X)$, $b_i Y^{m_i} - h_i(X)$ $(1 \leqslant i \leqslant r)$, and $h_i \in O_{\underline{K}}[X]$, $h_i^{\circ}(c) \neq 0$ for $1 \leqslant i \leqslant r$, as contended.

Q.E.D.

## 4. Proof of the main result

Denote by $\underline{L}$ the first order language of valued fields, whose vocabulary contains, besides logical symbols, constants 0,1, function symbols for the field opperations (+,-,.,-1) with convention $0^{-1}=0$) and one unary predicate which in a valued field is interpreted as the valuation ring. For $k \in \underline{N}$, let $\underline{L}_k$ be the first order many sorted language of mixed-k-structures, whose vocabulary is the disjoint union of the vocabularies of rings (+,-,.,0,1), groups (.,-1,1) and ordered groups (+,-,0,≤), augmented by function symbols standing for the inter-sorts maps $\Theta_k$ and $v_k$ defined in Introduction. The language $\underline{L}$ and the languages $\underline{L}_k$ ($k \in \underline{N}$) are related via the following translation procedure.

Lemma 4.1. Let $k \in \underline{N}$ and $\psi(x_1,\dots,x_n;y_1,\dots,y_m;\xi_1,\dots,\xi_\ell)$ be an $\underline{L}_k$ - formula, where the $x_i$'s are ring variables, the $y_i$'s are group variables, and the $\xi_i$'s are ordered group variables. One assigns efectively to $\psi$ an $\underline{L}$-formula $tr_k(\psi)(z_1,\dots,z_{n+m+\ell+1})$ in such a way that for each valued field $\underline{K}$ of characteristic zero and residue characteristic exponent p, and for arbitrary $a_i \in O_{\underline{K}}$ $(1 \leq i \leq n)$, $a_i \in K^{\times}$ $(n+1 \leq i \leq n+m+\ell)$,

$$\underline{K}_k \models \psi(a_i + \underline{m}_{\underline{K},2k} : 1 \leq i \leq n;\ a_i(1+\underline{m}_{\underline{K},k}) : n+1 \leq i \leq n+m;\ va_i : n+m+1 \leq i \leq n+m+\ell)$$

iff $\underline{K} \models tr_k(\psi)(a_1,\dots,a_{n+m+\ell},p)$.

The proof is immediate.

Now let $\underline{K}=(K,v)$ be a valued field of characteristic zero and residue characteristic exponent p, and $\underline{L}=(L,v)$ be a Henselian algebraic extension of $\underline{K}$. Let $\underline{T}(\underline{L}/\underline{K})$ be the existential $\underline{L}$-theory with parameters from $\underline{K}$ described in Introduction. We are ready to prove Theorem II stated in Introduction.

(4.2) Proof of Theorem II

The implications i) $\to$ ii) and ii) $\to$ iii) are trivial, so it remains to prove the implication iii) $\to$ i).

First we may assume without loss of generality that $\dot{\underline{K}}$ is Henselian. For, otherwise consider the Henselization $(K', \dot{v})$ of $\dot{K}$. As $\underline{L}$, $\underline{F}$ are Henselian, $\dot{\underline{L}}$, $\dot{\underline{F}}$ are Henselian too and hence $(K', \dot{v})$ is identified with a common valued subfield of $\dot{\underline{L}}$ and $\dot{\underline{F}}$. Moreover, as the residue field of $(K', \dot{v})$ is $K^{\circ}$, $K'$ with the valuation determined by $v_{K^{\circ}}$ and $\dot{v}_{K'}$ is identified with a common valued subfield $\underline{K}'$ of $\underline{L}$ and $\underline{F}$. Obviously, $Hom_{\underline{K}}(\underline{L},\underline{F}) \simeq Hom_{\underline{K}'}(\underline{L},\underline{F})$, so it remains to observe that $\underline{T}(\underline{L}/\underline{K})$ and $\underline{T}(\underline{L}/\underline{K}')$ are logically equivalent since $\underline{K}_k \cong \underline{K}'_k$ for each $k \in \mathbb{N}$. Thus we may assume that $\dot{\underline{K}}$ is Henselian.

According to Theorem 3.2, $\underline{L}$ is a directed union of finite nice extensions of $\underline{K}$. By a standard compactness argument it suffices to show that $Hom_{\underline{K}}(\underline{M},\underline{F})$ is non-empty for each finite nice extension $\underline{M}$ of $\underline{K}$ contained in $\underline{L}$. As by hypothesis, $\underline{T}(\underline{L}/\underline{K}) \subset \underline{T}(\underline{F}/\underline{K})$, the last fact follows immediately by Corollary 2.4.

Q.E.D.

## References


1 J.Ax, A metamathematical approach to some problems in number theory, AMS Proc.Symp.Pure Math.20(1971), 161-190.

2 S.Basarab, Relative elimination of quantifiers for Henselian valued fields, Preprint Series in Math. (1986), INCREST, Bucharest.

3 A.Prestel-P.Roquette, Formally p-adic Fields, Lecture Notes Math.1050(1984), Springer.